Algebra of loops in surface orbifolds and volumes in their moduli spaces

Christopher-Lloyd Simon

Abstract

Consider an orbifold F whose fundamental group π admits a faithful and discrete representation $\pi \to \mathrm{PSL}_2(\mathbb{R})$. The homotopy classes of unoriented closed loops in F correspond to the mutually inverse pairs conjugacy classes $\overline{\pi}$. Two such collections of loops α, β have a geometric intersection number $\mathrm{i}(\alpha, \beta)$. We explain how to recover the intersection pairing $\mathrm{i}\colon \overline{\pi} \times \overline{\pi} \to \mathbb{N}$ as some kind of Killing form over the fundamental group, which we define using its adjoint action on the graded Lie algebra associated to its descending central series. The geodesic representatives for the multiloops α and β lift to knots in the unit tangent bundle, which have a linking number $\mathrm{lk}(\alpha, \beta)$ provided they are homologically trivial. We relate this to the Poisson bracket $\{t_\alpha \mid t_\beta\}$ of their trace functions defined over the symplectic variety of $\mathrm{PSL}_2(\mathbb{R})$-characters. This character variety has a compactification by valuations, whose geometric part corresponds to the space of measured laminations $\mathcal{ML}$. The space $\mathcal{ML}$ has a natural piecewize linear integral structure, a privileged measure in the Lebesgues class, and an intersection function which extends the one defined on simple loops. We define the Newton polytope for the trace functions t_α as a finite collection $\Psi(\alpha)$ of integral measured laminations, and its the volume $\mathrm{vol}(\alpha)$ as that of the dual polytope with respect to the intersection form defined by $\Psi(\alpha)^* = \max\{\mathrm{i}(\psi, \cdot) \mid \psi \in \Psi(\alpha)\} \leqslant 1$. We relate the mixed volumes of the dual polytopes $\Psi(\alpha)$ and $\Psi(\beta)$ to the aforementioned Poisson bracket, intersection and linking numbers.

Keywords. Planar tiling group, character variety, Teichmüller space, symplectic structure; orbifold, Goldman Poisson bracket; skein algebra, quantization; geodesic currents, measured laminations, intersection form.

Contents

Introduction

Motivation and plan of the paper

Warning. Almost everything we shall say is well know when π is the fundamental group of a closed orientable surface F. One of the reasons for these notes is to try and understand to what extend those constructions and results generalize when π is a more general group like $\mathrm{PSL}_2(\mathbb{Z})$.

Notations

The symbol $\mathbb{N} = \{0, 1, \dots\}$ refers to the set of non-negative integers. A plus $+$ indexing a set of numbers means we only consider non-negative elements; a star $*$ (resp. a cross $\times$) in exponent means we have taken out 0 (restricted to invertibles for some multiplicative law). For instance $\mathbb{N} \setminus \mathbb{N}^* = \{0\}$ and $\mathbb{Z}^\times = \{\pm 1\}$ while $\mathbb{Q}_+ \cap \mathbb{Q}_- = \{0\}$ and $\mathbb{Q}^\times = \mathbb{Q}^*$. A *collection*, sometimes called multiset in the cominatorial literature, is a set in which elements may appear with multiplicity.

If unspecified, spaces are given their most natural topology, this holds for instance when performing products or taking quotients. Manifold topology is done in the smoothly differentiable category, with natural Fréchet topology over spaces of maps (like loop spaces $\mathbb{S}^1 \to F$). In what follow, F is a Fuchsian orbifold with fundamental group π; but for the time being we may think of a compact orientable surface with boundary and s special marked points instead of singularities.

We call an oriented h-*loop* in F the homotopy class of a smooth map from the circle $\mathbb{S}^1$ to F. The oriented h-loops correspond to conjugacy classes $\vec{\pi}$ of the fundamental group. An oriented h-loop is *primitive* when represented by an element in π generating a maximal cyclic subgroup; it is *torsion* when represented by a torsion element. It is *peripheric* when homotopic into a boundary component or hole. An oriented g-*loop* in F is a generic immersion of $\mathbb{S}^1$ into F, considered up to positive reparametrization. An immersion has no cusp-point singularities, and it is generic when the only singular points in the image correspond to transverse double intersections; in particular there are no self-tangencies or triple points. An oriented isotopy class of g-loops, called i-*loop* for short, corresponds to a connected component in the space of g-loops (hence all topological features of a g-loop are defined at the level of its isotopy class g-loops will serve only in this paragraph).

In each case, replacing the circle at the source by a finite disjoint union of $k \in \mathbb{N}$ circles, we obtain the concept of labelled oriented *multiloop with* k *strands*, which may be preceded by h, g, i. We shall always consider unlabeled multiloops, quotienting by the symmetric group action. The unoriented counterparts are defined similarly. For instance unoriented h-loops, correspond to the set $\overline{\pi} = \vec{\pi} / \mathrm{inv}$ of mutually inverse pairs of conjugacy classes. We sometimes forget the adjective (un)oriented or the prefix g, h, i when referring to an object whose nature has been previously defined or when it is clear from the context to which type of loops the assertion applies.

Two g-multiloops α and β are in *general position* if their union is generic, that is a g-multiloop with $k_\alpha + k_\beta$ strands; the same definition holds for i-multiloops. We denote $\mathrm{si}(\alpha)$ the geometric self intersection number of an i-multiloop, and $\mathrm{i}(\alpha, \beta)$ the geometric intersection number between two i-multiloops in general position. Self-intersection and intersection numbers are defined for h-loops by taking the minimum of the above quantities over i-loop representatives. An i-loop is *taut* if it has minimal geometric self-intersection number among elements in its homotopy class. A loop α is *simple* when $\mathrm{si}(\alpha) = 0$; it is *essential* when there exists another loop $\beta \neq \alpha$ such that $\mathrm{i}(\alpha, \beta) \neq 0$.

Simple i-multiloops are precisely the embeddings of closed one-dimensional manifolds into F, up to isotopy. We denote Φ the set of all such unoriented submanifolds including the emptyset $\emptyset$. A simple h-multiloop φ with (unlabeled and) non-trivial strands has a unique taut i-representative, and is tantamount to the collection $\{\varphi_j\}$ of its strands which are disjoint non-trivial simple loops; we say that φ_j belongs to φ and write $\varphi_j \in \varphi$. We call *states* and denote Ψ the set of unoriented submanifolds whose strands are neither trivial nor torsion (encircling one of the s marked points).

Finally, when F has boundary or punctures, all such loop definitions may be adapted to include arcs which are proper maps from an interval (with zero, one or two ends) mapping the (at most two) boundary points into the boundary of F (proper implies that the ends must diverge into punctures). The corresponding sets will be agremented with primes and their elements qualified as dual.

0 Fuchsian groups and orbifolds

0.1 Orbifolds

General orbifolds. We use here the notion of an orbifold, as defined in [Thu97, Sco83], that is a countable topological Hausdorff space locally modeled on quotients of the upper half space by finite groups : there is a countable covering by open sets U_j presented as quotients $\phi_j \colon \widetilde{U}_j \to \widetilde{U}_j/\Gamma_j \simeq U_j$ of open sets $\widetilde{U}_j$ in the upper half plane by finite groups Γ_j, such that every inclusion $U_i \subset U_j$ is covered by an equivariant embedding $\widetilde{U}_i \to \widetilde{U}_j$ with respect to an inclusion $\Gamma_i \to \Gamma_j$ so that the diagram commutes with projections. (in other words we have a presheaf of finite *group actions*). They form a category whose continuous maps must commute with the group actions in the local models. The *stabilizer* of a point x is the smallest group Γ_j appearing among the local models of its neighborhoods $U_j \ni x$ belonging to a maximal covering system (in presheaf language, it is the group acting on the stalk above x, and can be expressed as an inverse limit). A point is *singular* if it has non trivial stabilizer. From now on, orbifolds will be connected: one can define the *orbifold fundamental groups* either in terms of paths, or deck transformations (there holds an analog of covering theory with universal covers in the connected orbifold category). As usual, the fundamental groups are based, and two of them are related by an isomorphism which is unique up to post conjugation. We omit base points from notations, but keep them in mind. The orbifold has *finite type* when its fundamental group does. Finally an orbifold is *developable* (sometimes called good) when the orbifold universal cover is smooth (trivial stabilizers Γ_j), or equivalently when isomorphic (as an orbifold) to the quotient of a manifold by a subgroup of diffeomorphisms acting properly and discontinuously. In general the underlying topological space of an orbifold may fail to be a manifold (as it is the case for the quotient of a 3-ball by antipody, which is a cone on $\mathbb{RP}^2$), but this phenomenon does not occur in the 2-dimensional case which concerns us.

Type of a surface orbifold. Let F be a 2-dimensional connected orbifold with *compact* boundary components, in the end we shall only consider the finite type ones with negative Euler characteristic and no mirror reflectors, but let us explain those last three conditions starting from the general case. The orbifold F has *type* (g, h, b, c, d, m) where g is the genus of the underlying topological surface F_0 with boundary, h is the number of punctures, b is the number of (circular) boundary components, m is the number of mirror reflection lines (singular points with $\mathbb{Z}/2$ stabilizers, which are boundary components of F_0 but not considered as boundary of F), and $c = \{c_1, \dots, c_s\}$ (respectively $d = \{d_1, \dots, d_r\}$) is a possibly infinite collection of elements in $\mathbb{N}_{\geqslant 2}$ corresponding to the orders of the conical singularities (reflector corners). To be precise c_j (or $2d_i$) is the order of the cyclic (or dihedral) stabilizer of the point in a local parametrization. The compactness assumption on boundary components implies they are circles, so for instance we exclude the closed upper half space and any of its variations (adding handles, boundary, singularities, or taking connected sums). The orbifold F has finite type if an only if it has a finite number of singularities and boundary components meaning that $g, p, b, s, r, m < \infty$. The type does not determine the isomorphism type of F, unless we know the distribution of corner reflectors on the reflector lines. Punctures may sometimes be thought as conical singularities with infinite order, but also as boundary components with zero length (for some compatible metric structure). Indeed, punctures and boundary components are homotopic relatively to the complement of a neighborhood, they only remain distinguished up to homeomorphism ; and the *interior* of F is homeomorphic to an orbifold with circular boundaries replaced by punctures. We use the word *holes* for puncture or circular boundary, and set $h = p + b$.

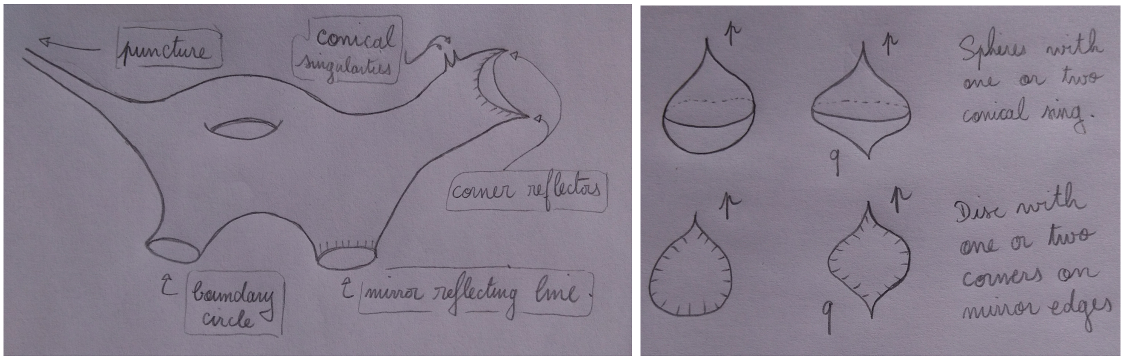


Figure 1: Different kind of singularities. The bad orbifolds.

Euler characteristic and uniformization. As for closed surfaces, developable orbifolds are uniformized by exactly one of the three planar geometries $\mathbb{G} \in \{\mathbb{S}, \mathbb{E}, \mathbb{H}\}$ according to the sign of the orbifold Euler characteristic:

$$\chi(F) = 2 - 2g - p - b - \sum_j \left(1 - \frac{1}{c_j}\right) - \sum_i \left(1 - \frac{1}{2d_i}\right)$$

More precisely F can be obtained as the quotient of a connected and contractible *submanifold with geodesic boundary* $\widetilde{F}$ in $\mathbb{G}$, by the action of a discrete subgroup of isometries $\pi \subset \mathsf{G}^{\pm} = \mathrm{Isom}^{\pm}(\mathbb{G})$ which is isomorphic to its orbifold fundamental group π. A sufficient condition for developability is to carry a non-positively curved metric (with appropriate angles at singularities), which in dimension 2 amounts to having non-positive orbifold Euler characteristic. More precisely, there are four infinite families of bad orbifold surfaces pictured on the right hand side of figure 1: the sphere with one, or two conical singularities of different orders; a disc whose boundary corresponds to one reflector line meeting at a reflector angle, or two reflector lines meeting at reflector angles with different orders.

So any orbifold with negative Euler characteristic is isomorphic to the quotient of the hyperbolic plane $\mathbb{H}$ by a discrete subgroup of $\mathrm{Isom}^{\pm}(\mathbb{H}) = \mathrm{PGL}_2(\mathbb{R})$. Now suppose it has no mirror reflecting lines (ergo no corner reflectors neither so that $r = 0 = m$, in particular it is *uniquely characterized* up to isomorphism by its type, which may be abbreviated (g, p, b, c)); and assume furthermore it is orientable. Those last two conditions (which can always be simultaneously achieved by taking a double cover) imply together that the orbifold F under consideration is homeomorphic to the quotient of a contractible subset $\widetilde{F}$ in $\mathbb{H}$ by a discrete subgroup π of $\mathrm{PSL}_2(\mathbb{R})$.

This submanifold $\widetilde{F}$ is the image by a developing map. In absence of boundary, it is the whole of $\mathbb{H}$, and F is homeomorphic to $\mathbb{H}/\pi$. In presence of boundary, $\widetilde{F}$ has geodesic boundary, and F can equivalently be obtained as the *convex core* inside the complete quotient $\mathbb{H}/\pi$. This complete quotient has trumpets whose collar geodesics project to the boundary components of F ;by tightening each trumpet to a cusp while shrinking the collar geodesic as it goes to infinity, we obtain an orbifold homeomorphic to both $\mathbb{H}/\pi$ and the interior of F, obtained from the latter by replacing each boundary component by a puncture. So under uniformization, geodesic boundary circles are the truncation of (infinite area) trumpets at the collar, and punctures are trumpets with zero length collar (finite area). We now replace F by its interior, having $h = p + b$ holes.

The hyperbolic metrics on F correspond (by their holonomy representations) to the discrete embeddings $\rho \colon \pi \to \mathrm{PSL}_2(\mathbb{R})$ up to conjugacy at the target, such that $\mathbb{H}/\rho(\pi)$ is homeo to F.

Fuchsian orbifolds and lattices. From now on, F will be a connected orientable 2-dimensional orbifold with negative Euler characteristic, no boundary and no mirror reflection lines. We baptise this a *Fuchsian orbifold*: its fundamental group π admits a *Fuchsian representation* $\pi \to \mathrm{PSL}_2(\mathbb{R})$, meaning faithful and discrete. Indeed, a hyperbolic metric on F defines holonomy representations which are Fuchsian, and uniquely determined up to conjugacy at the target.

When it has finite type, thus characterized up to homeomorphism by the triple (g, h, c), we call it a *lattice orbifold* as it is *homeomorphic* to the quotient of $\mathbb{H}$ by a *lattice* in $\mathrm{PSL}_2(\mathbb{R})$, which is a discrete subgroup with finite covolume. If furthermore compact (no holes), thus characterized up to homeomorphism by the pair (g, c), then all corresponding lattices in $\mathrm{PSL}_2(\mathbb{R})$ are cocompact.

Several orbifolds may have the same group (like a holed torus and a three holed sphere), so the group does not determine the orbifold. When it does, we may call it *uniform* (non-standard terminology) because this means exactly one homeomorphism type (one form) occurs among the quotients $\mathbb{H}/\rho(\pi)$ as $\rho\colon \pi \to \mathrm{PSL}_2(\mathbb{R})$ ranges over the set of all its Fuchsian representations.

Peripheral multiloop. Let F be an oriented Fuchsian orbifold with fundamental group π. Recall the identification between (inverse pairs) of conjugacy classes in π and (unoriented) homotopy classes of loops in F. The naturally oriented primitive peripheric loops (encircling one hole) together form the *peripheral multiloop* δ corresponding to a *set* of primitive conjugacy classes $\delta_j \in \overrightarrow{\pi}$. Since F has contractible universal cover, it behaves like a $K(\pi, 1)$, and group homology of the pair (π, δ) mimics that of the oriented embedding $\delta \to F$. In particular, the relative homology of (π, δ) is that of the fundamental group for the closed orbifold with each end δ_j contracted to a point. So the map in $\mathbb{Z}$-homology $\bigoplus \mathbb{Z}\delta_j \simeq H_1(\delta) \to H_1(\pi) \simeq \pi/[\pi, \pi]$ has rank one kernel generated by $[\delta] = \delta_1 \dots \delta_h$ (well defined thanks to the orientations of the δ_j regardless of the labeling) to the power $\mathrm{lcm}(c_j)$. By Mayer-Vietoris, this kernel is the image of $H_2(\pi, \delta)$, generated by the canonical class ω induced by the orientation of F. The rank-one kernel reflects a surjectivity condition saying that we have gone around *all* peripheric strands, while the order $\mathrm{lcm}(c_j)$ cokernel reflects the primitivity assumption. So we have a boundary map $H_2(\pi, \delta) \to H_1(\delta)$ sending $\omega \mapsto \mathrm{lcm}(c_j) \cdot [\delta]$, uniquely defined up to the symmetric group permuting the generators δ_j. Switching orientation reverses the signs of all generators ω and δ_j simultaneously. We call (π, δ) a *framed Fuchsian group*.

Modular groups. Denote $\mathrm{Mod}^{\pm}(F) = \pi_0 \mathrm{Diff}^{\pm}(F) = \mathrm{Diff}^{\pm}(F)/\mathrm{Diff}_0(F)$ the *full modular group*. Its index two orientation preserving subgroup $\mathrm{Mod}(F)$ is generated by Dehn twists, and half-twists along simple loops encircling two conical singularities or holes of the same order (c_j or ∞). It extends $B(F, (z_j, c_j)) \to \mathrm{Mod}(F) \to \mathrm{Mod}(\overline{F}_0)$ the modular group for the underlying topological surface with holes filled in, by the *surface braid group* on $h + s$ points z_j respecting their partition into same orders, modulo torsion relations $\sigma_{ij}^{m_{ij}}$ where $m_{ij} = 2 \gcd(c_i, c_j)$ and $\gcd(\infty, c_j) = c_j$.

It contains the *pure modular group* $\mathrm{PMod}(F) \to \mathrm{Mod}(F)$ whose elements induce the trivial permutation on the $h + s$ holes and singularities, with quotient a product of symmetric groups. The pure modular group fits in a similar extension $PB(F, (z_j, c_j)) \to \mathrm{PMod}(F) \to \mathrm{PMod}(\overline{F}_0)$. Intermediate *half*-pure modular subgroups can be defined, by fixing only the holes for instance.

