RESEARCH STATEMENT

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Abstract. The purpose of this document is to present my relationship with mathematics. After skimming over my education before graduation, I present my current investigations during thesis, and propose a few directions for future research, before closing with acknowledgements.

1. Mathematical Education

Generating functions and symmetric groups. My first mathematical interests manifested mostly in euclidean geometry (like Ptolemy's theorem) and combinatorics (especially graph theory).

Yet, my earliest accountable research activity goes back to my years in classes préparatoires. After learning about Cayley's theorem, stating that every group could be represented in the permutation group of its own elements, I set on trying to understand the poset of subgroups in $\mathfrak{S}_n$. One may ask about the asymptotic shape of that poset as n goes to infinity: how many subgroups of each cardinal, how many chains or antichains of each length, and so on; but these questions were out of my reach. In hindsight, some of them can be approached from the theory of incidence algebras as developed by Rota, by analysing the zeta function of the poset.

Hence my attention shifted to the asymptotic distribution of conjugacy classes and of the order. That led to my first autonomous research project, which I presented at the entrance for ENS Lyon. Most of the substantial results are subsumed by Polya's enumeration theory and the investigations of Erdös and Tùran on problems of statistical group-theory. Let me mention two.

It is easy to compute the number of elements a_c in $\mathfrak{S}_n$ with cycle structure $c = (c_1, c_2, \dots)$, corresponding to a partition of $n(c) = c_1 \times 1 + c_2 \times 2 + \dots$, and deduce their generating series:

$$a_c = \frac{n!}{\prod_{j\geq 1} j^{a_j} a_j!} \quad \text{so} \quad \sum_{c\in\mathbb{N}^{(\mathbb{N}^*)}} a_c X^c \frac{T^{n(c)}}{n(c)!} = \exp\left(\sum_{j\geq 1} \frac{X_j T^j}{j}\right) \quad \text{where} \quad X^c = X_1^{c_1} X_2^{c_2} \dots$$

The coefficient of $T^n X^c$ is the probability that an element in $\mathfrak{S}_n$ belongs to c, and one may deduce that for a subset $S \subset \mathbb{N}^*$ such that $\sum 1/s$ converges, the random variables $c_s \colon \mathcal{S}_n \to \mathbb{N}$ counting the number of s-cycles converge as $n \to \infty$ to independent Poisson distributions of parameters $1/s$.

Concerning the distribution of orders $W_n(\sigma)$ of elements $\sigma \in \mathfrak{S}_n$ (equal to $\operatorname{lcm}(c_j)$), a theorem of Erdös and Tùran states that the law of the random variable $\log W_n$ converges as $n \to \infty$ to a normal distribution of mean $\mu_n = \frac{1}{2}\log(n)^2$ and variance $\sigma_n^2 = \frac{1}{3}\log(n)^3$.

Arithmetic of L-series. During my first year at ENS Lyon, my interests drifted to arithmetics. I enjoyed applying the power of generating functions to enumerate structures, allying the symbolic method and the complex analysis, to the Dirichlet series arithmetic functions. Hence I went to Strasbourg in the summer 2016 for a short internship with J.-P. Wintenberger, and although it went well, I came back with the desire to reconnect with geometry and combinatorics.

Topology and combinatorics of singular curves in the real projective plane. Soon after, I received an email from Etienne Ghys (with whom we had exchanged a couple of times during the school year, in particular about what would later become the starting point of my thesis), proposing me to read a first draft of his singular mathematical promenade [Ghy17]. This opened a door onto many new subjects of mathematics, and announced the beginning of our fruitful collaboration.

The initial problem was to understand the topology of an algebraic curve of the real plane, in the neighbourhood of a singularity. To such a singularity one can associate two combinatorial invariants: a chord diagram, and its interleaving graph. In [GS20] we described which diagrams and which graphs come from singularities of algebraic curves.

I took up this study in [Sim22] under Etienne's direction during the summer 2018. First, I exploited the operad structure of chord diagrams to enumerate the isotopy classes of singularities by means of techniques from analytic combinatorics. Then I described the global topology of an algebraic curve on a smooth real surface: there are several algebraic singularities, connected by arcs of smooth curves. Which topological configurations are algebraically possible? In the sphere, there are no obstructions. I could finally propose an estimation for the number isotopy classes of singular algebraic curves in the real projective plane by combinatorial methods relating to maps in surfaces.