The Dehn-Nielen-Baer theorem gives an isomorphism between $\mathrm{Mod}^{\pm}(F)$ and the group $\mathrm{Out}(\pi, \overline{\delta})$ of relative outer automorphisms preserving the set $\overline{\delta} \subset \overline{\pi}$ of inverse-pairs of peripheral conjugacy classes as a whole. The orientation preserving mapping classes correspond to those acting trivially on $H_2(\pi, \delta)$, forming the subgroup $\mathrm{Out}(\pi, \delta)$ of relative outer automorphisms. This restricts to an isomorphism from the pure modular group acting identically over the set of unsessential loops to the subgroup of outer automorphisms acting identically on the peripheral and torsion classes.

0.2 Fuchsian groups.

We denote $\mathsf{G} = \mathrm{PSL}_2(\mathbb{R})$. An abstract group π is called *Fuchsian* if isomorphic to a discrete subgroup of G; in other words it admits a faithful and discrete representation $\pi \to \mathsf{G}$, called a *Fuchsian representation*. From the preceding discussion, any Fuchsian representation ρ of a (finite type) group π presents it, up to conjugation, as the orbifold fundamental group of a (finite type) Fuchsian orbifold $F = \mathbb{H}/\rho(\pi)$. Now denoting (g, h, c) its type, and F_∞ the manifold obtained from F_0 by removing the conical singular points, an immediate application of the Van-Kampen theorem shows that π is the quotient of $\pi_1(F_\infty)$ by the relations $\tau_j^{c_j}$ where τ_j correspond to the loops around the newly created holes. From this we deduce the following algebraic presentation:

$$\pi = \left\langle \lambda_1, \mu_1, \dots, \lambda_g, \mu_g; \delta_1, \dots, \delta_h; \tau_1, \dots, \tau_s \mid \prod [\lambda_j, \mu_j] = \prod \delta_j \prod \tau_j \quad \text{and} \quad \tau_j^{c_j} = 1 \right\rangle$$

This computation runs parallel to those performed on fundamental groups of 3-manifolds undergoing (or presented by) Dehn surgery operations. Actually, the 3-manifold setting somewhat contains the 2-orbifold one, by considering the unit tangent bundle $\mathrm{U}\,F \simeq \mathsf{G}/\rho(\pi)$. Indeed, the natural (Seifert) fibration presents its fundamental group $\mathbb{Z} \to \pi_1(\mathrm{U}\,F) \to \pi$ as the universal central extension of π.

Unless $h = 0$ or $h = 1$ and $g = 0$, the group does not determine the type (g, h) of the underlying surface, let alone the orbifold. Still one may recover the torsion data, and the Euler characteristic.

Torsion. We now explain why the group π uniquely determines the collection c, by considering its torsion elements. First $c = \emptyset$ if and only if π has no torsion, so now suppose it has. Applying Van-Kampen again but this time to the underlying surface with an additional hole F_0^*, one expresses the group π as an amalgam of $\pi(F_0^*)$ and $\mathbb{Z}/c_1 * \dots * \mathbb{Z}/c_s$ over the group $\mathbb{Z}$, mapping its generator in the former to the loop around the added hole represented by $\prod[\lambda_j, \mu_j] \prod \delta_j^{-1}$, and in the latter to the product of generators of the cyclic groups $\prod \tau_j$. Since $\pi_1(F_0^*)$ has no torsion (as it is free on $2g + h$ generators), the free amalgam $*\mathbb{Z}/c_j$ is the smallest subgroup of π which contains all the torsion, thus it is intrinsic to π (so is the quotient, canonically isomorphic to $\pi_1(F_0^*)$). But now the unique factorization of amalgams [SW79, Theorem 3.5] following from results of Gruško and Kuroš, ensures that the c_j are uniquely determined up to permutation.

Euler characteristic. Let us show how to also recover from π the Euler characteristic of the underlying surface $\chi_0 = 2 - 2g - h$, and the orbifold Euler characteristic χ which is equivalent once we know c; first using combinatorial invariants, and then by homological methods.

Of course, we may first attempt to extract the quantity $2g + h$ from the previous amalgam decomposition over $\mathbb{Z}$, but we should be careful of discussing according to the nature of the kernel and cokernel of the maps from $\mathbb{Z}$ into each factor, and this amounts to distinguishing the case when F_0 is closed on the one hand, and when c is empty on the other. In fact, one may directly identify the orbifold Euler characteristic χ as the *cost* of the group π: this numerical invariant defined for finitely generated groups is additive with respect to *free* amalgams (that is over nothing, unlike what we had above), and equals $1 - 1/c$ for a cyclic group of order c (including ∞), but we shall not say any more as this would lead us too far off. So after two incomplete attempts, we finally introduce the notion of deficiency which shall be used again later on to provide an intuitive computation for the dimension of the representation variety $\mathrm{Hom}(\pi, \mathsf{G})$. The *deficiency* of a finitely presented group is the minimal difference between the number of generators and relations among all its finite presentations. For a finitely generated Fuchsian group as above, one may check (distinguishing the closed case to the presence of ends) that its deficiency always equals $1 + |\chi_0|$ where $\chi_0 = \chi(F_0)$.

Alternatively, one may determine χ_0 from the group homology of π. Concretely, this can be done in two steps, first by examining the existence of cocompact Fuchsian representations, and then by computing the rank of the abelianization $\pi/[\pi,\pi] = H^1(\pi)$. If there are cocompact representations, then the rank of $H_1(\pi)$ is $|\chi_0| + 2$, otherwise it is $|\chi_0| + 1$. More conceptually, since F_0 is a manifold and an Eilenberg-Maclane space (because its universal cover is contractible), one may identify its homology with that of its fundamental group. In particular the Euler characteristic of both homology complexes (say with $\mathbb{Q}$ coefficients) coincide, and χ_0 is indeed computable from π. To recover the previous recipe, note that F is connected so $\dim H_0(F) = 1$, that $\dim H_0(F_0)$ is either 1 or 0 depending on whether F_0 is closed (by Poincaré duality) or not (homotopic to a wedge of circles), and that the rank of the abelianization is $2g$ when the surface is closed and $2g + h - 1$ otherwize; in particular the difference $\dim H_2(F_0) - \dim H_1(F_0)$ does not distinguish the closed case. In formula: $\chi_0 = \dim H_0(\pi_1(F_0)) - \dim H_1(\pi_1(F_0)) + \dim H_2(\pi_1(F_0)) = 2 - (2g + h)$.

Correspondences. To sum up the previous discussion, a Fuchsian representation $\rho \colon \pi \to G$ provides an isomorphism between π and any based fundamental group of $F = \mathbb{H}/\rho$. The homeomorphism type of the underlying topological surface may depend on the representation, but the singular data c and Euler characteristic χ_0 (or χ) are intrinsic to π.

This is enough to provide, in the closed case, a bijection between *homeomorphism* types of closed Fuchsian orbifolds and *isomorphism* classes of finite type Fuchsian groups with non-zero second homology, since both are determined by the type (g, c). In particular, a compact implies uniform. More generally, the finite type Fuchsian orbifolds are classified up to homeomorphism by the isomorphism class of the fundamental group π together with the additional numerical data h, as this uniquely determines the homeomorphism type (g, h, c) of F.

One may improve by recasting this in terms of the peripheral data, using the notion of framed lattices (π, δ), which we defined by describing the map between framed groups $H_2(\pi, \delta) \to H_1(\delta)$. From the previous discussion, oriented lattice orbifolds F up to *homeomorphism* correspond to framed lattices (π, δ) up to *automorphisms* $\mathrm{Aut}(\pi, \delta)$. Now fix equivalent types (g, h, c) and (π, δ). An oriented based lattice orbifold (F, x) up to the group $\mathrm{Diff}_0(F, x)$ of isotopies relative to the base point, corresponds to a framed lattice (π, δ); and those admit equivariant actions by the automorphism groups $\mathrm{Mod}(F, x) \simeq \mathrm{Aut}(\pi, \delta)$ for their respective structures. Birman's short exact sequence $\pi_1(F, x) \to \mathrm{Mod}(F, x) \to \mathrm{Mod}(F)$ says that forgetting the base point amounts to quotienting by conjugacy. So an oriented lattice orbifold F modulo isotopies $\mathrm{Diff}_0(F)$ corresponds to a framed lattice up to inner automorphisms $(\pi, \delta)/\mathrm{Inn}(\pi)$; those admit equivariant actions by the structural automorphism groups $\mathrm{Mod}(F) \simeq \mathrm{Out}(\pi, \delta)$. Forgetting orientations, a lattice orbifold up to isotopy corresponds to a framed lattice (π, δ) up to orientation reversal and conjugacy; those admit equivariant $\mathrm{Mod}^{\pm}(F) = \mathrm{Out}(\pi, \bar{\delta})$ actions. To be precise: $\mathrm{Out}(\pi, \bar{\delta})$ is the $\mathbb{Z}/2$ extension of $\mathrm{Out}(\pi, \delta)$ by *the* outomorphism simultaneously inverting all generators in a presentation containing the δ_j.

In categorical language, we started with a groupoid: a set of objects with some structure (based lattice orbifolds modulo relative isotopy), and *all isomorphisms* preserving the structure. After identifying connected components (given by the type), we focused on a connected groupoid, forming a torsor under the automorphism group of the structure. We then applied the fundamental group functor to find an equivalent category (framed lattices), this amounts to an equivalence between simply transitive group actions. Finally, considering various equivalent forgetful functors on both sides, we found weaker correpondences (between groupoids with less objects having more individual automorphisms), providing equivalent torsors under quotients of the initial automorphism group.

Automorphisms. The homeo-type of an oriented lattice orbifold F is almost determined by the modular group Mod(F) since the maximal braid normal subgroup recovers the number of holes, and the surface braid subgroup $\mathrm{BP}(F, (z_j, c_j))$ recovers the partition of conical singularities according to their order as well as the $\gcd(c_i, c_j)$. The latter determine the c_j.

In algebraic terms, a framed lattice (π, δ) up to isomorphism is equivalent to a group $\mathrm{Aut}(\pi, \delta)$ up to conjugacy. As before, this equivalence between the connected components of two groupoids can be refined in restriction to each component, to establish equivalencies of transitive group actions. The functor Aut yields an equivalence between framed lattices (π, δ) and groups $\mathrm{Aut}(\pi, \delta)$, under the transitive free actions of $\mathrm{Aut}(\pi, \delta)$, tautological on the former and by conjugacy on the latter. Quotienting by subgroups of the torsor group $\mathrm{Aut}(\pi, \delta)$, we get weaker equivalencies between framed lattices up to conjugacy and groups $\mathrm{Out}(\pi, \delta)$, or between framed lattices up to conjugacy and orientation reversal and groups $\mathrm{Out}(\pi, \bar{\delta})$.

Finally, let us note that given a framed lattice (π, δ) up to isomorphism, there is a unique group embedding $\mathrm{Aut}(\pi, \delta) \subset \mathrm{Aut}(\pi)$ up to conjugacy (at the target) and similarly for $\mathrm{Out}(\pi, \delta) \subset \mathrm{Out}(\pi)$. Hence the different framings of π (its many forms) are given by those finite index subgroups.

Fuchsian representations. Consider a finite type Fuchsian group π and denote $\mathsf{G} = \mathrm{PSL}_2(\mathbb{R})$. A representation $\rho \colon \pi \to \mathsf{G}$ can be described by considering a presentation for π, and then choosing a matrix per generator which satisfy the required relations. Hence the space of all representations $\mathrm{Hom}(\pi, \mathsf{G})$ is an algebraic subset in some Cartesian product G^N which is smooth (actually it is an affine variety but we shall wait until section 2.2 to study its algebraic structure). Let us compute its dimension, starting with that of the underlying surface F_0. The group G has dimension 3, so loosely speaking we have 3 parameters per generator and 3 equations per relation. Put differently this equals 3 times the deficiency of $\pi(F_0)$, that is the minimal difference between the numbers of generators and relations $2g + h - 1 = |\chi_0| + 1$ We deduce that the space of $\pi_1(F_0)$- representations has dimension $3|\chi_0| + 3$, and torsion elements are uniquely determined by the center of rotation so this adds $2s$ to the dimension of $\mathrm{Hom}(\pi, \mathsf{G})$: the result only depends on π indeed. Now the group G acts on the space of representations by conjugation at the target, but this action is not properly discontinuous and the quotient is a *singular* algebraic set of dimension $3|\chi_0| + 2s = 6g - 6 + 3h + 2s$.

The subset $\mathcal{F}$ of Fuchsian representations in the space $\mathrm{Hom}(\pi, \mathsf{G})$ endowed with the compact-open topology, is closed [Ota90, Proposition 53] and has non-empty interior. Moreover, it has two connected components: one for each possible orientation on the quotient (algebraic distinguished by comparing the maps induced on $H_2(\pi, \delta)$). The conjugacy action by G restricted to $\mathcal{F}$ is properly discontinuous, this defines a G-principal fibration over the *smooth* algebraic set $\mathcal{T}(\pi) = \mathcal{F}/\mathsf{G}$ of dimension $3|\chi_0| + 2s$. This quotient space can be partitioned into a finite number of cells $\mathcal{T}(2g, h)$ according to the homeo-type (g, h, c) of the quotients $\mathbb{H}/\rho(\pi)$, those are indexed by the partitions $2 - \chi_0 = 2g + h$ with $h > 0$ unless π cocompact. More precisely, each $\mathcal{T}(2g, h)$ has two connected components $\mathcal{T}^{\pm}(F)$ (one for each orientation), which are both homeomorphic to a closed ball (as we shall see). Each component of $\mathcal{T}(2g, h)$ contains in its boundary 2^h analytic subspaces of lower dimensions, defined by the simultaneous vanishing of some peripheral traces. This defines a stratification of $\mathcal{F}$ into a finite number of cells, whose local incidence combinatorics around a cell (telling which cells belong to its boundary) are given by the formation of cusps, and whose adjacency relations (which cells share a common cell in their boundaries) are dictated by the identification of boundary components or splittings along non separating simple closed loops, yielding creation and annihilation of handles. Finally, one can foliate each open cell in $\mathcal{T}(2g, h)$ by the $6g-6+2h+2s$ open cells defined by fixing the trace (hyperbolic lengths) for the collars geodesics arround trumpets.

0.3 Summary

Categorical equivalence. Consider the category whose objects are the based orbifolds without boundary up to relative isotopy with respect to base points, and whose morphisms are the isotopy classes of homeomorphisms preserving the orbifold structure. The full subcategory whose objects are the based lattice orbifolds (F, x) up to relative isotopy forms a groupoid (full means we consider all morphisms between the objects). Applying the based fundamental group functor yields an equivalence to the category of all Fuchsian lattices (π, δ), with group isomorphisms preserving the subset of conjugacy classes. The connected components of each groupoid is given by the type (g, h, c), and the homology sequence of the pair $H_\star(\pi, \delta)$. In restriction to a given component, this same functor yields an equivalence of torsors under the actions of $\mathrm{Mod}^{\pm}(F, x) \simeq \mathrm{Aut}(\pi, \overline{\delta})$.

Moreover, the uniformization theorem provides an equivalence between complete hyperbolic structures on F, which are better defined in terms of complete $(G, \mathbb{H})$-structures on orbifolds, and conjugacy classes of discrete and faithfull representations of its Fundamental group $\mathcal{F}(\pi)/G$. So with this equivalence in hand, we then concentrate the study on the space of all Fuchsian lattices $\pi \to G$. We first identify the isomorphism type of the group π in terms of its homology groups, then describe the embedding $\pi \to G$ in terms of the long exact sequence in homology for the pair.

Finding the group. First the isomorphism type of the group π is uniquely determined by the groups of $\mathbb{Z}$-homology functor $H_\star(\pi; \mathbb{Z})$ as follows. Either $H_2(\pi) = \mathbb{Z}$, in which case the lattice is cocompact and then $H_1(\pi) = \mathbb{Z}^{2g} \oplus \mathbb{Z}/c_j$ determines π as the free amalgam of $\mathbb{Z}\mu_j * \mathbb{Z}\lambda_j * \mathbb{Z}/c_j$ modulo a relation of the form $\prod[\lambda_j, \mu_j] = \prod \tau_j$. In particular any finite type abelian group with even rank can be realized as the abelianization of a cocompact Fuchsian lattice. Otherwize $H_2(\pi) = 0$, in which case the lattice has at least one end $\mathrm{card}(\delta) = p > 0$, and then $H_1(\pi) = \mathbb{Z}^{2g+h-1} \oplus \mathbb{Z}/c_j$ determines π as the free amalgam of $\mathbb{Z}\mu_j * \mathbb{Z}\lambda_j * \mathbb{Z}\delta_j * \mathbb{Z}/c_j$ modulo a relation of the form $\prod[\lambda_j, \mu_j] = \prod \delta_j \prod \tau_j$. In particular any finite type abelian group can be realized as the abelianization of a non-cocompact lattice. Hence the lattices π with the rational homology of 2-spheres ($H_2(\pi; \mathbb{Q}) = \mathbb{Q}$ and $H_1(\pi; \mathbb{Q}) = 0$, ergo finite abelianization) have type $(g = 0, h = 0, c_j)$, and are isomorphic to $*\mathbb{Z}/c_j$ modulo $c_1 \dots c_j = 0$. They can also be described by asking that the manifold $G/\pi = \mathrm{U}\,F$ is a rational homology 3-spheres. The lattices with trivial rational homology have type $(g = 0, h = 1, c_j)$, they are uniform but not cocompact, and isomorphic to $\pi = *\mathbb{Z}/c_j$.

Finding the type. **CHANTIER jusque fin de section** Then we use the long exact sequence in homology of the pair (G, π). The relative homology groups $H_\star(G, \pi)$ correspond to those of the quotient 3-manifold G/π which is the unit tangent bundle of the corresponding orbifold $\mathbb{H}/\pi$.

blablalba il faut que je comprenne l'homologie d'un résea dans un groupe de Lie et du quotient pour bien formuler le nombre de bouts purement en terme de la représentation discrète

In particular the group is uniform when it is a homology sphere blabla

and correspond to the orbifolds for which $h = 0$ or $h = 1$ and $g = 0$.

lift intersection pairing on $\pi/[\pi, \pi]$ to π and universal central extension

Studying the moduli. Now fix the embedding type $\pi \subset G$. This cuts out two connected components $\mathcal{F}(2g, h)$ in the space of Fuchsian representations $\mathcal{F}(\pi) \subset \mathrm{Hom}(\pi, G)$, projecting to a pair of open balls $\mathcal{T}(2g, h)$ in the base $\mathcal{T}(\pi) = \mathcal{F}(\pi)/G$ of the principal G-fibration. Each components $\mathcal{T}(2g, h)$ forms a torsor under the action $\mathrm{Out}(\pi, \delta)$, the quotient is the moduli space of hyperbolic structures on the oriented lattice orbifold F, and its study will occupy us in the sequel.

0.4 Geometric intersection pairing over the group

In this subsection, we consider a lattice Fuchsian orbifold F with type $(\mathfrak{g}, \mathfrak{h}, \mathfrak{c})$ up to isotopies relative to the ends, which is equivalent to a framed lattice (π, δ) up to conjugacy and orientation reversal with δ of cardinal $\mathfrak{h}$, we could denote this $(\pi, \overline{\delta})/\pi$. we shall abusively denote this $(\overline{\pi}, \overline{\delta})$.

The geometric intersection of $\mathfrak{h}$-loops on F is invariant under the modular group so it defines a pairing $i \colon \overline{\pi} \times \overline{\pi} \to \mathbb{N}$ which is invariant under the *diagonal* action of the *framed* outer automorphism group $\mathrm{Out}(\pi, \overline{\delta})$ (but not the full outomorphism group in general). We explain how to recover the framed lattice $(\pi, \overline{\delta})$ up to automorphism, and the isormorphism type of $\mathrm{Mod}^{\pm}(F) = \mathrm{Out}(\pi, \overline{\delta})$ from the knowledge of the intersection pairing i. Again, this comes from a equivariant with resepect to the natural action of $\mathrm{Mod}^{\pm}(F) = \mathrm{Out}(\pi, \overline{\delta})$. Finally, we will try to compute i solely in terms of the algebraic structure of the framed lattice.

Note that we could extend this to an actual symmetric bilinear form on the real vector space over the base $\overline{\pi}$ of unoriented $\mathfrak{h}$-loops, and discuss the properties of the infinite dimensional quadratic space $(\mathbb{R}^{(\overline{\pi})}, i(\cdot, \cdot))$; but we shall leave most of the linear analysis for subsection 1.3 on geodesic currents, and stick to the combinatorial point of view for now.

Essential, peripheric and torsion classes. A loop γ is unessential in F when it is homotopic to a point (smooth or singular) or hole ; equivalently when $\mathrm{tr}(\rho(\gamma)) \leqslant 2$ for some representation $\rho \in \mathcal{F}(F)$. Those are precisely the loops which have finite orbits under the modular group. The non-trivial $\mathfrak{h}$-loops partition themselves between essential, peripheric and torsion classes (respectively mapping to hyperbolic, parabolic and elliptic matrices under $\rho \in \mathcal{F}_F$).

Now consider the pairing i on $\overline{\pi}$. Note that its isotropy elements correspond to the simple loops, which must in particular be primitive. By definition, the pairing detects unessential loops as those α which intersect every *distinct* β trivially. It also detects the unessential and primitive classes as the kernel of the pairing : the loops γ such that $i(\gamma, \cdot)$ vanishes on $\overline{\pi}$. Those classes consist in the peripheral classes δ along with trivial and primitive torsion classes. Now assume we also know π: the trivial and torsion classes can be distinguished algebraically from the peripheral ones, so we know the set of inverse pairs of primitive peripheral conjugacy classes $\overline{\delta} \subset \overline{\pi}$. But the algebraic (first homology) structure of π determines an orientation for the δ_j up to simultaneous inversion.

Therefore, it is equivalent to know (π, i) and (π, δ), both up to conjugacy and inversion, and those sets admit equivariant actions from $\mathrm{Out}(\pi, \overline{\delta})$.

Recovering the modular group from the intersection. Now we explain why the only data of $\overline{\pi}$ and its pairing i, forgetting π and its group law up to conjugacy, determine the *isomorphism* type of the modular group $\mathrm{Out}(\pi, \overline{\delta})$. This is reminiscent to the fact in Lie group theory that the automorphism group of a simple Lie group is the isometry group of its Killing form.

Indeed, one may use the pairing to construct the so called *curve-complex* which is a finite dimensional cell-complex whose k-faces correspond to the $k+1$-tuples of mutually non intersecting loops (in particular the 0-faces, called vertices, correspond to simple loops). Its symmetries correspond to the bijections of $\overline{\pi}$ which preserve the pairing, they form the orthogonal group $\mathrm{O}(\overline{\pi}, i(\cdot, \cdot))$ of the quadratic space. A deep theorem of Ivanov [Iva97] says that when F is a closed surface manifold, the automorphism group of that complex is precisely the outer automorphism group $\mathrm{Out}(\pi) = \mathrm{Mod}^{\pm}(F)$. It has been extended by Luo to the holed manifold case [Luo00], except the result has a couple of exceptions with low $|\chi|$. Thus in absence of torsion, it is true that

$O(\overline{\pi}, i) = \mathrm{Out}(\pi, \overline{\delta})$. It is not hard to deduce the same result for a general lattice Fuchsian orbifold whose underlying surface has negative Euler characteristic.

blabla

Thus knowing $i \colon \overline{\pi} \times \overline{\pi} \to \mathbb{N}$ recovers the abstract group $\mathrm{Out}(\pi, \overline{\delta})$ and its action on simple conjugacy classes. But there is only one way to extend it to an action on the set of all conjugacy classes in such a way that it descends from an action by automorphisms of the group considered up to inner automorphisms. Hence knowing (π, i) and π up to conjugacy will recover $\mathrm{Out}(\pi, \overline{\delta}) \subset \mathrm{Out}(\pi)$.

Cyclic order structure and orthogonality. We know how the different orbifold structures are related to one another from the combinatorics of the aforementioned cellular decomposition for the discrete and faithful representation variety into $\mathcal{F}(F)$, however it is not clear how to compare the different intersection functions. Let us hint to an approach for understanding the combinatorics of the intersection pairing in terms of the cyclic order structure derived from a Fuchsian representation.

Since F has no boundary, a representation $\rho \in \mathcal{F}(F)$ provides a quasi-isometric embedding of π in $\mathbb{H}$ (by considering the orbit of an arbitrary point), and therefore an homeomorphism between their Gromov boundaries $\flat\pi \simeq \mathbb{S}^1$ (see [GlH90] for the Gromov boundary of a hyperbolic group). Note that $\overline{\pi}$ maps into the set of unordered pairs of distinct boundary points $(\flat\pi \times \flat\pi - \Delta)/\pm 1$ by considering the doubly-infinite periodisation of a conjugacy class. This in turn provides a cyclic order structure on $\flat\pi$ and thus on $\overline{\pi}$. This cyclic order structure encodes the orthogonality relation defined by the pairing $i \colon \overline{\pi} \times \overline{\pi} \to \mathbb{N}$, that is the property of having disjoint representatives.

One can also formulate this by composing the representation $\rho \colon \pi \to \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{Homeo}(\mathbb{S}^1)$ to obtain an action of π on the circle for which every essential element acts with two fixed points. Distinct primitive essential transformations correspond to disjoint loops if and only if their fixed points define unlinked pairs on the circle. When the elements are related by a power (in particular equal or inverse), the order structure does not determine their orthogonality relation so easily. Still, the order structure determines the pairing since any set of distinct two by two disjoint loops (which we cannot assume simple a priory) with maximal cardinal is necessarily composed of $2|\chi_0|+1-s-g$ *simple* loops, and thus determines g. This argument is far from computational : it seems hard to determine intersection numbers from the cyclic order or from the orthogonality relation.

Intersection pairing as killing form. We finally show how to express the intersection function $i \colon \overline{\pi} \times \overline{\pi} \to \mathbb{N}$ purely in terms of the group structure with peripheral data. This enables us to deduce some functorial properties and understand how the intersection function behaves under group morphisms (especially those which correspond to orbifold covers in the geometric setting). Again, we are inspired by the Killing form on a simple Lie group, which is given by the trace of the composition between the adjoint actions on the Lie algebra.

For this, consider the graded Lie algebra given by the descending central series $\mathfrak{gr}(\pi) = \bigoplus \mathfrak{gr}_k$ where $\mathfrak{gr}_k = \pi_k/\pi_{k+1}$ and π_k denotes the normal subgroup of π generated by commutators of depth k like $[\alpha_0, \dots [\alpha_{k-1}, \alpha_k] \dots]$. Its addition of classes follows from the group product of their representatives whereas the Lie bracket is given by their commutator $[\cdot,\cdot] \colon \mathfrak{gr}_i \times \mathfrak{gr}_j \to \mathfrak{gr}_{i+j}$. The group π acts by conjugation on its graded Lie algebra $\mathfrak{gr}(\pi)$ and this gives a linear representation $\mathrm{ad} \colon \pi \to \mathrm{End}(\mathfrak{gr})$. One may thus speak of the composition $\mathrm{ad}\,\alpha \circ \mathrm{ad}\,\beta$, and **provided this is traceable** then one can compare its trace to the intersection of their conjugacy classes.

Question 1 (Killing form versus intersection)**.** $\mathrm{tr}(\mathrm{ad}\,\alpha \circ \mathrm{ad}\,\beta) = 2 \cdot i(\alpha, \beta)$

NONONON LE GROUPE EST RESIDUELLEMENT NILPOTENT LA FORME DE KILLING EST NULLE

Goldman Poisson algebra as Lie algebra of the modular group REVOIR LE MAIL ENVOYE A JULIEN DEPLACER CE PARAGRAPHE EN SECTION POISSON

hey mais est ce que les différents F **ne correspondraient pas (grace au tangent unitaire) avec les différentes classes d'isomorphismes d'extension centrales universelles de** π **? Il faut préciser et faire intervenir les données pérphériques parce que tel quel il y a unicité de l'extension centrale à unique iso près. Par exemple dans le diagramme on peut rajouter des morphismes provenant des sous groupes périphériques (des cercles en bas et des tores en haut). Si oui une fois correctement formulé alors on doit pouvoir déduire l'intersection de l'extension centrale (puisqu'elle est lisible étant donné la surface), or clairement l'extension centrale (groupe fonda du fibré unitaire tangent) encode tout ce qu'il faut pour pour l'enlacement, hehehe !! Mais alors... ce ne serait par ailleurs pas surprenant que la positivité de l'enlacement découle des phénomènes de positivité dand l'algèbre skein du tangent !!!**

1 Teichmüller space, measured laminations, geodesic currents

In this section, F is a smooth oriented connected surface of genus $g \in \mathbb{N}$ with $h \in \mathbb{N}$ circular boundary components, and negative Euler characteristic $\chi = 2 - 2g - h$. We label the boundary components from 1 to h. Denote π its fundamental group, $\widetilde{F}$ it universal cover; and $G = \mathrm{PSL}_2(\mathbb{R})$ the group of orientation preserving isometries of the hyperbolic plane $\mathbb{H}$. Finally, $\mathrm{PMod}^{\pm}(F)$ is the pure modular group which fixes each boundary component, and $\mathrm{PMod}(F)$ its index two subgroup of orientation preserving mapping classes, the latter being generated by Dehn twists.

For a closed surface, the Dehn-Nielen-Baer theorem provides an isomorphism between the full modular group $\mathrm{Mod}^{\pm}(F)$ and the outer automorphism group $\mathrm{Out}(\pi)$, sending $\mathrm{Mod}(F)$ to the stabilizer of the fundamental class in $H_2(\pi;\mathbb{Z}) \simeq \mathbb{Z}$. In presence of boundary, one must consider the subgroup of outer automorphisms fixing the conjugacy classes δ_j which correspond to the boundary components (or simultaneously permuting them all with their inverses).

1.1 Teichmüller space and length functions

Teichmüller space $\mathcal{T}(F)$ consists of isotopy classes of complex (or conformal) structures on F. Precisely, consider the space of complex markings, which are smooth homeomorphisms $m\colon F \to X$ to a Riemann surface X with boundary, up to the equivalence relation given by $m_1 \sim m_2$ whenever $m_2 \circ m_1^{-1}\colon X_1 \to X_2$ is isotopic to a biholomorphism.

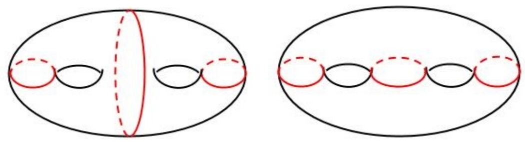
By Alfors-Bers, every conformal structure contains a unique complete Riemann metric of constant curvature -1 with geodesic boundary components of finite positive length. We call this a hyperbolic metric on F, and one can similarly define the points of Teichmüller space $\mathcal{T}(F)$ as the isotopy classes of markings $m\colon F \to X$ to a hyperbolic surface with geodesic boundary.

Being locally isometric to the hyperbolic plane, a hyperbolic metric is equivalent to a $(G, \mathbb{H})$-structure in the sense of Klein and Thurston [Thu97]. The associated developing map $\mathrm{Dev}\colon \widetilde{F} \to \mathbb{H}$ and holonomy representation $\rho\colon \pi \to G$, are well defined up to the action of G at the target, respectively by translation and conjugation. Such hyperbolic structures are thus uniformized as the quotient $\widetilde{F}/\rho(\pi)$ of a connected and simply connected closed subset with geodesic boundary in the hyperbolic plane, under the properly discontinuous action of a discrete subgroup of isometries uniquely defined up to conjugation (one may alternatively restrict to the so called convex core inside the complete quotient $\mathbb{H}/\rho(\pi)$, of which the interior is a retract by deformation). Notice that the developing map is surjective only when the surface is closed.

We just anachronistically recovered from Alfors-Bers the Klein-Poincaré-Koebe uniformization theorem, giving an equivalence between isotopy classes of hyperbolic structures on F and conjugacy classes of (necessarily faithful and discrete) representations $\rho\colon \pi \to \mathbb{G}$ with quotient $\mathbb{H}/\rho(\pi)$ homeomorphicic to F. Consult [dSG18] to know more. We shall come back to this setwize identification in a moment after describing the topology (and symplectic structure) on $\mathcal{T}(F)$ thought as parametrizing isotopy classes of hyperbolic structures.

We may partition $\mathcal{T}(F)$ into strata according to the length of the labelled boundary components. For $L = (L_1, \dots, L_n) \in (\mathbb{R}_+^*)^h$, denote $\mathcal{T}(F, L) \subset \mathcal{T}(F)$ the subset of hyperbolic structures on F whose (labeled) geodesic boundary components have lengths $L_j \in \mathbb{R}_+^*$; together they form a foliation by submanifolds. The pure modular group $\mathrm{PMod}(F)$ acts by homeomorphisms $m \mapsto m \circ \varphi$ on Teichmüller space $\mathcal{T}(F)$, preserving each leaf $\mathcal{T}(F, L)$. This properly discontinuous actions has finite stabilizers, and the quotient orbifold $\mathcal{M}(F)$ is the *moduli space* of hyperbolic metrics with geodesic boundaries, containing the subset $\mathcal{M}(F; L)$ of those with prescribed boundary-lengths.

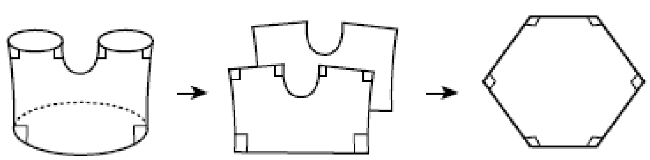
Fenchel-Nielsen coordinates. A pants (or three-holed sphere) decomposition of F, consists of a simple $\mathfrak{i}$-multiloop γ with $d = 3g-3+h$ distinct components γ_j (that is a disjoint collection of d homotopy classes of simple loops non-isotopic to one another) which are non trivial or homotopic to the boundary. An Euler characteristic count shows indeed that the complement is a disjoint union of pants. The following diagram represents two pants decompositions of a genus 2 surface.



Recall that on a hyperbolic surface X with geodesic boundary, every essential h-loop (non-trivial homotopy class of closed loop) has a unique geodesic representative. Now fix $L \in (\mathbb{R}_+^*)^h$, and let us explain how to recover a hyperbolic metric $X \in \mathcal{M}(F; L)$ from its restriction to the hyperbolic pants defined by cutting along the geodesic representatives in γ_j, along with and some additional gluing parameters.

First notice that in a hyperbolic pair of pants there are unique shortest geodesic arcs connecting the boundary components, so every boundary component has a privileged pair of points (those closest to each of the two others), and by symmetry they must be antipodal (half way distance across the geodesic boundary). Out of the two, mark the point which is closet to the shortest of the other boundary components, this choice is well defined for X in a dense open set of $\mathcal{M}(F; L)$, and uniquely extends by continuity. As one glues two pants along a boundary component, the angle $\theta_j \in \mathbb{R}/2\pi\mathbb{Z}$ is defined as the renormalized distance between those privileged points. The metric X on F is uniquely determined by the metric on each pair of pants along with the angles $\theta_j \in \mathbb{R}/\pi\mathbb{Z}$ at which they are glued.

Now it is a amusing to show that there is a unique hyperbolic metric on a pair of pants with prescribed lengths l_1, l_2, l_3 on the boundary. For this one cuts along the shortest geodesic arcs connecting the boundary components to obtain a pair of isometric rectangular hexagons of which three non-adjacent sides of lengths are given by $l_j/2$, and there is a unique such hyperbolic hexagon up to isometry. Consequently, the Fenchel-Nielsen coordinates (l_j, θ_j) associated to the pants decomposition γ uniquely determine the hyperbolic metric $X \in \mathcal{M}(F)$.



Finally, by lifting the angles θ_j to the real line, one obtains a global coordinate system on $\mathcal{T}(F, L)$, and therefore a homeomorphism with $(\mathbb{R}_+^* \times \mathbb{R})^d$ for $d = 3g-3+h$. By letting the lengths of the boundary components vary in $\mathbb{R}_+^*$ we obtain coordinates $\mathcal{T}(F)$ which is thus homeomorphic to an open ball of dimension $3|\chi|$. Note that every pants decomposition yields such coordinates, but the graph over the vertex set of pants decompositions whose edges correspond to some elementary moves (see A.1) is connected, and the elementary coordinate changes from a Fenchel-Nielsen system to a neighboring one preserve some geometric features, like the symplectic form $\sum_j dl_j \wedge d\theta_j$...

Weil-Peterson symplectic form. Wolpert [Wol82] expressed the Weil-Peterson symplectic form on Teichmüller space in terms of the Fenchel-Nielsen coordinates associated to any pants decomposition: $\omega_{wp} = \sum dl_j \wedge d\theta_j$. This enabled Goldman to show [Gol84] that ω_{wp} is invariant under the modular group, and thus descends to moduli space.

Mirzakhani then introduced her topological recursion method [Mir07a] to compute integrals over moduli spaces against the volume form $\mathrm{vol}_{WP} = \omega^d/d!$, and applied it in two ways. She first computed the volumes $V_g(L)$ of moduli spaces $\mathcal{M}(F, L)$ as a (symmetric) polynomial in the L_j of degree $2d = 6g - 6 + 2h$, and related its coefficients in [Mir07b] to certain intersection numbers (line integrals of Chern classes for the cotangent bundle over moduli space).

This enabled her to integrate other functions over moduli space $\mathcal{M}(F; L)$, such as the number of loops in a given modular group orbit $\mathrm{Mod}(F) \cdot \gamma$ whose geodesic representative has length smaller than L_0. Again, this is a polynomial expression in L_0 whose coefficients depend both on volumes $V_{g'}(L')$ of smaller moduli spaces and on constants associated to the class γ.

Length functions Recall that on a hyperbolic surface X, an essential h-loop has a unique geodesic representative, and that unoriented h-loops in F correspond to the set $\overline{\pi}$ conjugacy classes up to inversion. Thus we can define for every $\alpha \in \overline{\pi}$, its *length function* $l_\alpha \colon \mathcal{T}(F) \to \mathbb{R}_+^*$ which associates to $m \colon F \to X$ the length of the unique geodesic homotopic to $m(\alpha) \subset X$. This is related to the well defined value of the trace $t_\alpha(\rho) = \mathrm{tr}\, \rho(\alpha)$ for the representations ρ corresponding to m, by the formula $|t_\alpha(\rho)| = 2\cosh(l_\alpha(m)/2)$.

Denote Σ the set of all unoriented simple loops in F, including those parallel to the boundary. Thurston showed that given a pants decomposition γ, the Fenchel-Nielsen twist coordinates θ_j were uniquely determined by the lengths of the γ_j together with an additional set of $6g-6$ simple loops. In particular the lengths of all simple loops determine the hyperbolic metric (compare [Ota90] for an analog in variable negative curvature). More precisely (see [FLP79, FLP12]) the *length functional* $m \mapsto (l_\alpha(m))_{\alpha \in \Sigma}$ defines a proper embedding of $\mathcal{T}(F, L)$ in $\mathbb{R}^\Sigma$ with product topology, and after composing with the natural projection $\mathbb{R}^\Sigma \to \mathbb{P}(\mathbb{R}^\Sigma)$, it maps homeomorphically onto an open ball of dimension $6g - 6 + 2h$. One has a similar results for $\mathcal{T}(F)$.