This adventure also led to the encounter with my thesis advisor Patrick Popescu-Pampu, a specialist in the theory of complex singularities and their Milnor fibrations.

Curves in surfaces and Fricke polynomials. In the meantime, I learnt more about some of the topics I had discovered through my interaction with Etienne, including the topology of knots in three-manifolds, the actions of groups on hyperbolic spaces, and the so-called dessins d'enfants. My 2017 summer internship with Dennis Sullivan and Moira Chas was an opportunity to apply those and make progress on the following question.

Consider a closed orientable surface S of negative Euler characteristic χ. For every hyperbolic metric there is a unique geodesic in each homotopy class, these correspond to the set $\hat{\pi}$ of conjugacy classes in $\pi_1(S)$. One may thus define two equivalence relations on $\hat{\pi}$: two loops are hyperbolic equivalent if they have the same length for every hyperbolic metric, and are simply equivalent if they have the same intersection number with every simple loop. One may show, for instance using the "collar lemma", that hyperbolic equivalence implies simple equivalence. Moreover, we knew from work of Fricke that hyperbolic equivalence classes are finite but can be arbitrarily large. We thus wondered about the difference between those two relations, in particular if the simple equivalence classes are finite. Moreover, we asked for topological interpretations of these equivalence relations.

We obtained a complete picture for the case of the sphere with three punctures and the punctured torus by examining the monomials appearing in the Fricke polynomial of an element in the free group on two generators. A representation $\rho\colon F_2 \to \mathrm{SL}_2(\mathbb{C})$ is uniquely determined up to $\mathrm{SL}_2(\mathbb{C})$-conjugation at the target by the traces $x = \operatorname{Tr} a$, $y = \operatorname{Tr} b$, $z = \operatorname{Tr} ab$ of two generators and their product, and such traces can take all values in $\mathbb{C}^3$. In other terms $\mathrm{Hom}(F_2; \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ is an algebraic variety isomorphic to $\mathbb{C}^3$. The trace of an element c in F_2 defines a regular function on this variety $\operatorname{Tr} \rho(c) \in \mathbb{Z}[x, y, z]$, called its Fricke polynomial. We knew that two elements are hyperbolic equivalent when they have the same Fricke polynomial, and we showed that they are simply equivalent when their Fricke polynomials have the same Newton polytope.

Then I grasped the ideas for addressing closed surfaces of higher genera by recovering something like a Fricke polynomial, which is an invariant of curves in surfaces analogous to the Kauffman bracket of knots. I will say more about this train of thought in the next paragraph as I took it up during my last internship in 2019 with Julien Marché on character varieties, and this remains a source of inspiration for my current research.

2. Current research

Character varieties of surface groups and their compactifications. The hyperbolic metrics on S form its Teichmüller space $\mathcal{T}$ homeomorphic to an open ball of real dimension $3|\chi|$. It embeds into the space of real functions $\mathcal{F}$ over $\hat{\pi}$ with the product topology, by sending a metric m to its length function denoted $\gamma \mapsto i(\gamma, m)$. The set $\hat{\pi}$ also embeds into $\mathcal{F}$ by sending a loop α to its intersection function $\gamma \mapsto i(\gamma, \alpha)$. The weighted simple curves map to a cone whose completion identifies with the space of measured laminations $\mathcal{ML}$, a cone on a sphere of dimension $3|\chi| - 1$. In the projective space $\mathbb{P}(\mathcal{F})$, we find that $\mathbb{P}(\mathcal{T}) = \mathcal{T}$ is compactified by $\mathbb{P}(\mathcal{ML})$. The precise construction, which we owe to Thurston and Bonahon [Bon88], involves geodesic currents, that is Radon measures on the space of all geodesics in S, instead of functions on $\hat{\pi}$. The set of all loops has dense image in the space of currents and the intersection pairing extends to a symmetric bilinear pairing. The whole picture is analogous to the Lorentzian model for hyperbolic space with a cone $\mathcal{ML}$ tangent to a hyperboloid $\mathcal{T}$, and the intersection of currents plays the role of the metric. We deduce from this discussion that hyperbolic equivalence and simple equivalence amount to the equality of the function $i(\alpha, \cdot)$ and $i(\beta, \cdot)$ restricted to $\mathcal{T}$ and its boundary $\mathcal{ML}$ respectively.

From the algebraic viewpoint, the character variety $\mathcal{X}$ of S is defined as a quotient of the space of representations $\mathrm{Hom}(\pi_1(S), \mathrm{SL}_2(\mathbb{C}))$ by the conjugacy action of $\mathrm{SL}_2(\mathbb{C})$ at the target. It is a complex affine variety, whose ring of functions is generated by the trace functions $t_\alpha\colon \rho \mapsto \mathrm{Tr}\,\rho(\alpha)$ for $\alpha \in \pi$ with relations generated by $t_1 = 2$ and $t_\alpha t_\beta = t_{\alpha\beta} + t_{\alpha\beta^{-1}}$. One may use the latter relation to resolve intersections in α and express t_α as a linear combination $\sum n_\mu t_\mu$ of trace functions of multicurves, that is disjoint unions of simple loops $\mu = \mu_1 \cup \dots \cup \mu_k$, the trace of a union being the product of traces. In fact the trace functions of multicurves are linearly independent, so they form a basis of the algebra $\mathbb{C}[\mathcal{X}]$. This unique decomposition in the basis of multicurves is precisely the aforementioned Fricke polynomial of loops, analogous to the Kauffman bracket, and the collection of multicurves appearing as the "monomials" of such a polynomial defines a "polytope" in $\mathcal{ML}$.

Now the Teichmüller space of S identifies with a Zariski dense real component of $\mathcal{X}$, on which the trace functions restrict to $\mathrm{Tr}\,\rho(\alpha) = 2\cosh(i(\alpha, m)/2)$. Thus two loops are hyperbolic equivalent if and only if they yield the same trace functions on the character variety, that is the same Fricke polynomial. The idea is now to read off simple equivalence in terms of the highest degree "monomials" of those Fricke polynomials, which loosely speaking can be defined as extremal points of the "polytope" in $\mathcal{ML}$. To do this we consider the Riemann-Zariski compactification of the affine variety $\mathcal{X}$, which consists in the space of all valuations of its field of functions $\mathbb{C}(\mathcal{X})$ endowed with the Zariski topology. In [MS20], we focus on the valuations $v\colon \mathbb{C}(X) \to \mathbb{R} \cup \{\infty\}$ with are monomial with respect to the multicurve decomposition: those which satisfy $v(t_\alpha) = \max v(t_\mu)$. We call them simple because every such valuation arises from a unique $\gamma \in \mathcal{ML}$ by extending the expression $v_\gamma(t_\mu) = -i(\alpha, \mu)$. Now reformulating the simple equivalence of α and β as the equality of $v_\gamma(t_\alpha) = v(t_\beta)$ for all $\gamma \in \mathcal{ML}$, and using the max distributivity of v_γ, we find that it corresponds to the equality between the set of dominant monomials in the Fricke polynomials of α and β, or geometrically of the extremal points of the polytope in $\mathcal{ML}$ defined by their support.

These questions about equivalence of loops in surfaces served as a motivation for understanding the compactification of the character variety by simple valuations. Let me mention other results which ensued from this investigation. We introduced this valuative model in [MS20] and related it with Morgan-Shalen's compactification [Ota15] to show that the algebraic automorphism group of $\mathcal{X}$ is a finite extension of the modular group $\mathrm{Mod}(S)$. Then we explored it in [MS] to show the unitarity of Fricke polynomials, and recover Thurston's symplectic structure at $\gamma \in \mathcal{ML}$ in terms of residual values of Goldman's Poisson bracket $\{t_\alpha, t_\beta\}$ at v_γ (compare with [PP91, Gol86]).