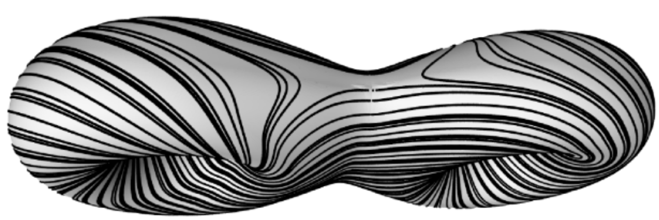
Representations. Recall the Klein-Poincaré-Koebe uniformization theorem as it was described in the first paragraph, giving an equivalence between isotopy classes of hyperbolic structures on F and conjugacy classes of representations $\rho \colon \pi \to G$ with quotient $\mathbb{H}/\rho(\pi)$ homeomorphic to F. We may improve this setwize identification to a topological (and analytic) one. Consider, in the space of representations $\mathrm{Hom}(\pi, G)$ endowed with the compact-open topology, the subset $\mathcal{F}(F)$ of those ρ for which the quotient $\mathbb{H}/\rho(\pi)$ is homeomorphic to F. The group G acts on $\mathcal{F}(F)$ by conjugation at the target of ρ, and the quotient $\mathcal{F}(F)/G$ has two connected components distinguished by the orientation induced on the quotients $\mathbb{H}/\rho(\pi)$. One of those may thus be identified with $\mathcal{T}(F)$. One can make analog statements with prescribed boundary lenghts (or in presence of orbifold singularities and cusps) while fixing the traces for peripheral conjugacy classes.

The dimension count $3|\chi|$ for Teichmller space can now be interpreted as follows. The appropriate notion of dimension for a finitely presented group is the *deficiency*, that is the minimal difference between the number of generators and relations among all finite presentations. It is not hard to show, distinguishing the case of closed surfaces to those with boundary, that $\pi_1(F)$ has deficiency $|\chi| + 1$. The group G has dimension 3. So loosely speaking, the space of representations has dimension $3|\chi| + 3$ and considering them up to conjugacy leaves $3|\chi|$ parameters.

1.2 Measured geodesic laminations

Recall that in presence of boundary, the various notions of loops may be adapted to include proper arcs whose boundary points belong to the boundary of F. The sets for these dual notions will primed: the dual states Ψ' are simple multiloops or arcs up to homotopy preserving the boundary (so one can get rid of arcs and loops homotopic into the boundary of F).

Measured geodesic laminations. A *geodesic lamination* on a hyperbolic surface X is a closed subset partitioned by complete simple geodesics (including those ending in the boundary). The most elementary geodesic lamination consists of (the geodesic representative for) a dual simple loop; a little more general are the dual essential states. But typically, a geodesic lamination Λ contains a non countable set of non closed geodesics, each of them being dense in Λ. A transversal cut would then look like a Cantor set. Here is a graphical suggestion of a geodesic lamination in the genus two closed surface, along with its lift to the hyperbolic plane.



A *transverse measure* for a geodesic lamination Λ is a family of positive Radon measures defined on every free arc $k\colon [0,1] \to X$ transverse to Λ, which is compatible with restrictions to subarcs, and equivariant with respect to homotopies of k respecting the intersection with Λ. This equivariance property implies that the measures have support included in Λ. A *measured geodesic lamination* λ consists of a geodesic lamination Λ together with a fully supported transverse measure.

For a lamination given by a simple geodesic arc or loop ψ and a weight $r \in \mathbb{R}$, define the measured lamination which assigns to any subset of a transverse arc, the counting measure of its intersection with ψ multiplied by r; some call this a weighted simple loop. A little more generally, a state ψ with (any real multiple r of) its counting measure yields a measured lamination (recall from the introductory notations that the components of ψ need not be homotopically distinct, thus it may have several parallel copies of the same loop ψ_j and we take this into account by multiplying the counting measure supported on ψ_j as many times as it occurs). This map, from unweighted essential states (with $r = 1$) to measured laminations, is injective.

We note $\mathcal{ML}(F)$ the set of measured geodesic laminations endowed with the weak $*$ topology whose test functions are the continuous $k\colon [0,1] \to \mathbb{R}$ with compact support contained in a generic geodesic free arc (i.e. transverse to all simple complete geodesics, or equivalently, not contained in any simple complete geodesic). Its contains the subspace $\mathcal{ML}(F, L)$ of those which attribute the measures $L_j > 0$ to the boundary components. Denote $\mathbb{P}\mathcal{ML}(F)$ and $\mathbb{P}\mathcal{ML}(F, L)$ the corresponding projectivized sets. When a statement holds with or without prescribed data on the boundary, we omit F and L from the notations.

Piecewize linear topology. Inspired by Fenchel-Nielsen coordinates, Thurston introduced train tracks [Thu02, chapter 8] to provide the charts for a piecewize linear structure on $\mathcal{ML}(F)$. Let us say, without delving into their definition, that to every train track corresponds a linear chart, which is the positive cone over a closed polyhedral cell in $\mathbb{R}^{3|\chi|}$ together with a linear coordinate system, and they are related by piecewize linear transition maps.

This yields a stratification of $\mathcal{ML}(F)$ according to the dimension of these open cells, which depend on the combinatorics of their associated train tracks. The maximal dimension is attained for cells corresponding to the so called maximal train tracks : the union of their interior is open and dense. Moreover, in these charts the $\mathcal{ML}(F, L)$ are defined by some linear conditions imposed by the measures for the boundary components, and together they form a foliation of $\mathcal{ML}(F)$ by piecewize linear submanifolds. Finally, the piecewize linear transition maps between train track coordinates have *integral* coefficients, so one can speak of the integral points $\mathcal{ML}(F; \mathbb{Z})$: those are precisely given by the halves $\frac{1}{2}\psi$ of essential dual states $\psi \in \Psi'$ with trivial homology class $[\psi] \in H_1(F, \partial F; \mathbb{Z}/2)$.

This *piecewize linear integral* structure only depends on the topology of F, not on the hyperbolic structure used to speak of geodesic laminations. In particular the topology but also the stratification, foliation by piecewize linear submanifols and integral points are all intrinsic to $\mathcal{ML}(F)$ without any reference to a hyperbolic metric. Moreover it is preserved by the modular group action. A clear exposition of the topology on measured laminations from that perspective has been written out by Hatcher in [Hat88], and [PH92] gives a thorough investigation of train track combinatorics.

Compactifying Teichmuüller space. The aforementioned injection from Ψ' into $\mathcal{ML}$ composed with the projectivization map to $\mathbb{P}\mathcal{ML}$ remains injective. We may also inject Ψ' into $\mathbb{R}^{\Sigma}$ by sending ψ to the intersection functional $(i(\psi, \alpha))_{\alpha \in \Sigma}$, and again this remains injective after projection into $\mathbb{P}(\mathbb{R}^{\Sigma})$. Similarly, the space $\mathcal{ML}$ embeds properly in $\mathbb{R}^{\Sigma}$ by $\lambda \mapsto (i(\lambda, \alpha))_{\alpha \in \Sigma}$ where $i(\lambda, \alpha)$ denotes the minimal λ measure of transverse representatives for α; the composition $\mathcal{ML} \to \mathbb{P}(\mathbb{R}^{\Sigma})$ remains an embedding. After those embeddings into $\mathbb{P}(\mathbb{R}^{\Sigma})$, the image of Ψ' has completion that of $\mathcal{ML}$, in particular the weighted states $\mathbb{Q}\Psi'$ are dense in $\mathcal{ML}$; see [FLP79, FLP12] for proofs. Moreover, as a weighted state $r \cdot \psi$ converges to $\lambda \in \mathcal{ML}$, the intersection function $r \cdot i(\psi, k)$ between the weighted state and any arc k transverse to λ (like a non-simple closed loop) tends to the λ measure of k denoted $i(\lambda, k)$. In particular, the intersection function between weighted simple loops and states extends continuously to a symmetric bilinear pairing on the cone of measured laminations.

Now recall Thurston's proper embedding of $\mathcal{T}$ into $\mathbb{P}(\mathbb{R}^{\Sigma})$. After projectivization, we obtain an homeomorphism from $\mathbb{P}\mathcal{ML}(F, L)$ to a $6g - 7 + 2h$ dimensional sphere in $\mathbb{P}(\mathbb{R}^{\Sigma})$ which realizes the disjoint compactification of Teichmüller space $\mathcal{T}(F, L)$ into a closed ball. The same goes for the pair $(\mathcal{T}(F), \mathbb{P}\mathcal{ML}(F))$ which gets map homeomorphically to $(\mathbb{B}^{3|\chi|}, \mathbb{S}^{3|\chi|})$. This is Thurston's compactifiaction of Teichmüller space by projective measured laminations, and it is equivariant with respect to the modular group action. Thurston proved that a sequence of elements m_j in $\mathcal{T}(F)$ converges to a projective measured lamination λ if and only if, for every pair of simple loops α_1 and α_2, such that $i(\lambda, \alpha_2) \neq 0$, the ratio of their m-lengths converges to ratio of their intersection numbers with λ:

$$\lim_{j \to \infty} \frac{l_{m_j}(\alpha_1)}{l_{m_j}(\alpha_2)} = \frac{i(\lambda, \alpha_1)}{i(\lambda, \alpha_2)}$$

This last property will play a key role when, following Bonahon [Bon88], we shall revisit Thurston's compactification of Teichmüller space from the point of view of geodesic currents.

Thurston's compactification of $\mathcal{T}$ by $\mathcal{ML}$ in $\mathbb{R}^{\Sigma}$ also coincides with (a component of) Morgan-Shalen's compactification of the character variety as we shall briefly explain in 2.2 following [Ota12].

Thurston's pairing and volume. Using train tracks, one may define as in [PH92, section 3.2] a symplectic form ω_{Th} named after Thurston over $\mathcal{ML}(F, L)$: that is a piecewize-linearly varying family of non-degenerate skew-symmetric bilinear pairings on the tangent spaces above the interior points of maximal train track linear charts, whose monodromy around codimension one cells is trivial (this accounts for usual the closure condition). The actual construction of ω_{Th} is derived from the algebraic intersection product in homology: in a chart corresponding to some train track t, by pulling back the symplectic structure on the first homology group $H_1(F_t; \mathbb{R})$ of the double cover F_t ramified at one point in each complementary region of t.

Question 2. *Can we formulate a precise relationship between the symplectic structures on* $\mathcal{ML}(F)$ *and* $H_1(F, \mathbb{R})$ *(going further than merely observing that one is constructed using the other)?*

The volume form associated to this symplectic pairing yields Thurston's measure vol_{Th} which can actually be defined only in terms of the piecewize linear integral structure of $\mathcal{ML}(F)$, forgetting about the intersection pairing. Indeed, Masur showed that up to scaling, there is only one $\mathcal{M}(F)$-invariant measure on $\mathcal{ML}(F)$ in its Lebesgue class. Therefore it is proportional to the Borelian measure which assigns to every open set U, the limit as $r \to \infty$ of $r^{3|X|}$ times the cardinal of the set $r \cdot U \cap \Psi'$ of integral measured laminations in the dilated set $r \cdot U$. The analog statement for the leaves $\mathcal{ML}(F, L)$ holds provided this time that the L_j are integral : one needs enough integral points, a necessary and sufficient condition being that the Ψ' be dense in $\mathbb{P}\mathcal{ML}(F, L)$.

Symplectic pairings : from Weil-Petersen to Thurston. Let us now say a word about the relation between the piecewize linear topology with symplectic pairing on measured laminations, and the symplectic topology of Teichmüller space. There are two ways to think about this : locally and at infinity. Indeed the projective space of measured laminations $\mathbb{P}\mathcal{ML}$ can be (topologically) identified with the boundary of $\mathcal{T}$, but also with its projective (co)tangent spaces as we shall briefly explain; this is reminiscent of what happens in hyperbolic geometry. Concerning the relationship at infinity, let us just mention that Papadopoulos and Penner [PP91] showed that when F has at least one boundary component, the symplectic pairing ω_{Th} can be recovered as a limit of ω_{wp}.

We now turn to the local relationship. The *cotangent* space at a point $X \in \mathcal{T}(F, L)$ can be identified with measured geodesic laminations as can be seen by transiting through quadratic differentials and measured foliations. Also, Thurston [Thu98] provides an explicit embedding of $\mathbb{P}\mathcal{ML}(F, L)$ into $T^*_m \mathcal{T}(F, L)$ as the boundary of a convex polyhedron containing the origin in its interior, through the map $d_m \log i(\lambda, \cdot)$. Here $i(\lambda, \cdot)$ is the length function over Teichmüller space, which we have defined for simple loops, but extends by linearity and continuity to all measured laminations. The earthquake map defines a similar embedding of $\mathbb{P}\mathcal{ML}(F, L)$ into the *tangent* bundle of $\mathcal{T}(F, L)$; by sending a measured lamination λ to the unit speed potential vector field E_λ for the Weil-Petersen symplectic pairing over Teichmüller space, associated to the length function l_λ. The unormalized earthquake map $\mathcal{ML}(F) \to T\,\mathcal{T}(F)$ is equivariant with respect to Thurston's and Weil-Petersen's symplectic pairings.

Note that one can enrich the identification between the space $\mathcal{ML}(X)$ of geodesic measured laminations on a hyperbolic surface X and its cotangent space $T_X \mathcal{T}$ by considering various structures depending on X. This includes for instance the differentials of length functions associated to measured laminations.

1.3 Geodesic currents and intersection forms

Bonahon [Bon88] taught us how to recast the compactification of Teichmüller space by measured laminations in terms of geodesic currents. The object of this subsection is to summarize this picture for closed surfaces with an eye towards the more general case of finite type Fuchsian orbifolds. Recall from section 0 and keep in mind that a Fuchsian orbifold F modulo isotopy is equivalent to a pair (π, δ) up to conjugacy, and that such a pair thus defines an intersection pairing $i\colon \overline{\pi} \times \overline{\pi} \to \mathbb{N}$.

Motivation. Consider the vector space $\mathbb{R}^{\overline{\pi}}$, of real functionals over the set of all unoriented $\mathfrak{h}$-loops in F, with the product topology; this extends the product vector space $\mathbb{R}^{\Sigma}$ considered in previous subsection. The proper embedding of Teichmüller space $\mathcal{T}(F) \to \mathbb{R}^{\Sigma}$ lifts to all of $\mathbb{R}^{\overline{\pi}}$ by extending the length functional to all loops $\mathfrak{m} \mapsto (\mathfrak{l}_{\mathfrak{m}}(\gamma))_{\gamma \in \overline{\pi}}$; again its projection to $\mathbb{P}(\mathbb{R}^{\overline{\pi}})$ remains an embedding. We may consider, dual to the obvious injection of $\overline{\pi}$ by the indicator function, the map given by the geometric intersection pairing $\alpha \mapsto i(\alpha, \cdot)$ for $\alpha \in \overline{\pi}'$ and its projectivization $\overline{\pi}' \to \mathbb{P}\mathbb{R}^{\overline{\pi}}$, which are both *injective*. In particular, the image of dual simple loops $i\colon \Sigma' \to \mathbb{P}\mathbb{R}^{\overline{\pi}}$ has completion a copy of projective measured laminations embedded by the projectivized intersection functional $\lambda \mapsto i(\lambda, \cdot)$. After those identifications, the sphere of measured laminations realizes the disjoint compactification of Teichmüller space into a closed ball $\mathbb{P}\mathcal{ML} \sqcup \mathcal{T}$; a marked metric $\mathfrak{m}$ converges to a projective measured lamination λ if and only if for every pair of loops, the ratio of their $\mathfrak{m}$-lengths converges to the ratio of their intersection numbers with λ (provided it is defined). We may similarly consider the completion of the image of all $\mathfrak{h}$-loops : this yields the space of geodesic currents $\mathcal{C}$ on which the intersection pairing can be extended by continuity.

This introductory discussion shows that that the projective space of real functionals $\mathbb{P}\mathbb{R}^{\overline{\pi}}$ is very handy, as it brings together Teichmüller space, projective measured laminations and loops through their length and intersection functionals. Still it remains quite cumbersome as it does not take into account the geometry of $\overline{\pi}$ since we have initially considered its discrete topology. Moreover the relation between loops and metrics, intersection and length, remains quite mysterious, and it is hard to assess how big is $\mathcal{C}(F)$; for instance does it contain Teichmüller space, is it all of $\mathbb{R}^{\overline{\pi}}$? Bonahon's approach deals with both questions providing a unified account for metrics, laminations and loops by adding some topological structure to the space of closed geodesics $\overline{\pi}$. More precisely, this topology will be inherited by its injection as a subset in the space of all geodesics. After presenting this, we shall come back to this naïve functional approach, and explain how a systematic reflection on the intersection pairing over $\overline{\pi}$ can recover and shed light upon the space of currents.

Remark. *Let us make an elementary remark belonging to the realm of topological vector spaces, in order to emphasize why the functional point of view may be called naïve. The space of continuous maps $\overline{\pi} \to \mathbb{R}$ with compact open topology when $\overline{\pi}$ is given the discrete topology, corresponds to $\mathbb{R}^{\overline{\pi}}$ with product topology : that is the space of real functionals we have been considering. When the topology considered at the source $\overline{\pi}$ is made coarser (with less open and closed sets but more compact sets and converging sequences), the space of continuous functions becomes smaller, and so does the associated topological dual vector space.*

For instance if we endow $\overline{\pi}$ with the cofinite topology (whose closed sets are the finite sets), then the continuous functions are those with finite support, and they form the vector space $\mathbb{R}^{(\overline{\pi})} = \bigoplus_{\overline{\pi}} \mathbb{R}\gamma$ over the countable base $\overline{\pi}$ of all $\mathfrak{h}$-loops (the compact open topology coincides here with the direct limit topology). One may consider intermediary spaces between $\mathbb{R}^{(\overline{\pi})}$ and $\mathbb{R}^{\overline{\pi}}$, either by considering another meaningful topology on $\overline{\pi}$ as we shall following Bonahon, or by completing the former with respect to a natural norm derived from the intersection paring on $\overline{\pi}$.

Note that the commonly accepted wisdom in functional analysis according to which reducing a topological vector space enlarges the topological dual is incorrect if interpreted literally (as can already be seen in the finite dimensional case). Indeed if $V \to W$ is a continuous injection, then we have the correspondingly dual continuous surjection $W^* \to V^*$. What may happen however is a competition between enlarging the space and coarsening the topology and this is often the case when considering the "natural topology" on the space of functions over a given space.

For instance, one may consider on a fixed a closed smooth manifold M, the space of smooth functions $C^\infty(M, \mathbb{R})$ with natural Fréchet topology (measuring the partial derivatives), and the larger space of continuous functions $C^0(M, \mathbb{R})$ whose natural topology is given by the supremum norm. The dual of the former (distributions) is much larger than the dual of the latter (Radon measures). But if we pull back the supremum topology on $C^\infty(M) \to C^0(M)$, then the topological dual space becomes smaller in comparison, and is now included in the space of Radon measures (actually equal since smooth functions are dense in continuous ones).