Linking numbers of modular knots. Conjugacy classes in the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ appeared in the story of mathematics with the investigation of classes of real quadratic irrationals and integral binary quadratic forms with positive discriminant. An important invariant is given by the Rademacher function, which is an integer initially computed in terms of Dedekind sums. In a famous paper [Ati87] on the logarithm of the Dedekind η function, Atiyah showed that seven other functions appearing in diverse areas of mathematics coincide with Rademacher's: in this "Omnibus" theorem figures in particular the special value of a Shimizu L-series, further studied in [ADS83].

The hyperbolic conjugacy classes in Γ correspond to the hyperbolic geodesics in the modular orbifold $\mathbb{H}/\Gamma$. They lift to the periodic orbits for the geodesic flow in its unit tangent bundle $\mathrm{PSL}_2(\mathbb{R})/\Gamma$, called modular knots. This three-manifold is homeomorphic to the complement of the trefoil knot in the sphere, and Ghys identified the Rademacher number of a hyperbolic class with the linking number between that trefoil knot and the corresponding modular knot. His online article with Jos Leys [GL06] provides a visual introduction to the modular flow (it was the subject of my first discussion with Etienne Ghys in 2015).

Moreover [Ghy07] identified the modular flow in $\mathrm{PSL}_2(\mathbb{R})/\Gamma$ with the dynamical system arising from Lorenz equations, thus connecting the study of modular knots with the work of Birman and Williams [BW83]. In particular, he proved that the collection of all periodic orbits for the modular flow is isotopic to a link inside the Lorentz template. He concluded by asking for the meaning of the linking number between two modular knots, and the goal of my thesis is to provide something like an omnibus theorem for linking numbers of modular knots.

I obtained a first result in this direction by considering the action of Γ on the Farey tree. A hyperbolic element acts by translation along an oriented geodesic called its axis. For two hyperbolic elements $a, b \in \Gamma$ with distinct translation axes, define $\mathrm{cross}(a, b) \in \{-1, 0, 1\}$ to be their algebraic intersection (given by the cyclic order of the endpoints on the boundary), and if non zero let $\mathrm{cosign}(a, b) \in \{-1, 1\}$ compare their orientations along their intersection. Then I show that the linking number of the modular knots corresponding to the hyperbolic conjugacy classes A, B in Γ is given by the following sum:

$$\mathrm{lk}(A, B) = \tfrac{1}{2} i(A, B) + \frac{1}{2} \sum |\mathrm{cross}(a, b)| \times \mathrm{cosign}(a, b) \tag{1}$$

indexed by the pairs (a, b) in $A \times B$ modulo the diagonal conjugacy action of Γ, or equivalently by fixing a, b and then conjugating b by the elements in the double coset $\mathrm{Stab}(a)\backslash\Gamma/\mathrm{Stab}(b)$. The sum has finite support because $\mathrm{cross}(a, b) = 0$ except for a finite subset of indices.

Moreover, notice that the sum $\sum \mathrm{cross}(a, b)$ equals the symplectic intersection pairing between the modular geodesics, which identically zero for such an orbifold whose underlying topological space has no homology. We also recover that $\mathrm{lk}(A, B) + \mathrm{lk}(A, B^{-1}) = \sum |\mathrm{cross}(a, b)|$ equals the geometric intersection $i(A, B)$ between the modular geodesics, which is topologically obvious. Finally, the sum $\sum \mathrm{cross}(a, b) \times \mathrm{cosign}(a, b)$ resembles one of the descriptions we gave in [MS20] for the residual Goldman-Poisson bracket of loops A, B at a valuation (which yields an action on a simplicial tree in the theory of Bass-Serre [Ser77], and on a real tree in the generalisation by Morgan-Shalen).

Denote $\mathrm{len}(c)$ the minimum displacement length of an element $c \in \Gamma$ acting on the Farey tree. For instance if c has infinite order and is primitive, this corresponds to the sum of the integers appearing in the minimal even period for the continued fraction expansion of its fixed points. Observe that $\mathrm{cosign}(a, b) = \mathrm{sign}(\mathrm{len}(ab) - \mathrm{len}(ab^{-1}))$, and more precisely that if $\mathrm{cosign}(a, b) = 1$ then $\mathrm{len}(ab) = \mathrm{len}(a) + \mathrm{len}(b) > \mathrm{len}(ab^{-1})$. This expresses the linking in terms of the continued fractions arithmetic, and will help to analyse the series (3) in the next paragraph.

Limiting value of a function on the character variety. In the character variety X of the modular group Γ, the component coming from irreducible representations is isomorphic to an affine line, with function ring $\mathbb{C}[t]$. There is one valuation centered at infinity: it is minus the degree in the variable t, and in terms of Morgan-Shalen's correspondence it corresponds to the action of Γ on the Farey tree. Up to conjugation and change of variable, there is a unique one-parameter algebraic family of representations $\rho_t \colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$. Denote $a_t = \rho_t(a)$.

The real locus of the character variety contains the Teichmüller space of the modular orbifold, corresponding to the (faithful and) discrete representations of Γ in $\mathrm{PSL}_2(\mathbb{R})$. Those act on the hyperbolic plane and quotient to an orbifold obtained by opening the cusp of the modular orbifold into a trumpet, whose collar geodesic has length λ given by $t = 2\cosh(\lambda/2)$. The conjugacy class A of $a \in \Gamma$ yields an algebraic function on the character variety: its Fricke polynomial $\operatorname{Tr} A_t \in \mathbb{Z}[t]$, and it is now an elementary matter to show that it is unitary.

The second result of my thesis investigations consists in recovering formula (1) for the linking number of the modular knots A and B as the limiting value as $t \to \infty$ of a series $F_t(A, B)$ defined on the character variety, by summing an appropriate function $f_t(a, b)$ for (a, b) in $A \times B$ modulo the diagonal conjugacy action of Γ (naturally indexed by the double cosets $\operatorname{Stab}(a)\backslash\Gamma/\operatorname{Stab}(b)$).

Choosing $f_t(a,b) = [\![\operatorname{bir}(a_t, b_t) > 1]\!]/\operatorname{bir}(a_t, b_t)$ to be the inverse of the cross ratio $\operatorname{bir}(a_t, b_t)$ between the fixed points of a_t and b_t subject to the condition $[\![\operatorname{bir}(a, b) > 1]\!] \in \{0, 1\}$ that the axes intersect, we find a finite sum which can be written in terms of the cosine $\cos(a_t, b_t)$ of the angle between the oriented translation axes of $a_t, b_t \in \mathrm{PSL}_2(\mathbb{R})$ for the action on the hyperbolic plane:

$$\operatorname{Bir}_t(A, B) = \sum \frac{[\![\operatorname{bir}(a_t, b_t) > 1]\!]}{\operatorname{bir}(a_t, b_t)} = \tfrac{1}{2} i(A, B) + \frac{1}{2} \sum |\operatorname{cross}(a_t, b_t)| \times \cos(a_t, b_t) \tag{2}$$

Deforming the representation by letting t go to infinity amounts to opening the cusp of the modular orbifold: the angles tend to $0 \bmod \pi$, and $\cos(a_t, b_t) \to \operatorname{cosign}(a, b)$, so we recover the series (1).

Note that if we take $f = \operatorname{cross} \times \operatorname{cosign}$, then we obtain an analog of Wolpert's cosine formula computing the Poisson bracket of two loops in a closed hyperbolic surface, and the limiting result can be compared with Bonahon's interpretation for Wolpert's cosine formula at a measured lamination.

Of course other choices for $f_t(a, b)$ can yield $F_t(A, B)$ with $\operatorname{lk}(A, B)$ as limit, and the obvious one is not necessarily the only interesting one when considering the behaviour at the finite places of X, or when it comes to relationships with topology, hyperbolic geometry, or arithmetics of 3-manifolds. I suggest other examples motivated by the study of Poincaré series as future research projects.

The following series defines an algebraic function on the character variety, all of whose terms have degree $\operatorname{len}(a) + \operatorname{len}(b)$, and which evaluates to $\operatorname{lk}(A, B) - \frac{1}{2} i(A, B)$ at the infinite place $v = -\deg$:

$$\mathrm{P}_t(A, B) = \frac{1}{2} \sum |\operatorname{cross}(a, b)| \left(\operatorname{Tr} a_t b_t - \operatorname{Tr} a_t b_t^{-1}\right). \tag{3}$$

Note that, if we omit the factors restricting the series (2&3) to the pairs of axes which intersect, and sum instead over all double cosets, then we obtain an infinite series, but it diverges...