Geodesic currents. In this paragraph which is a subset of [Bon88], F is a closed surface. We first make sense of a topological space containing all unoriented complete hyperbolic geodesics in F independently on the metric. For this we define a topology on the Gromov boundary $\flat\widetilde{F}$ of the universal cover of F. Fix L an array of lengths for the boundary components of F. The developing map $\widetilde{F} \to \mathbb{H}$ associated to any metric $X \in \mathcal{M}(F, L)$ embeds the universal cover of F into the hyperbolic plane. This defines a hyperbolic metric on $\widetilde{F}$ and identifies its Gromov boundary $\flat\widetilde{F}$ with a subset of $\flat\mathbb{H} = \mathbb{S}^1$. When F has no boundary $\flat\widetilde{F} = \mathbb{S}^1$, otherwise it is a cantor set. Two such metrics define hyperbolic structures on $\widetilde{F}$ which map to one another (using developing maps) by a π-equivariant quasi-isometry. Thus the topology (and the the Hölder structure [GlH90]) induced on the Gromov boundary $\flat\,F$ does not depend on the chosen metric (but the differentiable one does).

Now let $\mathcal{G}(\widetilde{F}) = (\flat\widetilde{F} \times \flat\widetilde{F} - \Delta)/(\mathbb{Z}/2)$ be the space of unoriented pairs of distinct points in the Gromov boundary of $\widetilde{F}$, which identifies with the space of unoriented geodesics on $\widetilde{X}$ for any hyperbolic structure. When F is closed, $\mathcal{G}(\widetilde{F})$ is homeomorphic to a Möbius band, otherwise it is some kind of Sierpinski carpet inside a Möbius band. Following Bonahon [Bon88], we define a <u>*geodesic current*</u> on the surface F as a π-invariant positive Radon measure over $\mathcal{G}(\widetilde{F})$ (Radon means Borel, locally finite and regular). The set $\mathcal{C}(F)$ of geodesic currents on F is endowed with the (metrizable) weak $*$ uniform structure given by the family of semi-distances $d_f(\alpha, \beta) = |\alpha(f) - \beta(f)|$ for continuous functions $f\colon \mathcal{G} \to \mathbb{R}$ with compact support. This makes it into a complete space. In other terms, the signed π-invariant Radon measures, which are formal differences in $\mathcal{C}(F)$, correspond to the dual of the space of π-invariant continuous functions on $\mathcal{G}(\widetilde{F})$.

Any primitive h-loop α in F lifts to a π-invariant set in $\widetilde{F}$ and summing the Dirac measures at each of its elements defines a geodesic current. To the n-th power α, we associate n times that measure. One could also consider weighted h-loops $r\beta$ and the corresponding multiples of the associated Dirac measures. This injection from the set $\mathbb{R}_+\overline{\pi}$ of weighted h-loops into $\mathcal{C}(F)$ has a dense image. The intersection form on $\mathbb{R}_+\overline{\pi}$ extends to a continuous symmetric bilinear function $i\colon \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}_+$. Bonahon then shows how to embed properly Teichmüller space $\mathcal{T}(F)$ in the space of currents $\mathcal{C}(F)$ through the <u>*Liouville current*</u> λ_m associated to a marked hyperbolic metric $m\colon F \to X$. Instead of recalling its usual definition, we present another construction of the λ_m based on Thurston's idea of a random geodesic which helps to understand the relation between the embeddings of $\mathcal{T}(F)$ and $\mathbb{R}_+\overline{\pi}$. Choose a random vector in the unit tangent bundle $\mathrm{U}\,X$ according to the measure induced by the volume form, and consider the length t geodesic arc launched in that direction. If we close it up by any segment whose length is bounded the diameter of the surface (say the shortest) we obtain a homotopy class which pulls back by m to give an element $\alpha_t \in \overline{\pi}$. Then the weighted loop α_t renormalized by its length $l_m(\alpha_t)$ converges almost surely as $t \to \infty$ to the Liouville current λ_m renormalized by the volume $\pi\sqrt{|\chi|}$ of $T_1 X$. As usual, this map projects to an embedding $\mathcal{T}(F) \to \mathbb{P}\,\mathcal{C}(F)$. He also embeds the space of measured laminations $\mathcal{ML}(F)$ in $\mathcal{C}(F)$, the idea is more natural (and was the starting point in his circle of ideas) it is not to difficult to interpret a measured lamination as a measure on the space of geodesics. Of course, one may projectivize in $\mathbb{P}\,\mathcal{C}$ to recover Thurstons compactifiction of Teichmüller space.

The intersection form evaluated on loops or measured laminations α, β gives their geometric intersection number $i(\alpha, \beta)$; on a marked hyperbolic metric m and a loop or a measured lamination α it gives the length $l_m(\alpha)$ (after composing by the marking); and between two marked metrics m, m' it is less than $2\pi|\chi|$ with equality if and only if $m = m'$. Moreover, the space of measure laminations is the null cone of the intersection form: $i(\alpha, \alpha)$ if and only if $\alpha \in \mathcal{ML}(F)$. This picture looks very similar to the hyperboloid model in Minkowski space and its projectivization to Klein's ball model : the quadratic form has unit level surface a double sheeted hyperboloid, and restricts to the tangent space defining a constant negative curvature Riemannian metric; this projectifies to Klein's ball model. The isotropy cone contains the rays asymptotic to the hyperboloid, and projects to the boundary sphere. Bonahon pushed this analogy until he understood how to recover the Weil-Petersen *metric* using the intersection form i.

Back to the functional approach. If we endow $\overline{\pi}$ with the coarsest separated topology, that is the cofinite topology (whose closed sets are the finite sets), then the continuous functions are those with finite support, and form the vector space $\mathbb{R}^{(\overline{\pi})} = \bigoplus_{\overline{\pi}} \mathbb{R}\gamma$ over the base $\overline{\pi}$ of all h-loops. Its topological dual vector space with weak topology corresponds to the product $\mathbb{R}^{\overline{\pi}}$, which is the space of real functionals we have been considering. We may recover the intermediary space of geodesic currents by completing $\mathbb{R}^{(\overline{\pi})}$ appropriately (instead of varying the topology as we have just done).

Indeed, the intersection pairing on the set of h-loops extends to a continuous symmetric bilinear form on the vector space $\mathbb{R}^{(\overline{\pi})}$. When the orbifold F is a closed manifold, that is when $H_2(\pi; \mathbb{Z}) \neq 0$, the restriction of $i(\cdot,\cdot)$ to the codimension one subspace over the base of *non trivial* h-loops is non-degenerate. More generally it has kernel the subspace generated by unessential unoriented loops. Hence a loop which is neither trivial, encircling a conical singularity or hole, must be essential, and denoting ϖ the set of essential loops, we have a direct sum decomposition $\mathbb{R}^{(\overline{\pi})} = \mathbb{R}^{(\varpi)} \oplus \mathbb{R}^{h+s+1}$ the right factor is precisely the kernel of the intersection form i, which therefore restricts to a non-degenerate symmetric bilinear form on $\mathbb{R}^{(\varpi)}$. This defines a continuous injection $i\colon \mathbb{R}^{(\varpi)} \to \mathbb{R}^{\varpi}$ of the space into its dual, and the completion of the image can be defined as the space of signed currents $\mathcal{C}^{\pm}(F)$, which can be thought as formal differences of positive currents. The cone of positive currents $\mathcal{C}(F)$ is obtained by intersecting the countable number of half spaces $i(\cdot, \alpha) \geqslant 0$ for $\alpha \in \overline{\pi}$. So it is a cone on an infinite dimensional polytope (dual to the convex hull of the $\alpha \in \overline{\pi}$ with respect to the intersection form) with a piecewize linear fractal-shaped boundary.

The modular group $\mathrm{Mod}^{\pm}(F) = \mathrm{Out}(\pi, \overline{\delta})$ acts on ϖ and this provides a faithful linear representation into the orthogonal group $O\left(\mathcal{C}(\pi), i\right)$ of the intersection form (which is an indefinite analog to the orthogonal group of Hilbert space). In fact, Ivanov's theorem [Iva97] on the automorphism group of the curve complex (which has been extended by Luo to the holed case [Luo00]), implies that $\mathrm{Out}(\pi, \overline{\delta}) = O\left(\mathcal{C}(\pi), i\right)$.

Currents over the group. Recall from section 0 that when F has holes or boundary, that is when $H_2(\pi;\mathbb{Z})=0$, the group π does not determine the homotopy type of its orbifold F, and there is no way of telling which subset of $\overline{\pi}$ correspond to simple loops. In this paragraph we consider a lattice Fuchsian orbifold F which is equivalent to a framed lattice (π,δ). In particular such a pair defines an intersection pairing $i\colon \overline{\pi}\times\overline{\pi}\to\mathbb{N}$, who's isotropic elements correspond to simple loops. Thus it is reasonable to hope for a definition of currents and the intersection form in terms of (π,δ) only. We refer to [GlH90] for the definitions and properties of Gromov-hyperbolic spaces and groups.

Since F has no boundary, any finite volume hyperbolic structure (thought as a discrete and faithful representation $\pi\to G$) provides a quasi-isometric embedding of π in $\mathbb{H}$ (by considering the orbit of an arbitrary point), and therefore a homeomorphism between their Gromov boundaries $\flat\pi\simeq\mathbb{S}^1$. So in the finite volume case, geodesic currents could be defined as positive Radon measures on the space $\mathcal{G}(\flat\pi)$ of unoriented pairs of distinct points in $\flat\pi$ with the aforementioned topology.

Two equivalence relations on loops. Two conjugacy classes α, β in $\overline{\pi}$ are called *trace equivalent* when the absolute value (or the square) of their trace functions are equal $|t_\alpha(\rho)|=|t_\beta(\rho)|$ at every representation $\rho\colon\pi\to PSL_2(\mathbb{R})$. This implies that their geodesics have the same length for any hyperbolic metric. Leininger showed the converse [Lei03, Theorem 3.2] when F is a closed surface, so trace equivalence and hyperbolic length equivalence are the algebraic and geometric facets of the same concept. Let us sketch his argument, anticipating the next section. It boils down to the following facts : any point in Teichmüller space corresponds to a Fuchsian representation which lifts to $SL_2(\mathbb{R})$, but those are dense in the real affine variety $\mathrm{Hom}(\pi,SL_2(\mathbb{R}))$, whose complexification $\mathrm{Hom}(\pi,SL_2(\mathbb{C}))$ is an irreducible algebraic variety. Hence the equality of the the squared trace functions (which are algebraic) on the component of the set of Fuchsian representations $\mathrm{Hom}(\pi,G)$ corresponding to Teichmller space, implies their equality on an open set of the whole complex character variety, and thus everywhere including the non discrete real representations. In particular, one can reformulate trace equivalence as the equality of the intersection functions $i(\alpha,\cdot)$ and $i(\beta,\cdot)$ when restricted to the image $\mathcal{T}(F)\subset\mathcal{C}(F)$ of Teichmüller space in the space of currents.

Two loops $\alpha,\beta\in\overline{\pi}$ are called *simple intersection equivalent* if they have the same intersection number with any simple dual loop $\gamma\in\Sigma'$. By continuity of the intersection form in the space of currents and density of weighted simple loops in measured laminations, this is equivalent to asking for the equality of $i(\alpha,\cdot)$ and $i(\beta,\cdot)$ when restricted to $\mathcal{ML}$. In his article, Leininger proves that this topological notion has a geometrical counterpart: α and β have the same length for so called branched flat metrics. Such a metric amounts to a quadratic differential, or a measured lamination, so his assertion is not surprising in light of the previous comment.

Notice that Thurston's sequential description of convergence to the boundary of Teichüller space shows immediately that length equivalence implies simple intersection equivalence. The converse is not true and Leininger constructs a counter-example. We shall see in the next section how to understand the discrepancy between simple intersection and trace equivalence. This idea stemmed during a collaborative work with Moira Chas as we found a systematic way to test simple intersection and trace equivalence in the hole torus, which enabled us to compute plenty of counter-examples. It relies on a striking analogy between a state-sum formula for the trace functions and Dylan Thurston's intersection formula with a simple loop. After the algebraic tools have been set up, we shall be able to interpret simple intersection equivalence as a tropical limit for trace equivalence.

2 Character variety and Goldman Poisson algebra of loops

The previous sections focused on hyperbolic structures over surface orbifolds and their moduli spaces, which amount to Fuchsian representations of their fundamental group considered up to conjugacy and their deformation spaces. The representation point of view is more supple for several reasons : we can release the discreteness assumption, vary the Lie group PSL_2 and extend the ring of scalars to $\mathbb{C}$. Before delving into the algebraic study of character varieties over the complex numbers, let us review a couple of facts concerning the geometry of representations into $\mathrm{PSL}_2(\mathbb{C})$.

Recall that the complex projective line $\mathbb{CP}^1$ (also called the Riemann sphere as it is conformally diffeomorphic to $\mathbb{S}^2$), contains the real projective line $\mathbb{RP}^1$ as the set of fixed points under complex conjugation (an equatorial circle of the Riemann sphere). The automorphism group $\mathrm{PGL}_2(\mathbb{C})$ of the complex projective line $\mathbb{CP}^1$, contains the stabilizer $\mathrm{PGL}_2(\mathbb{R})$ of the real projective line $\mathbb{RP}^1$. The subgroup $\mathrm{PSL}_2(\mathbb{C})$ of index 2 in $\mathrm{PGL}_2(\mathbb{C})$ consists of elements preserving the orientation of $\mathbb{CP}^1$. It contains $\mathrm{PSL}_2(\mathbb{R})$, also a subgroup of index 2 in $\mathrm{PGL}_2(\mathbb{R})$, which preserves both the orientation and the equator, and acts on the upper hemisphere. This upper sphere $\mathbb{H}$ yields a conformal model for the hyperbolic plane : two points x and y in the hemisphere define a geodesic which intersects the equator in two other points x' and y', and half the logarithm of the cross ratio $[x, y; y', x']$ defines $\mathrm{PSL}_2(\mathbb{R})$-homogeneous metric with constant negative curvature on $\mathbb{H}$.

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Quasi-Fuchsian representations and Bers-uniformization A group π is <u>*Kleinian*</u> if it admits a faithful and discrete representation $\rho\pi \to \mathrm{PSL}_(\mathbb{C})$ called a <u>*Kleinian representation*</u>. Those are the orbifold fundamental groups of orientable and locally orientable 3-dimensional orbifolds which admit a hyperbolic metric. We shall not go into the details, as they are similar to those expounded in section 0 for 2-dimensional orbifolds, and we are interested in Fuchsian groups.

A representation $\rho\colon \pi \to \mathrm{PSL}_2(\mathbb{C})$ is <u>*Fuchsian*</u> if is conjugated to a Fuchsian representation in $\mathrm{PSL}_2(\mathbb{R})$, in other terms the action of π on $\mathbb{CP}^1$ through ρ preserves a circle S_ρ. The quotient $\mathbb{CP}^1/\rho(\pi)$ by the image of a Fuchsian representation consists of two surface-orbifolds of negative Euler characteristic endowed with conformal classes of hyperbolic riemannian metrics. The inversion of $\mathbb{CP}^1$ in the circle S_ρ descends to an isomorphism between the orbifold structures which is anticonformal with respect to the underlying Riemann surfaces (smooth complex curves).

A representation $\rho\colon \pi \to \mathrm{PSL}_2(\mathbb{C})$ is <u>*quasi-Fuchsian*</u> if it is faithful and discrete $\rho(\pi)$ acts in $\mathbb{CP}^1$ preserving a Jordan curve.

Let $\mathcal{QF}_F$ be the set of all quasi Fuchsian representations.

It may be non compact : denote h the number of ends. If the representation is not quasi Fuchsian, then F is connected, otherwize it is a pair of isomorphic Riemann surfaces $\mathbb{H}/\rho(\pi)$.

Stratification and foliation Denote $\mathcal{F}(\mathbb{C})$ the space of Kleinian representations (it contains the space $\mathcal{F}(\mathbb{R})$ of Fuchsian representations, whose orbits under conjugacy in $\mathrm{SL}_2(\mathbb{C})$ consists of all quasi-Fuchsian representations) topology of the quotient $\mathbb{CP}^1/\rho(\pi)$ does not distinguish

The variety $\mathrm{X}(\pi)$ is partitioned and stratified into the subvarieties $\mathrm{X}(\pi, \delta)$ consisting of representations (up to conjugation) which map the peripheral elements to parabolic matrices. Each $\mathrm{X}(\pi, \delta)$ being algebraically foliated by the subvarieties $\mathrm{X}(\pi, \delta, T)$ of representations whose peripheral traces are fixed by T, an family of h complex numbers indexed by δ.

To sum up this discussion, we may say that representations in $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$

2.1 Representation spaces, connected components and coverings

When studying the algebraic geometry of character varieties, it is far more convenient to land in the double cover SL_2 of PSL_2 and work over the complex numbers. But before defining and studying $\mathrm{SL}_2(\mathbb{C})$-character varieties from an algebraic standpoint, we relate the topology of SL_2 representation spaces to those of PSL_2, over the real and complex numbers.

In this subsection, π is the fundamental group of a genus $\mathsf{g} \geqslant 2$ closed orientable surface, and T is a connected 3-dimensional Lie group. The goal is to describe the connected components of $\mathrm{Hom}(\pi, \mathsf{T})$, how they lift and cover each other as the the group T varies under coverings, relying mostly on [Gol81, Gol82]. Let us announce the general picture without further due, before specifying it in each context of our interest.

General picture. Let S be a connected 3-dimensional Lie group which is simple (like $\mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$), and U its universal cover. The simplicity assumption on S ensures it minimal with respect to Lie group coverings. Consider an intermediate connected cover $\mathsf{U} \to \mathsf{T} \to \mathsf{S}$, which corresponds to a natural inclusion between fundamental groups $\pi_1(\mathsf{T}) \subset \pi_1(\mathsf{S})$. First recall that the fundamental group of a connected Lie group identifies with a discrete subgroup of its universal cover which is central (in particular it is abelian). It follows that more generally, the automorphism group $\mathsf{K} \simeq \pi_1(\mathsf{S})/\pi_1(\mathsf{T})$ for a finite cover of connected Lie groups $\mathsf{T} \to \mathsf{S}$ also corresponds to the kernel of a central extension $\mathsf{K} \to \mathsf{T} \to \mathsf{S}$.

We define the <u>*Euler class*</u> $\mathrm{eu} \in \mathsf{H}^2(\pi, \pi_1(\mathsf{T}))$ by its evaluation on a representation $\rho \colon \pi \to \mathsf{T}$ as follows. Given any standard presentation of π as in 0, lift the images by ρ of the $2\mathsf{g}$ generators in T to the universal cover U : the product of their commutators in U projects to the identity in T, so it lies in the center $\pi_1(\mathsf{T}) \subset \mathsf{U}$. It is well defined independently of the chosen presentation and lifts and is called the Euler class of ρ denoted $\mathrm{eu}(\rho)$. It is invariant by T-conjugacy at the target.

The Euler class is a continuous function on $\mathrm{Hom}(\pi, \mathsf{T})$, hence locally constant, so it might be useful to distinguish connected components. In fact, the level sets of $\mathrm{eu} \colon \mathrm{Hom}(\pi, \mathsf{T}) \to \pi_1(\mathsf{T})$ are connected, so the values of the Euler class $\mathrm{im}(\mathrm{eu}) \subset \pi_1(\mathsf{T})$ parametrize the connected components of $\mathrm{Hom}(\pi, \mathsf{T})$. This implies in particular that when U is a connected simply connected 3-dimensional Lie group as above, the space $\mathrm{Hom}(\pi, \mathsf{U})$ is connected. Beware that $\mathrm{eu} \colon \pi_0(\mathrm{Hom}(\pi, \mathsf{T})) \to \pi_1(\mathsf{T})$ may not be surjective, as we shall see in the examples. (In fact, when T is an algebraic group, the space $\mathrm{Hom}(\pi, \mathsf{T})$ is an algebraic variety so it must have finitely many connected components, and if in addition T has infinite fundamental group, then eu cannot be surjective.)

We now explain how to understand the Euler class as an obstruction to lifting representations, which is the only one for surface fundamental groups. If a representation $\rho \colon \pi \to \mathsf{S}$ lifts to T, its Euler class $\mathrm{eu}(\rho) \in \pi_1(\mathsf{S})$ must belong to the subgroup $\pi_1(\mathsf{T}) \subset \pi_1(\mathsf{S})$, in other terms be trivial in the kernel $\mathsf{K} = \pi_1(\mathsf{S})/\pi_1(\mathsf{T})$ of the central extension $\mathsf{K} \to \mathsf{T} \to \mathsf{S}$. It turns out (by general arguments in obstruction theory) that this is a sufficient condition to ensure a lift. So the connected components of $\mathrm{Hom}(\pi, \mathsf{S})$ which lift to $\mathrm{Hom}(\pi, \mathsf{T})$ are parametrized by the intersection of $\mathrm{im}(\mathrm{eu}) \subset \pi_1(\mathsf{S})$ with the subgroup $\pi_1(\mathsf{T}) \subset \pi_1(\mathsf{S})$. Moreover when $\rho \in \mathrm{Hom}(\pi, \mathsf{S})$ lifts to $\rho \in \mathrm{Hom}(\pi, \mathsf{T})$, we can multiply the lift by a central representation $\pi \to \mathsf{K}$ to obtain another one. Again, by obstruction theory, the set of lifts is transitive (a torsor) under this multiplicative action of $\mathrm{Hom}(\pi, \mathsf{K}) = \mathsf{H}^1(\pi; \mathsf{K})$. More precisely, every connected component X of $\mathrm{Hom}(\pi, \mathsf{S})$ which lifts, is covered by a union of components Y in $\mathrm{Hom}(\pi, \mathsf{T})$ and we have a Galois covering $\mathsf{Y} \to \mathsf{X}$ with group $\mathsf{H}^1(\pi; \mathsf{K}) \simeq \mathsf{K}^{2\mathsf{g}}$.

Of course this discussion holds without the simplicity assumption on S which makes it minimal with respect to covering, and one may discuss lifts from T_1 to T_2 in the tower $\mathsf{U} \to \mathsf{T}_2 \to \mathsf{T}_1 \to \mathsf{S}$.

Automorphisms of the complex projective line. In this paragraph $S = \mathrm{PSL}_2(\mathbb{C})$, the automorphism group of the complex projective line, also the orientation preserving isometry group for 3-dimensional hyperbolic geometry. The central extension $\mathbb{Z}/2 \to \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$ corresponds to its universal covering. Indeed, $\mathrm{SL}_2(\mathbb{C})$ retracts by deformation on its maximal compact subgroup $\mathrm{SU}(2)$ which is homeomorphic to the 3-sphere and thus simply connected.

We deduce from the general that the representation variety $\mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbb{C}))$ is connected. Moreover, there are two components in $\mathrm{Hom}(\pi, \mathrm{PSL}_2(\mathbb{C}))$ parametrized by the values of the Euler class $\mathrm{eu}\colon \mathrm{Hom}(\pi, \mathrm{PSL}_2(\mathbb{C})) \to \mathbb{Z}/2$ which is onto. The projection map from $\mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbb{Z}))$ to $\mathrm{Hom}(\pi, \mathrm{PSL}_2(\mathbb{C}))$ defines a connected covering above the component having trivial Euler class, whose Galois group $H^1(\pi, \mathbb{Z}/2)$ acts as multiplication by central representations $\pi \to \{\pm \mathrm{id}\}$.

A representation with non trivial Euler class must send the product of commutators for a standard set of generators to $-\mathrm{id}$, and this implies it cannot be reducible **because ?**. So the connected component that does not lift only contains irreducible representations. This implies that the conjugacy action by $\mathrm{PSL}_2(\mathbb{C})$ has separated orbits (**is properly discontinuous ?**) and the quotient is a smooth analytic variety. This is a noticeable fact, as it never happens for the component with trivial Euler class (which contains the identity).

Automorphisms of the real projective line. In this paragraph $S = \mathrm{PSL}_2(\mathbb{R})$, the automorphism group of the real projective line, also the orientation preserving isometry group for 2-dimensional hyperbolic geometry. Its double cover $\mathrm{SL}_2(\mathbb{R})$ retracts by deformation on its maximal compact subgroup $\mathrm{SO}(2)$ which is homeomorphic to the circle with fundamental group $\mathbb{Z}$. Hence $\pi_1(\mathrm{PSL}_2(\mathbb{R})) = \mathbb{Z}$ also, and is the kernel of the central extension $\mathbb{Z} \to \widetilde{\mathrm{PSL}}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$.

The representation space $\mathrm{Hom}(\pi, \widetilde{\mathrm{PSL}}_2(\mathbb{R}))$ is connected. It follows from works of Milnor and Wood (see [Gol82]) that the image of the Euler class $\mathrm{eu}\colon \mathrm{Hom}(\pi, \mathrm{PSL}_2(\mathbb{R})) \to \mathbb{Z}$ is equal to the interval $[\chi, -\chi]$, so $\mathrm{Hom}(\pi, \mathrm{PSL}_2(\mathbb{R}))$ has $1 + 2|\chi|$ connected components. Only the trivial component $\{\mathrm{eu} = 0\}$ lifts, and again the projection map $\mathrm{Hom}(\pi, \widetilde{\mathrm{PSL}}_2(\mathbb{R})) \to \mathrm{Hom}(\pi, \mathrm{PSL}_2(\mathbb{R}))$ defines a connected covering above it, with Galois group $H^1(\pi, \mathbb{Z})$ acting as multiplication by central representations $\pi \to \mathbb{Z}$. Moreover, Goldman showed that a representation $\rho\colon \pi \to \mathrm{PSL}_2(\mathbb{R})$ is Fuchsian (recall faithful and discrete) if and only if it has extremal Euler class $\mathrm{eu}(\rho) = \pm\chi$. They form two components distinguished by the orientation induced on the quotient spaces $\mathbb{H}/\rho(\pi)$. The action of $\mathrm{PSL}_2(\mathbb{R})$ by conjugation preserves each connected component, and is properly discontinuous over the Fuchsian components which project modulo conjugacy on two copies of Teichmüller space.

Now let $T = \mathrm{SL}_2(\mathbb{R})$ and consider the intermediate covering $\mathbb{Z}/2 \to \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$. A representation $\rho\colon \pi \to \mathrm{PSL}_2(\mathbb{R})$ lifts that central extension when its Euler class is even, in particular the Fuchsian representations can be lifted. Each component $X_{2k} \subset \mathrm{Hom}(\pi, \mathrm{PSL}_2(\mathbb{R}))$ with even Euler class $2k$ is the base of a Galois covering $Y_{2k} \to X_{2k}$ with group $H^1(\pi; \mathbb{Z}/2) \simeq (\mathbb{Z}/2)^{2g}$ acting as multiplication by central representations $\pi \to \mathbb{Z}/2$. There is however a difference between the lifts of the Fuchsian components $X_{\pm\chi}$ whose total spaces both have 2^{2g} components permuted by the Galois action whereas the others have a connected lift Y_{2k}, so in total there are $2.2^{2g} + 2g - 3$ connected components in $\mathrm{Hom}(\pi, \mathrm{SL}_(\mathbb{R}))$.

More generally, denote G_n the unique n-fold connected covering of $G_1 = \mathrm{PSL}_2(\mathbb{R})$. When m divides n the central extension $\mathbb{Z}/(n/m) \to G_n \to G_m$ yields a canonical inclusion $\pi_1(G_m) \subset \pi_1(G_n)$. Now a component of $\mathrm{Hom}(\pi, G_m)$ lifts to G_n precisely when its Euler class, which we know is zero $\bmod m$, remains trivial $\bmod n$. Above each component which lifts, we have a Galois covering map with group $H^1(\pi; \mathbb{Z}/(n/m))$, whose total space is a union of components in $\mathrm{Hom}(\pi, G_n)$.

2.2 The character variety, its ring of functions and its topology

We shall use elementary concepts in complex algebraic geometry, always dealing with affine algebraic varieties defined over $\mathbb{Z}$. The study of such an affine variety is equivalent to that of its ring $\mathbb{C}[V]$ of polynomial functions $V \to \mathbb{C}$, whose field of fraction $\mathbb{C}(V)$ correspond to the rational maps. We also denote by $\mathbb{C}[E]$ the free polynomial (also called the symmetric) algebra over a set of variables indexed by E (this does not enter in contradiction with the previous notation if we consider E as the algebraic variety whose open subsets in the Zariski topology are the cofinite sets).

Let π be a finite type Fuchsian group. By section 0, it is the quotient of a free amalgam of cyclic groups $\mathbb{Z}^{*(2g+h)} * \mathbb{Z}/c_1 * \cdots * \mathbb{Z}/c_s$ by a relation of the form $\prod[\lambda_j, \mu_j] = \prod \delta_j \prod \tau_j$, and every torsion element can be conjugated in one of the $\mathbb{Z}/c_j$ so that up to conjugacy π contains exactly s distinct maximal torsion subgroups isomorphic to the $\mathbb{Z}/c_j$. From now on, $G = \mathrm{SL}_2(\mathbb{C})$.

Algebro-geometric quotient. The set of representations $\mathrm{Hom}(\pi, G)$ is a finite dimensional algebraic variety on which the group G acts by conjugation at the target, and the invariant functions are precisely the polynomial expressions in the characters $t_\gamma \colon \rho \mapsto \mathrm{tr}(\rho(\gamma))$ for $\gamma \in \pi$. Those trace functions satisfy the following algebraic trace-relations which follow from those in π and SL_2 and properties of the trace.

The first pair of relations follows from invariance of the trace by conjugation and the fact that in SL_2 a matrix is conjugate to the transpose of its inverse. In particular t_α only depends on $\alpha \in \overline{\pi}$.

$$\forall \alpha, \beta \in \pi : \qquad t_{\alpha\beta\alpha^{-1}} = t_\beta \qquad \& \qquad t_\alpha = t_{\alpha^{-1}} \qquad\qquad \text{(conjugation \& inverse)}$$

A torsion element τ_j of order c_j must be sent into a conjugate of the discrete subgroup $\mathbb{Z}/c_j$ of unitary matrices whose orders divide c_j, so its trace must satisfy $\mathrm{PT}_{c_j}(t_{\tau_j}) = 2$ where PT_{c_j} denotes the Tchebychev polynomial such that $2\cos(c_j\theta) = \mathrm{PT}_{c_j}(2\cos(\theta))$.

$$\forall j \in \{1, \ldots, s\} : \qquad t_1 = 2 \qquad \& \qquad \mathrm{PT}_{c_j}(t_{\tau_j}) = 2 \qquad\qquad \text{(unit \& torsion)}$$

Finally, for $A, B \in \mathrm{SL}_2(\mathbb{C})$, multiply the Cayley-Hamilton relation $B + B = \mathrm{tr}(B)\,\mathrm{id}$ by A and take the trace to obtain the famous skein relation $\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\,\mathrm{tr}(B)$.

$$\forall \alpha, \beta \in \pi : \qquad t_{\alpha\beta} + t_{\alpha\beta^{-1}} = t_\alpha t_\beta \qquad\qquad \text{(skein)}$$

It is a folklore result mainly due to Procesi [CP17], that every relation among the characters t_α can be deduced algebraically from those five. In other words, the ring $\mathbb{C}[\mathrm{Hom}(\pi, G)]^G$ of invariant functions under conjugacy is the symmetric algebra (free polynomial algebra) over π modulo the ideal generated by these trace relations. This corresponds to the ring of functions over its spectrum $\mathrm{Spec}\left(\mathbb{C}[\mathrm{Hom}(\pi, G)]^G\right)$, also referred as the algebraic quotient $\mathrm{Hom}(\pi, G)//G$; which is by definition the affine algebraic variety with $\mathbb{C}[\mathrm{Hom}(\pi, G)]^G$ as ring of functions, and whose complex points are the ring morphisms $\mathbb{C}[\mathrm{X}(\pi)] \to \mathbb{C}$.

This algebraic quotient is called the <u>*character variety*</u> and is often denoted $\mathrm{X}(\pi)$, so the quotient ring under consideration may consistently be written $\mathbb{C}[\mathrm{X}(\pi)]$. So for now we have the following algebraic presentation (which actually holds for any finitely generated group π).

Theorem (Algebraic presentation)**.** *The algebra $\mathbb{C}[\mathrm{X}(\pi)]$ is generated by the t_α for $\alpha \in \overline{\pi}$ with ideal of relations generated by conjugation & inverse & unit & torsion & skein relations for all $\alpha, \beta \in \pi$.*

Remark ($\mathrm{PSL}_2(\mathbb{C})$-characters.)**.** *One may similarly define the* $\mathrm{PSL}_2(\mathbb{C})$*-character variety of* π *as an algebraic quotient of the representation space* $\mathrm{Hom}(\pi, \mathrm{PSL}_2(\mathbb{C}))/\!/ \mathrm{PSL}_2(\mathbb{C})$*. The invariant functions are polynomial expressions in the* $t_\alpha t_\beta$ *for* $\alpha, \beta \in \pi$ *equal in homology* mod2*, among which we have the squares of trace functions (but the trace functions are not well defined because of the sign). The relations are more complicated, so the algebra of* $\mathrm{PSL}_2(\mathbb{C})$*-characters is not so easy to handle and we prefer to work in* $\mathrm{SL}_2(\mathbb{C})$*.*

Recall that quasi-Fuchsian representations form a connected component in $\mathrm{Hom}(\pi, \mathrm{PSL}_2(\mathbb{C}))$ *which lifts to* $\mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbb{C}))$*, and quotienting by conjugacy (which only depends on the matrix up to a sign) on finds that the geometric component of* $\mathrm{PSL}_2(\mathbb{C})$*-character variety lifts to* $\mathrm{X}(\pi)$*.*

Notice that the polynomial trace-relations in the previous presentation are unitary with integral coefficients, so $\mathrm{X}(\pi)$ is defined over $\mathbb{Z}$ and one may consider points over any ring, but the algebraic geometry becomes much more intricate over non algebraically closed fields.

Remark ($\mathrm{SL}_2(\mathbb{R})$-characters)**.** *Defining the* $\mathrm{SL}_2(\mathbb{R})$*-character variety as an algebraic quotient requires more care because* $\mathbb{R}$ *is not algebraically closed, and many pitfalls await the non specialists in real algebraic geometry. Still, we know that any representation* $\rho\colon \pi \to \mathrm{SL}_2(\mathbb{C})$ *on which all trace functions take real values can be conjugated into a real form of* $\mathrm{SL}_2(\mathbb{C})$*, that is* $\mathrm{SL}_2(\mathbb{R})$ *or* $\mathrm{SU}(2)$ *(and possibly even into* $\mathbb{R}$ *or* $\mathbb{S}^1$*).*

Moreover, this should not prevent us from thinking about real representations, but simply to prefer the geometric point of view when the algebraic one is less well suited. Recall for instance that the Teichmüller space of a closed orientable surface identifies with a pair of connected components in the space of $\mathrm{PSL}_2(\mathbb{R})$*-representations considered up to conjugation, which can both be lifted in* 2^{2g} *ways to the conjugacy classes of* $\mathrm{SL}_2(\mathbb{R})$*-representations, and those lifts are Zariski-dense.*

Components, stratification, foliation. When π has no torsion, the ring $\mathbb{C}[\mathrm{X}(\pi)]$ is an integral domain (see [PS19, Theorem 12]), so the character variety is irreducible (see also [RBKC96]), so its complex points are connected. In general there are several irreducible components arising from the factorisation of the Tchebychev polynomials PT_{c_j} appearing in the torsion relations. Indeed, the trace of a torsion element τ_j with order c_j must satisfy $t_{\tau_j} = 2\cos(2\pi C_j/c_j)$ for some $C_j \in \mathbb{Z}/c_j$. Accordingly, the ring $\mathbb{C}[\mathrm{X}(\pi)]$ splits into a product of $c_1 \dots c_s$ factors corresponding to the possible values of the C_j; we index them by an element C in the product of cyclic groups with orders c_j, that is the torsion of the abelianization $\pi/[\pi,\pi]$, also equal to $\mathsf{TH}_1(\pi;\mathbb{Z}) \simeq \mathsf{H}^1(\pi;\mathbb{Q}/\mathbb{Z})$). Each factor $\mathbb{C}[\mathrm{X}(\pi, C)]$ is an integral domain admitting the same algebraic presentation as $\mathbb{C}[\mathrm{X}(\pi)]$ with the additional relations $t_{\tau_j} = 2\cos(2\pi C_j/c_j)$ (which may replace the $\mathrm{PT}_{c_j}(t_{\tau_j}) = 2$): it corresponds to the ring of functions over an irreducible component $\mathrm{X}(\pi, C)$ of the character variety, which is the connected component over which $t_{\tau_j} = 2\cos(2\pi C_j/c_j)$. They all have complex dimension $3|\chi_0| + 2s$.

The faithful representations are contained in the $\phi(c_1)\dots\phi(c_s)$ components indexed by the primitive elements C in the group $\mathsf{TH}_1(\pi;\mathbb{Z})$ (invertibles of the ring). Every such component contains a dense open subset of discrete and faithful representations $\mathcal{QF}_C$ which is further partitionned into $\lfloor(2g+h-1)/2\rfloor$ open analytic subsets $\mathcal{QF}_C(\mathrm{F})$ consisting of representations ρ such that $\mathbb{CP}^1/\rho(\pi)$ is homeomorphic to a double copy of F. When F has type (g,h,c), the set $\mathcal{QF}_C(\mathrm{F})$ contains in its boundary 2^h analytic subspaces of lower dimensions each, defined by the simultaneous vanishing of some peripheral traces. This defines a stratification for the space of all quasi-Fuchsian representations (lifted to $\mathrm{SL}_2(\mathbb{C})$) which is compatible with the analytic foliation whose leaves are the intersections of level hypersurfaces for the trace functions on peripheral components.

Compactifications. The following discussion, which relies mostly on [Ota12, sections 2 and 3], will apply to any finite dimensional affine alebraic variety with a fixed generating set for its algebra of functions, but we shall deal with $\mathrm{X}(\pi)$ generated by the t_α for $\alpha \in \pi$.

It is a classical fact in algebraic geometry going back to Zariski [Zar44] that such an affine variety with Zariski topology is compactified by the Riemann-Zariski space of Krull valuations for its function field. Morgan-Shalen explained how to recover this compactification by considering limits of *valuative sequences* whose elements $\rho_n \in \mathrm{X}$ are generic points with respect to the field $\mathbb{Q}$ of definition, and which converge for the weak $*$ topology given by the evaluations $t_\alpha(x_n) \in \mathbb{CP}^1$.