Let me finish by saying that several works have been published on this theme. For instance [DIT17] considers the linking numbers $\operatorname{lk}(A + A^{-1}, B + B^{-1})$ between cycles obtained by lifting a geodesic and its inverse: this number amounts to the geometric intersection $i(A, B)$ of the modular geodesics. Furthermore [DV21] considers deformations of an arithmetic nature for these intersection numbers. Unfortunately, none of these address the actual linking number. Moreover, their approach is motivated by the arithmetic of modular forms, while mine is inspired by the geometry of the character variety (but the latter introduces very similar series, which may be related).

3. Future projects

I now present several directions of research, which I believe interesting, and that I wish to pursue. Some of the material presented to introduce them is part of my current research, which leads to open questions still out of my reach.

Topology of Gauss composition & arithmetic of Fricke polynomials. In his *Disquisitiones Arithmeticae*, Gauss discovered a composition law between integral binary quadratic forms with a same discriminant Δ, which descends to classes under the action of Γ by change of variables, and defines a finite abelian group $\mathrm{Cl}(\Delta)$.

There is a dictionary between hyperbolic matrices of trace Tr in the modular group and integral binary quadratic forms with discriminant $\Delta = \mathrm{Tr}^2 - 4 > 0$, which is equivariant with respect to the action of Γ by conjugacy, and change of variable. Hence the classes in $\mathrm{Cl}(\Delta)$ correspond to modular geodesics or modular knots with a given length, and one may ask for interpretations of Gauss composition in terms of their geometry and topology.

A first step would be to partition the classes in $\mathrm{Cl}(\Delta)$ according to the value of their Fricke polynomial, or equivalently the Alexander polynomial $\Delta(A_t) = (\mathrm{Tr}\, A_t)^2 - 4$ whose name will be explained in a moment. We are now back to considerations about trace equivalence or length equivalence, but this time for loops in the modular orbifold.

The unit tangent bundle of the modular orbifold is a Seifert circle-fibration $\mathbb{S}^1 \to \mathbb{U} \to \mathbb{M}$, inducing at the level of fundamental groups a central extension $\mathbb{Z} \to B_3 \to \Gamma$, in which $B_3 = \pi_1(\mathbb{U})$ is isomorphic to the braid group on three strands. A hyperbolic conjugacy class A in Γ corresponds to a modular geodesic, which lifts to a modular knot corresponding to a conjugacy class k_A in B_3. Such a lift is uniquely determined by its abelianisation, which is given by the linking with the trefoil knot, and must therefore equal the Rademacher invariant of A. Note that from the viewpoint of braids, a conjugacy class in B_3 is given by its closure, that is a link $\hat{\beta}_A$ in a solid torus.

Now recall that up to conjugation and change of variable, there is a unique one-parameter algebraic family of irreducible representations $\rho_t \colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$. Hence the well known Burau representation $B_3 \to \mathrm{SL}_2(\mathbb{C}[t])$ quotients to (a privileged choice for) the representation ρ_t. From this coincidence (which I am surely not the first to notice) I may show that up to a change of variable, the aforementioned Alexander polynomial $\Delta(A_t) = (\mathrm{Tr}\, A_t)^2 - 4$ of the hyperbolic class A is equal to the Alexander polynomial of the braid closure $\hat{\beta}_A$ multiplied by a power of t given by half the Rademacher number of A.

Alexander polynomials of links (especially closures of three stranded braids) are the subject of many results and conjectures: all these may thus be applied to the Alexander polynomials $\Delta(A_t)$ of hyperbolic modular classes. In particular, I hope to relate their divisibility properties by cyclotomic polynomials with the arithmetics of Gauss composition.