This Riemann-Zariski space is quite large, and one can define intermediary compactifications. For instance sending a point $\rho \in \mathrm{X}$ to $(\log(|t_\alpha(\rho)| + 2))_{\alpha\in\pi}$ defines a map $\mathrm{X} \to \mathbb{R}_+^\pi$ whose image after projectivization $\mathrm{X} \to \mathbb{P}(\mathbb{R}_+^\pi)$ has compact closure. Since this map may not be injective, this does not always define a compactification of X. However one can remedy this discrepancy by first considering the product with the natural injection into the one point compactification $\mathrm{X} \to \hat{\mathrm{X}}$, and then taking the closure of the image inside $\mathbb{P}(\mathbb{R}_+^\pi) \times \hat{\mathrm{X}}$. This defines a compact space $\overline{\mathrm{X}}_\pi$ containing a dense open set homeomorphic to X and $B_\pi(\mathrm{X}) = \overline{\mathrm{X}}_\pi \setminus \mathrm{X}$. Using their valuative sequences, Morgan-Shalen also recovered the points in the boundary $B_\pi(\mathrm{X}(\pi))$ of this new compactification in terms of the larger Riemann-Zariski compactification by valuations. Yet the proposition is not as precise as one could expect : they define a continuous surjection from a subset of of rank 1 valuations centered at infinity onto $B_\pi(\mathrm{X})$.

Moreover, it turns out that if π has no torsion, then $B_\pi(\mathrm{X})$ coincides with Thurston's space of projective measured laminations. This can be shown without any valuative considerations and follows from the definition of $B_\pi(\mathrm{X})$ and what we said in section 1. Indeed, the identity $t_\alpha = 2\cosh(l_\alpha(\rho)/2)$ shows that the length functions behave like the logarithms appearing in the definition of $B_\pi(\mathrm{X})$. This might seem surprising as we also know that when π has no torsion torsion, the real points coming from $\mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbb{R}))$ of the $\mathrm{SL}_2(\mathbb{C})$-character variety also embedd by the logarithm of trace functional into $\mathbb{P}(\mathbb{R}_+^\pi)$, and that both components with extremal euler class are simultaneously compactified by projective measured laminations. Moreover, Maxime Wolff showed in his thesis [Wol07] that every other connected component with non extrmal euler class has a boundary which strictly contains $\mathbb{P}\mathcal{ML}$.

We shall provide later in this subsection yet another way if understanding the compactification of the character variety by $B_\pi(\mathrm{X}) \simeq \mathbb{P}\mathcal{ML}$ in terms of valuations.

2.3 Algebra of characters : simple presentation and linear basis

Untill now we have mostly considered the group π abstractly. Now we consider an oriented Fuchsian obifold F (up to isotopy) with finite type $(\mathfrak{g}, \mathfrak{h}, \mathfrak{c})$ and fundamental group π (up to conjugacy). Denote (π, δ) the corresponding framed lattice (up to conjugacy), presented as in section 0 for some choice of λ_j, μ_j (unique up to the action of $\mathrm{Mod}(F) = \mathrm{Out}(\pi, \delta)$). We now have an identification between the (inverse pairs) of conjugacy classes in π and the (unoriented) $\mathfrak{h}$-loops in F. In this subsection, we use this topological information (amounting to the intersection pairing $\mathrm{i}\colon \overline{\pi} \times \overline{\pi} \to \mathbb{N}$) to *simplify* the presentation of $\mathbb{C}[X(\pi)]$ and derive a linear basis depending on F.

Simple algebraic presentations. Recall that $\mathbb{C}[X(\pi)]$ has been presented as the symmetric algebra $\mathbb{C}[\overline{\pi}]$ over unoriented loops in F, modulo the ideal generated by the unit & torsion & skein relations for $\alpha, \beta \in \pi$. Note that it makes sense to index our variables t_α by the set $\overline{\pi}$ of conjugacy classes paired by inversion, while considering all skein relations induced by any choices of α and β in the group π itself.

We may use the skein relation to show that $\mathbb{C}[X(\pi)]$ is generated by the t_γ where $\gamma \in \Sigma$ ranges over all *simple* unoriented loops. Indeed, suppose the loop $\gamma \in \overline{\pi}$ is represented by an oriented $\mathfrak{i}$-loop with a self intersection $x \in F$. Then basing the fundamental group at that point, the corresponding element $\gamma \in \pi_1(F, x)$ splits as a product of two elements $\alpha\beta$, so the skein relation expresses t_γ as a linear combination of the traces of the three loops corresponding to α, β and $\alpha\beta^{-1}$ each having strictly less self-intersections than γ.

Corollary (Simpler presentation). *The character algebra* $\mathbb{C}[X(\pi)]$ *is the quotient of the symmetric algebra* $\mathbb{C}[\Sigma]$ *over the set of simple loops by the ideal generated by the unit & torsion & skein relations where this time* α *and* β *only range among elements of* π *corresponding to simple loops in* F.

This presentation is still quite cumbersome as the skein relations may involve pairs of simple loops which intersect many times (and their product will decompose as a large combination of trace functions), but of course there is a lot of redundancy. For simple loops $\alpha, \beta \in \Sigma$ denote $\alpha \perp_0 \beta$ when $\mathrm{i}(\alpha, \beta) = 0$, $\alpha \perp_1 \beta$ when $\mathrm{i}(\alpha, \beta) = 1$, and $\alpha \perp_2 \beta$ when $\mathrm{i}(\alpha, \beta) = 2$ and their algebraic intersection is zero (meaning that the two geometric intersection points have opposite signs). When either of these occur, the loops are called *incident* and we denote $\alpha \top \beta$.

Corollary (Simplest presentation). *The algebra* $\mathbb{C}[X(\pi)]$ *is the quotient of* $\mathbb{C}[\Sigma]$ *by the ideal generated by the unit & torsion & skein relations for all* $\alpha, \beta \in \pi$ *representing* incident *simple loops.*

Proof sketch. This follows from Luo's generalization [Luo98a, Luo00, section 3] of the continued fraction expansions (the case of the torus) which we recall. For incident loops $\alpha \top \beta$, call $\partial(\alpha, \beta)$ the collection of simple loops forming the boundary of a regular neighborhood of (a taut representative for) $\alpha \cup \beta$. Those can be expressed as products of α and β or their inverses in the fundamental group based at their intersection points.

Consider ξ_{-1} a maximal set of disjoint simple $\mathfrak{h}$-loops, and ξ_0 the (finite) set of simple loops which are incident to every one in ξ_{-1}. Now for all $n \in \mathbb{N}$, let ξ_{n+1} be the set of simple loops represented by $\gamma \in \pi$ such that $\gamma = \alpha\beta$ where either $\mathrm{i}(\alpha, \beta) = 1$ and $\{\alpha, \beta, \beta\alpha\} \subset \xi_n$ or $\mathrm{i}(\alpha, \beta) = 2$ and $\{\alpha, \beta, \beta\alpha\} \cup \partial(\alpha, \beta) \subset \xi_n$. Then $\Sigma \subset \bigcup_{n \in \mathbb{N}} \xi_n$: so one can choose ξ_0 as generators for $\mathbb{C}[X(\pi)]$ and consider only the skein relations for α, β representing incident simple loops. $\square$

Remark. *We have analogous simple presentations of* $\mathbb{C}[X(\pi)]$ *for any* $\mathsf{C} \in \mathsf{TH}_1(\pi; \mathbb{Z})$, *primitive or not, by adding the* C*-torsion relations* $\mathrm{tr}(t_{\tau_j}) = 2\cos(2\pi \mathsf{C}_j / \mathfrak{c}_j)$.

Linear basis of states. We now use our geometrical model to derive a linear basis for $\mathbb{C}[X(\pi)]$.

First recall the topological procedure employed as we reduced the number of intersection points of an $\mathfrak{i}$-loop γ in order to derive our simpler presentation. This was an instance *smoothing* [Thu09] an $\mathfrak{i}$-multiloop γ at the intersection point x which consists in modifying the multiloop γ in a small neighborhood of x like in Figure 4 to produce either one of the two resolutions γ^+ or γ^-.

When the strands of the multiloop γ intersecting at x have been oriented, one may distinguish the positive smoothing γ^+ as the one respecting orientations. If like in the previous paragraph, the point x is a self-intersection of a strand γ_i, which splits as the product $\alpha\beta$ in $\pi_1(F, x)$, then the positive smoothing $\gamma_i^+ = \alpha \cup \beta$ for any orientation has one more strand whereas the other $\gamma_i^- = \alpha\beta^{-1}$ has the same number as γ. Otherwize, the intersection point belongs to different strands α, β, and then both resolutions $\gamma^\pm$ have one strand less than γ, as the intersecting strands have been merged by considering the products $\alpha\beta^{+1}$ and $\alpha\beta^{-1}$ in the fundamental group $\pi_1(F, x)$.

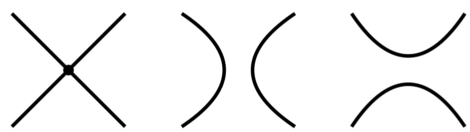

Figure 2: Smoothing the intersection point of an $\mathfrak{i}$-multiloop : $\gamma, \gamma^+, \gamma^-$.

Now define for every $\mathfrak{h}$-multiloop γ with strands $\{\gamma_j\}$, and thus in particular for any state, the element $f_\gamma = \prod_j(-t_{\gamma_j}) \in \mathbb{C}[X(\pi)]$. This quantity only depends on the homotopy class. It yields $f_\emptyset = 1$ for the empty state and $f_\gamma = -t_\gamma$ for a single loop. Of course, everything we said concerning the t_γ for single loops $\gamma \in \overline{\pi}$, namely the generation properties and relations they satisfy in the algebra $\mathbb{C}[X(\pi)]$, remain valid for the f_γ, except for the skein relation which now takes the following neater form $f_{\alpha\beta} + f_{\alpha\cup\beta} + f_{\alpha\beta^{-1}} = 0$ and this is the reason for introducing the sign which accounts for the number of strands. Moreover, this enables to properly recast the skein relation for any $\mathfrak{i}$-multiloop γ with smoothings γ^+ and γ^- at a self-intersection as in the following identity which remains the same when the crossing arcs belong to the distinct strands:

$$f_\gamma + f_{\gamma^+} + f_{\gamma^-} = 0$$

Now representing $\gamma \in \overline{\pi}$ by an $\mathfrak{i}$-loop γ in F, a recursive application of the skein relation $f_\gamma = -(f_{\gamma^+} + f_{\gamma^-})$ decomposes f_γ into the sum of f_φ for φ belonging to a collection of $2^{\mathrm{si}(\gamma)}$ one-submanifolds $\varphi \in \Phi(\gamma) \subset \Phi$ depending on the isotopy representative. Each one-submanifold is the disjoint union $\varphi = \psi \sqcup \theta$ of a state and a collection θ of trivial and torsion components.

$$f_\gamma = (-1)^{\mathrm{si}(\gamma)} \sum_{\varphi\in\Phi(\gamma)} f_\varphi = (-1)^{\mathrm{si}(\gamma)} \sum_{\phi\in\Phi(\gamma)} f_\theta f_\psi$$

Thus $\mathbb{C}[X(\pi, C)]$ is linearly spanned by the t_ψ for ψ ranging over the set of all states.

Theorem (Linear basis). *For every* $C \in \mathsf{TH}_1(\pi; \mathbb{Z})$*, the family* (t_ψ)*, where* $\psi \in \Psi$ *ranges over the set of states, forms a linear basis of* $\mathbb{C}[X(\pi, C)]$.

Proof. The linearly independance is know when π is torsion free (see [PS00] or [CM12]). In general, one can pass to a finite cover or mimic the ideas of the proof in [CM12]. $\square$

Remark (Not a monomial basis)**.** *Note that our linear basis is not stable by multiplication. In particular it does not replace the simple algebraic presentations we gave : to recover the algebraic structure one must add not only the unit & torsion relations but also the skein relations for* α, β *ranging among all pairs simple loops.*

Remark (Dependendence on the topological model)**.** *Recall from section 0 that* π *has a unique topological model when* $H_2(\pi) = \mathbb{Z}$ *or* $H_2(\pi) = 0$ *and* $H_1(\pi; \mathbb{Q}) = \mathbb{Q}$*, or equivalently from the point of view of the model's homeo-type,* $h = 0$ *or* $h = 1$ *and* $g = 0$*. In that case the linear basis is unique up to the action of* $\mathrm{Mod}(F) = \mathrm{Out}(\pi, \delta)$ *where* δ *has cardinal* $h \leqslant 1$*.*

Otherwise, denoting $2 - \chi_0 = 2g + h$*, we have* $\lfloor (2g + h - 1)/2 \rfloor$ *linear bases at our disposal, and it is compelling to study how they depend on the model, or whether one is better than the others. If any, it seems like the model with* $g = 0$ *deserves special interest : there are less simple curves, less relations in the simplest presentation (compare the one-holed torus and the three holed sphere) and the modular group is a braid group. Since we are often interested by the study of an orbifold with one hole, this method could be used to study its character variety in terms of the genus* 0 *one with same fundamental group.*

2.4 Compactifications of the character variety and intersecting loops

Support, extremal support and dual support. We still fix F a topological model for π and $C \in \mathsf{T}H_1(\pi; \mathbb{Z})$. In particular the states ψ inject naturally in the space of measured laminations $\mathcal{ML}(F)$ and the trace functions of torsion elements are the real numbers tr $f_{\tau_j} = -2\cos(2\pi C_j / c_j)$.

Definition 2.1 (Support of regular functions)**.** *Let* $f \in \mathbb{C}[X(\pi, C)]$ *be decomposed in the linear basis of states as* $f = \sum \langle f \mid \psi \rangle f_\psi$*. The support of* f *is the finite set of points in the space of measured lamination defined by* $\Psi(f) = \{\psi \mid \langle f \mid \psi \rangle \neq 0\}$*.*

Recall that $\mathcal{ML}'$ is BALBLABL the intersection form i: $\mathcal{ML} \times \mathcal{ML}' \to \mathbb{R}_+$

Definition 2.2 (Boundary, Extremal and dual Support)**.** *Let* $f \in \mathbb{C}[X(\pi, C)]$*. The extremal points of its support are the elements* $\psi_0 \in \Psi(f)$*, such that there exists* $\lambda' \in \mathcal{ML}'$ *satisfying the strict inequality* $i(\psi_0, \lambda') > i(\psi_1, \lambda')$ *for every other element* $\psi_1 \in \Psi(f)$ *in the support. The boundary support is defined similarly but with a large inequality* $i(\psi_0, \lambda') \geqslant (\psi_1, \lambda')$*. The dual support is* $\Psi(f)' = \{\lambda \in \mathcal{ML}' \mid \max i(\psi, \lambda) \leqslant 1\}$*, here the maximum is taken over all* $\psi \in \Psi(f)$*.*

Question 3. *By analogy, these correspond to the boundary and extremal points of the polytope defined as the "convex hull" of* $\Psi(f)$*, and to its dual polytope with respect to the intersection form.*

Is there a convex structure or the space of measured laminations, which behaves well enough with respect to the integral piecewize linear structure and the intersection form in order to turn the previous remark in a rigorous statement ? The space of currents might be more appropriate.

Question 4. *How to recognize the support, boundary and extremal support of trace functions coming from loops (one strand), in other terms what is the image* $\Psi(\overline{\pi}) \subset 2^{(\Psi)}$ *? We may filter this image according to the number of self intersection of the loops, or the cardinal of their supports.*

Example : the free group on two generators

Comparing length and simple intersection equivalence. Recall that two loops α, β are length equivalent when they have the same length $l_m(\alpha) = l_m(\beta)$ for any marked hyperbolic structure m on F, or equivalently when their trace functions are equal as elements in the ring $\mathbb{C}[X(\pi)]$ of regular functions of the charater variety (by Zariski-density of $\mathcal{F}(F)$ in $X(\pi)$). They are simple intersection equivalent when they have the same geometric intersection number with any other dual loop or measured lamination $\gamma \in \mathcal{ML}'$.

We now provide a unified description of trace equivalence and simple intersection equivalence for homotopy classes of loops in the orbifold F. This will be algorithmic enable us to say precisely when two loops are trace equivalent, when they are simple intersection equivalent. It will follow in particular that if a loop is filling, then there are finitely many loops in its simple intersection equivalence class. The idea is to combine relate the notion of extremal support with a formula for intersecting taut i-multiloops with dual states, due to Dylan Thurston [Thu09].

Let γ be an i-multiloop and $\lambda \in \Psi'$ a simple dual sate, considered simultaneously in F such that *their union is taut.* Suppose that γ has an intersection point x, and denote γ^+ and γ^- the two possible smoothings. Dylan's smoothing lemma says that $i(\gamma, \lambda) = \max\{i(\gamma^+, \lambda), i(\gamma^-, \lambda)\}$. He then deduces an expression for the intersection pairing with γ defined on the set Ψ'.

$$i(\gamma, \cdot) = \max\{i(\varphi, \cdot) \mid \varphi \in \Phi(\gamma)\}$$

Of course the trivial and torsion components in the one-manifolds $\varphi \in \Phi(\gamma)$ do not contribute to the intersection, so one may restrict to the torsion free states

$$i(\gamma, \cdot) = \max\{i(\varphi, \cdot) \mid \varphi \in \Phi(\gamma)\} = \bigvee_{\varphi \in \Phi(\gamma)} \sum_{\alpha \in \psi} i(\alpha, \cdot)$$

His proof uses subtle geometrical and combinatorial methods.

In particular this implies that if one chooses a taut i-multiloop γ, then any complete resolution $\phi \in \Phi(\gamma)$ which is extremal with respect to the intersection form must belong to the support of f_γ (it cannot cancel with any other).

Proposition. *Two loops $\alpha, \beta \in \overline{\pi}$ are simple intersection equivalent is and only if they have the same extremal support.*

2.5 Compactification by simple valuations and Newton polytopes

Consider the space of valuations on $\mathbb{C}[(\pi)]$

Those are precisely the functions $\nu \colon \mathbb{C}[X(\Sigma)] \to \{-\infty\} \cup [0, +\infty)$ which are null on $\mathbb{C}^*$, take finite values except for $\nu(0) = -\infty$, and satisfy for all f, g the relations $\nu(fg) = \nu(f) + \nu(g)$ and $\nu(f+g) \leqslant \max(\nu(f), \nu(g))$. We endow with the topology given by pointwize convergence, for which the action of $\mathrm{Aut}(X(\Sigma))$ is continuous.

Definition 2.3 (Simple valuations)**.**

Remark (Not a monomial basis)**.** *Note that our linear basis is not stable by multiplication.*

2.6 Goldman bracket and Poisson algebra of loops

In this subsection, F is a Fuchsian orbifold with fundamental group π, but we shall work with the subset of real points of the $SL_2(\mathbb{C})$-character variety coming from representations in $SL_2(\mathbb{R})$.

Goldman bracket for oriented loops We recall the Goldman Poisson-bracket on the symmetric algebra $\mathbb{Z}[\vec{\pi}]$ over the base of oriented loops in F. By the Leibnitz rule, it is enough to define the Lie bracket on the $\mathbb{Z}$-module with basis $\vec{\pi}$. Thus consider conjugacy classes $\alpha, \beta \in \vec{\pi}$ represented by two oriented loops in F, still denoted α and β, and such that $\alpha \cup \beta \colon \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \Sigma$ is a generic immersion (no singular points, no tangencies and no triple intersections). So α and β meet transversely, at a finite number of double points $p \in \alpha \pitchfork \beta$. At every such point, the ordered pair of tangent vectors along α and β defines an orientation; which compared to that of the ambient surface yields a sign $\epsilon(p; \alpha, \beta)$. Moreover, denoting α_p, β_p the corresponding representants in $\Pi(\Sigma, p)$, one may consider the conjugacy class $|\alpha_p \beta_p|$ of their composition. Geometrically, this corresponds to the homotopy class obtained by smoothing (resolving) the intersection while preserving orientations. Goldman showed in [**?**, section 5] that for closed surfaces the following expression does not depend on the chosen homotopy classes:

$$[\alpha, \beta] = \sum_{p \in \alpha \pitchfork \beta} \operatorname{sign}(p; \alpha, \beta) \cdot |\alpha_p \beta_p|$$

and that it satisfies, once extended bilinearly to the $\mathbb{Z}$-module over $\vec{\pi}$, the alternate and Jacobi identities; thus defining a *Lie bracket*. Chas and Gadgil generalized this in [CG16] to Fuchsian orbifolds.