Special values of Poincaré Series. As mentionned just before relation (3), other diagonally invariant functions $f_t(a,b)$ may yield interesting series defining functions $F_t(A,B)$ on the character variety whose limit at the infinite place recovers the linking number. Various motivations (special values of Poincaré series, and McShane's identity) suggest to chose $f_t(a,b) = (x + \sqrt{x^2-1})^{-s}$ where $x = -\frac{1}{2}\mathrm{Tr}(a_t b_t)$. This can also be written $e^{-i\theta}$ where θ is the angle between the oriented geometric axes of a_t and b_t when they intersect and e^{-sl} where l is the length of the ortho-geodesic arc γ connecting the geometric axes of a_t and b_t when they are disjoint. In formula:

$$L_t(A,B) = \sum (x + \sqrt{x^2-1})^{-s} = \sum_{[a]\perp\gamma\perp[b]} \exp(-sl_\gamma) - \sum_{p\in[a]\cap[b]} \exp(-si\theta_p). \tag{4}$$

So the sum over all double cosets splits as a finite sum computable in terms of 2 above, and an infinite series which converges for $\Re(s) > 1$ (the topological entropy for the action of Γ on the hyperbolic plane). The infinite sum resembles the univariate Poincaré "theta-series" which also appeared in the works of Eisenstein: those admit meromorphic continuation to $s \in \mathbb{C}$ and their special values in the variable s have been of interest for arithmetics and dynamics. The earliest appearance of such bivariate series I could find is in [For29, Section 50].

When $t = \infty$ and $s = 1$, the real part of the finite sum evaluates to $\mathrm{lk}(A, B) - \frac{1}{2} i(A, B)$, but one may wonder about the infinite series (now the order in which we take limits in s and t may import). More generally, one strategy to relate modular topology and quadratic arithmetics is to choose f with appropriate symmetries and analyticity properties so that the sum over all double cosets can be understood: then one deduces a relationship between a topologically meaningfull finite sum, and the infinite series whose special values may be of interest in arithmetics.

Generalisations to other Fuchsian groups. Of course, some of these considerations may be subject to generalisations for other Fuchsian groups instead of the modular group.

In particular, I have found a natural generalisation of the Rademacher invariant for conjugacy classes in those Fuchsian groups which are isomorphic to an amalgam of cyclic groups. I note that in this direction, a recent preprint [MU21] extends the Rademacher function to the triangular orbifolds (p, q, ∞) and investigates arithmetic analogs of the Dedekind eta function: this comes close to some of my considerations but my approach is topological and group theoretical.

For a general Fuchsian group Γ, one may similarly define bivariate functions on the character variety such as (2&3) and ask for interpretations of their limiting values at a measured lamination on the boundary. This connects to the work [MS] in which we considered the Poisson bracket. However one tantalising question is to appropriately define the linking numbers of geodesics lifted in the unit tangent bundle of the Fuchsian orbifold $\mathbb{H}/\Gamma$, and relate them to such limiting values. More generally, a vast program consists in reading topological invariants of loops and algebraic properties of character varieties in terms of the Newton polytopes for certain functions on the character variety, such as the univariate $\mathrm{Tr}\, A_t$ or the bivariate $F_t(A, B)$.

In another direction, I wish to extend such bivariate functions $F_\rho(A, B)$ for semi-conjugacy classes of representations $\rho \colon \Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ and for A, B in the space of all geodesic currents of Γ, and say in what sense such a function could be interpreted as a "differential form" on (an appropriate subspace in) the first bounded cohomology group $H^1_b(\Gamma; \mathbb{R})$.

4. Apology & Acknowledgements

I wish to apologise for the constant use of the first person, which is more natural from the narrative standpoint and serves to highlight my contributions. Yet, none of those would have been possible without the support and interaction with my teachers and mentors, so I wish to thank them all: my teachers in école préparatoire Roger Mansuy and Yves Duval, my thesis directors Étienne Ghys and Patrick Popescu-Pampu, and my other mentors or collaborators Dennis Sullivan, Moira Chas and Julien Marché. Progress in mathematics often arises from a collision of ideas between several people in the present, based on what they learnt from the past. It is both hard and vain to disentangle these interactions and claim ownership for ideas: what remains are stories with several actors, and the result of their common achievements.

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