Unoriented loops. The inversion of elements in the group yields an involution $\iota \colon \bar{\pi} \to \bar{\pi}$ on the set of conjugacy classes which extends to a linear involution on th symetric algebra $\mathbb{Z}[\vec{\pi}]$. This turns out to preserve the Goldman bracket so it is a Lie algebra automorphism, and its subalgebra of fixed points thus inherits a Lie bracket. But this fixed algebra, linearly generated by the $\alpha + \alpha^{-1}$, can be identified with the symmetric alebra $\mathbb{Z}[\bar{\pi}]$ over the set of unoriented loops. A simple computation shows that the bracket

$$[\alpha, \beta] = \sum_{p \in \alpha \pitchfork \beta} \operatorname{sign}(p; \alpha, \beta) \cdot |\alpha_p \beta_p|$$

Multiplicative structure and Poisson algebra The formal multiplication in $\mathbb{Z}[\vec{\pi}]$

Symplectic

Proposition. *There is a Lie algebra morphism from* $\mathbb{Z}[\vec{\pi}]$

Theorem (Chas Gadgil)**.** *Bracket determines intersection in an orbifold.*

Final remark. As a side remark, let us briefly explain how Goldman discovered this Poisson bracket. He was investigating the symplectic structure he defined in [Gol84] over the (smooth loci of) character varieties for closed orientable surfaces with values in a complex or real Lie group G satisfying very general conditions (it must preserve a non degenerate symmetric bilinear form on the Lie algebra, but concretely he works with classical linear algebraic reductive Lie groups and their standar real embeddings in $GL_N(\mathbb{R})$).

In short, his construction goes as follows. The cotangent space of $\mathrm{Hom}(\pi, G)/G$ at ρ can be identified with the first cohomology group $H^1(\pi; \mathfrak{g}_\rho)$ with coefficients in the lie algebra $\mathfrak{g}$ seen as a π-module through ρ composed with the adjoint action of $\pi \xrightarrow{\rho} G \to \mathrm{Aut}(\mathfrak{g})$. Combining the antisymetric cup-product on π with the symmetric bilinear $\mathrm{Ad}\, G$-invariant pairing on $\mathfrak{g}$, one deduces a pairing $H^1(\pi; \mathfrak{g}_\rho) \times H^1(\pi; \mathfrak{g}_\rho) \to H^2(\pi; \mathfrak{g}_\rho)$, which composed with the Poincaré duality lands in $H^0(\pi; \mathfrak{g}_\rho) = \mathbb{R}$. This pairing defines a smooth section of the second exterior power of the cotangent bundle above the smooth points of the character variety. He proves (switching to de Rham cohomology with local coefficients) that this smoothly varying non-degenerate antisymetric pairing on the tangent space of the character variety, is indeed closed.

For the $SL_2(\mathbb{R})$-character variety, that symplectic form coincides with the Weil-Petersen symplectic form over the components which are lifts of Teichmüller space. This symplectic form induces a Poisson bracket on the real subalgebra $\mathbb{R}[\mathrm{Hom}(\pi, SL_2(\mathbb{R})]^{SL_2(\mathbb{R})}$ of polynomial functions $\mathbb{C}[X(\pi)]$. He defined the Poisson bracket on the algebra of loops $\mathbb{Z}[\vec{\pi}]$ so that the map $\alpha \to t_\alpha$ extends to an epimorphism from the subalgebra of unoriented loops $\mathbb{Z}[\overline{\pi}]$ onto the Poisson algebra of polynomial functions over the $SL_2(\mathbb{R})$-character variety.

3 Skein algebra and quantization of the Golman algebra

3.1 Skein module of a 3-manifold and Kauffman bracket

A *banded link* in a compact orientable three manifold M is the embedding of a disjoint union of $n \in \mathbb{N}$ bands $\mathbb{S}^1 \times [-1, 1]$ into M, considered up to isotopy (homotopy in the space of embeddings). This is the same as a link with a trivialisation for the normal bundle of each component. Let $\Lambda = \mathbb{Z}[A^{\pm 1}]$ denote the ring of Laurent polynomials in the free variable A. The skein module $\mathrm{SS}(M)$ of M is the free Λ-module generated by banded links modulo the local relations depicted in figure REFIGURE.

FIGURE

The skein module of the three sphere is isomorphic to Λ, generated by the class of the empty set $[\emptyset]$ (a union of zero bands). It is clear that $P \in \Lambda \mapsto P \cdot [\emptyset] \in \mathrm{SS}(\mathbb{S}^3)$ defines an injective morphism and the Kauffman bracket provides the inverse. To construct a reciprocal, first represent a banded link L in the sphere as a planar link diagram where the plane provides the so called blackboard framing. By plane diagram we mean (as usual in knot theory [Kau01]) an i-loop whose intersections come with a binary symbol encoding whether they correspond to over or under crossings in the ambient space. Splicing its intersections all at once yields a state sum formula for its Kauffman bracket $\langle L \rangle \in \Lambda$ (see [Kau87, Kau01]) and defines the reciprocal homomorphism $\mathrm{SS}(\mathbb{S}^3) \to \Lambda$. In particular, the skein module of the sphere has a commutative algebra structure where multiplication is obtained by taking the union of two banded links and smoothing intersections, the result being independent on their relative positions.

Now if M is the thickened surface $F \times [-1, 1]$, its skein module is freely generated over Λ by the set of all *states*: non-trivial simple multiloops of F along with the empty loop. Again, this is shown by representing a band L as a diagram drawn on the surface and splicing all intersections to obtain its Kauffman bracket $\langle L \rangle \in \mathbb{Z}[\Sigma]$; a common generalisation of the spherical case and the partition function for generic loops in F. Following Kauffman, we denote $\langle L \mid z \rangle$ the coefficient of L corresponding to the state z, that $\langle L \rangle = \sum_z \langle L \mid z \rangle \cdot z$. An isotopy between banded links in M can be realised on the associated i-loops in the surface as a composition of regular homotopy moves (isotopy plus the local Reidemeister bigon move II and triangle move III) and an additional move III' which corresponds performing twice the monogon move I with opposite crossings. REFIGURE

FIGURE REGULARHOMOTOPY

Denote $\mathrm{SS}_{-1}(M)$ the tensor product $\mathrm{SS}(M) \otimes_\Lambda \mathbb{Z}$ where the action of A on $\mathbb{Z}$ is by -1 multiplication; this corresponds to evaluating A at -1. As a $\mathbb{Z}$-module, it is freely generated by the set of states. But when $A = -1$, we have both $A = A^{-1}$ so that the splicing of an intersection does not depend on whether it is an over or under crossing, and invariance by the three Reidemeister moves, so we can define a commutative multiplication on $\mathrm{SS}(M)$ by considering the union of two planar diagrams followed by the splicing procedure. After what we have said in the first section and a few graphical manipulations one eventually discovers that this algebra structure on $\mathrm{SS}_{-1}(M)$ is isomorphic to the skein algebra of characters $\mathbb{Z}[\Sigma]/I_\Sigma$ (see CITE for a proof).

When M is the unit tangent bundle of a surface F, an element in the skein module $\mathrm{SS}(M)$ can be represented as an i-multiloop in F whith the "blackboard" framing, and banded link isotopies in M coincide with regular homotopies of i-multiloops on F. As before, $\mathrm{SS}(M)$ is the free module over simple h-multiloops and the same procedure yields an algebra structure on $\mathrm{SS}_1(M)$. However, the invariance in the Reidemeister moves involves different signs which are relevant for the first move, and this time the Kauffman bracket recovers the regular partition function. UNSURE

3.2 Skein algebra and character variety for the unit tangent bundle

Character variety of the unit tangent bundle We can define

We have a natural epimorphism of fundamental groups $\vec{\pi} \to \pi$ which yields a monomorphism $\mathbb{C}[X(\pi)] \to \mathbb{C}[X(\vec{\pi})]$ into the subvariety coming from representations $\rho\colon \vec{\pi} \to \mathrm{SL}_2(\mathbb{C})$ such that ρ sends the fiber to id.

Remark. *Isotopy classes and homotopy classes of knots in* $\mathrm{U\,F}$ *can be represented by immersed loops in* F *up to the third Reidemeister move and regular homotopy respectively. Moreover, a knot is the lift of a primitive geodesic for some riemannian metric on* F *if [NC01] and only if [MFS82] its projection is taut. And by [HS94] the set of taut* i*-representatives of an* h*-loop are connected by the third Reidemeister move.*

3.3 Quantization and linking knots

JMARCHE derived skein module

Special case: a note on the quantum torus. We explain following [FG00] an easy way to compute the Poisson Lie bracket of simple loops $\{t_\alpha \mid t_\beta\}$ in the (non holed) torus by interpreting its skein algebra as functions over the quantum torus. The algebra of functions over the classical torus is isomorphic to the ring of bivariate Laurent polynomials $\mathbb{C}[l^{\pm 1}, m^{\pm 1}]$. A deformation of the multiplicative law yields an algebra over the ring $\Lambda_{\mathbb{C}} = \mathbb{C}[A^{\pm}1]$ of complex Laurent polynomials called the noncommutative torus, which contains the skein algebra of the torus. One may think of the elements in this noncommutative algebra as functions over some space called the quantum torus. All this discussion can be led over $\mathbb{Z}$ instead of $\mathbb{C}$.

To be more precise, define the *noncommutative torus* as the Λ-algebra generated by two variables m, l satisfying the relation $lm = A^2 ml$. As a Λ-module, it is freely generated by the $e_{p,q} = A^{-hq} l^p m^q$, which verify the relations $e_{p,q} e_{r,s} = A^d e_{h+r,q+s}$ where $d = ps - qr$. Then it is shown in [FG00, Theorem 4.3] that the map from the skein algebra $\mathrm{SS}_{-1}[^2]$ of the torus to the quantum torus sending the loop (p, q) to $e_{p,q} + e_{-h,-q}$ defines an injective morphism of Λ algebras with image the even subalgebra of the quantum torus. The proof relies on their product-to-sum formula [FG00, Theorem 4.1] which in our notations states that the multiplication in the skein algebra of the bands (p, q) and (r, s) with blackboard framing is equal to $A^d(h+r, q+s) + A^{-d}(p-r, q-s)$.

Knowing this one can compute, using Turaev's quantization, the Poisson bracket between the trace functions $t_{p,q}$ and $t_{r,s}$ of simple loops in the torus, by deriving at $A = -1$ the product-to-sum formula: $\{t_{p,q} \mid t_{r,s}\} = (-1)^{d-1} d\,(t_{h+r,q+s} - t_{p-r,q-s})$. One may wish to deduce from this a formula computing Poisson brackets in the holed torus... yet naïve approaches involving continued fractions lead to computational challenges from which I have not managed to extract anything.

4 Mixed volumes, intersecting loops and linking geodesics

4.1 Newton polytope of a trace function

Intersection form and duality in $\mathcal{ML}$ The pairing $i(\cdot,\cdot)$ together with the integral structure dual lattice to

$$\mathcal{ML}(F;\mathbb{Z}) = \{\frac{1}{2}\psi \mid \psi \in \Psi', \quad \text{and} \quad [\psi] = 0 \in H_1(F, \partial F; \mathbb{Z}/2)\}$$

with respect to the intersection form is the set of with half integral weights on the seprating components. Recall that a simple loop is separating when it disconnects the surface, equivalently when its homology class in $H_1(F;\mathbb{Z}/2)$ is trivial.

The pairing $i(\cdot,\cdot)$ together with the piecewize linear structure provides a notion of duality for poygons

Counting modular group orbits of loops via symplectic geometry In [Mir16], Mirzakhani estimates the number of elements in the modular group orbit of an h-multiloop γ, as a bound on their complexity tends to infinity. In particular she obtains, for every multiloop γ in a surface F of type (g,n) endowed with a complete hyperbolic structure $X \in \mathcal{M}_{g,n}$, the following asymptotic formula:

$$\lim_{L\to\infty} \frac{\#\{\alpha \in \mathrm{Mod}\cdot\gamma \mid l_X(\alpha) \leqslant L_0\}}{L_0^{6g-6+2n}} = \frac{n_\gamma m(X)}{m_g}$$

in which n_γ, $m(X)$ and m_g depend respectively only on the orbit of the homotopy class γ, the hyperbolic metric X, and the genus m_g. More precisely, $m(X)$ is the volume for the Thurston measure in $\mathcal{ML}_{g,n}$ of the set $\{\lambda \in \mathcal{ML} \mid i(X,\lambda) \leqslant 1\}$ and m_g is the integral of the proper function $m(X)$ over moduli space $\mathcal{M}_{g,n}$ against the Weil-Peterson volume. She suggests a qualitative description of the n_γ in terms of volumes in moduli spaces, which remains a little mysterious and of no computational use.

Counting loops via current analysis. Following her work, Rafi and Souto [RS17] generalized this formula from the perspective of geodesic currents, providing an expression for the constants n_γ which puts them on a similar footing to the $m(X)$. If α is a filling current, denote $m(\alpha) = \mathrm{vol}_{Th}\{\lambda \in \mathcal{ML} \mid i(\alpha,\lambda) \leqslant 1\}$. This is coherent with $m(X)$ when the hyperbolic metric X is interpreted as the Liouville current on the space of geodesic. Then for any two filling currents α and γ:

$$\lim_{L\to\infty} \frac{\#\{\varphi \in \mathrm{Map}(S) \mid i(\alpha, \varphi(\gamma)) \leqslant L_0\}}{L_0^{6g-6+2n}} = \frac{m(\alpha)m(\gamma)}{m_g}.$$

In particular if γ is a filling loop, they show that $n_\gamma = m(\gamma)$. This is another key point for deriving a combinatorial expression of n_γ in terms of the states of γ.

Remark. *Note that Mirzakhni's result holds for any loop as she counts elements in an orbit instead of mapping classes. However, as long as cardinals of stabilizers are dealt with carefully, one may perform topological recursion until all the loops left are filling, and the problem of counting orbits boils down to that of counting mapping classes.*

A combinatorial computation of the $m(\gamma)$. In this paragraph we explain how one can compute the constants $m(\gamma)$ from the states of a filling loop γ in F. It is a simple application Dylan Thurston's intersection formula to Rafi-Souto's result, using the expression of Thurston's symplectic pairing in Hatcher's standard train track charts for $\mathcal{ML}$.

By Dylan's formula **??**, the borelian $\{\lambda \in \mathcal{ML} \mid i(\gamma, \lambda) \leqslant 1\}$ is the intersection over the collection of states $\zeta \in \sigma_\gamma$ of the $B_\zeta = \{\lambda \in \mathcal{ML} \mid i(\zeta, \lambda) \leqslant 1\}$. Now fix a Fenchel-Nielsen decomposition $\{s_j\}$ of the surface (defined in A.1), and consider the associated cellulation of $\mathcal{ML}$ provided by the standard train tracks (see [PH92, Chapter 2] or [Hat88, figure 7]: there are four possible combinatorics by pants, and two for each cuff (depending on the inequalities satisfied by the weights). Those make up a finite number of charts T_t, disjoint up to a negligible set, each one being defined by a finite number of inequalities in $\mathbb{R}^N$ where $N = 6g - 6 + 2h$ is the total number of edges of a standard train track. The Fenchel-Nielsen coordinates are given by the weights over the edges of the train track, that is the intersection numbers l_j and twisting numbers θ_j with the loops in the pants decomposition. The convex sets $B_\zeta \cap T_t$ are defined by the triangular inequalities defining C_τ and a finite number of inequations on the coordinates l_s The symplectic pairing in each chart is given by $\sum dl_j \wedge d\theta_j$ where the l's are the intersection numbers with the loops defining the pants decomposition, and the θ's are the twisting numbers appearing in the annuli inserted between two pants. We are thus looking for the standard euclidean volume in $\mathbb{R}^N$ of the intersection over s's and union over t's of the $T_s \cap T_t$; for a computer, this is a an easy chore.

If γ is not filling, one can make the same computation by considering a smaller subsurface F' filled by γ. HOW and WHY ?

4.2 Intersection numbers and mixed volumes in $\mathcal{ML}$

4.3 Linking numbers and Poisson bracket

A Modular structures and Teichmüller tower

A.1 Modular structures

Luo observes that in a hyperbolic surface F, the set Σ' of its essential simple loops as well as the set $\mathcal{F}$ of its Fenchel-Nielsen decompositions bear modular structures; that is a $(\mathrm{PSL}_2(\mathbb{Z}), \mathbb{QP}^1)$-structure in the sense of Thurston [Thu97] with the additional compacity assumption that the automorphism group of the structure acts with finite orbits. Although to my knowledge he has not used them as such, they provide a better intuition for the relations between simple loops, and rely on ubiquitous principles which are the object of this section.

A Fenchel-Nielsen system, also called pants (or three holed sphere) decomposition, of $F_{g,n}$, consists of a simple multiloop with $3g-3+n$ components. An Euler characteristic count shows indeed that the complement is a disjoint union of pants. The *level* l of a surface is the number of loops in any of its pants decompositions. At level 0 we have the sphere with three holes, at level 1 the sphere with four holes and the once holed torus, and at level 2 the sphere with five holes and the twice holed torus. Note that two essential level 1 subsurfaces of F which intersect in a level 0 subsurface have union a level 2 subsurface.

Level one surfaces. We explain first the modular structure on simple loops in the holed torus $F_{1,1}$. The fundamental group Γ is the free group F_2 over two generators α, β. Essential simple loops Σ' are in bijective correspondence with $\mathbb{QP}^1$. They correspond to the integral points in the space of measured laminations with no boundary parallel components, which corresponds to the positive cone over $\mathbb{RP}^1$, or if you prefer the quotient of the plane $\mathbb{R}^2 = H_1(F_{1,1}; \mathbb{R})$ by the symmetry with respect to the origin.

Since two simple loops corresponding to different slopes p/q and r/s in $\mathbb{QP}^1$ intersect $ps - qr$ times, an essential simple multiloop contains at most one slope p/q with non zero multiplicity $m_{p/q}$. Denoting $s = m_{p/q}(p, q) \in \mathbb{Z}^2$ such a multiloop, its intersection with another one s' is $|\det(s, s')|$. Let C be a conjugacy class in Γ, represented by an i-loop c, which we may choose to be taut. Each one of its $2^{\mathrm{si}(c)}$ states defines (by forgetting unessential components) a unique vector $z = m_{p/q} p/q \in \mathbb{Z}^2$, so the set of *essential states* σ'_v is just encoded by a finite set of integral points in the cone $\mathbb{R}^2/\pm 1$. Dylan's formula thus provides a neat expression for the intersection function with c over Σ: $i(c, s) = \max\{|\det(z, s)| \mid z \in \sigma'_c\}$.

Note that if two simple loops α and β intersect once, which we denote $\alpha \perp_1 \beta$, then the two resolutions of the intersection correspond to their products $\alpha\beta$ and $\beta\alpha$ in the fundamental group based at that point. The graph whose vertices consists of essential simple loops $p/q \in \mathbb{QP}^1$ and whose edges are given by the relation $\perp_1$, is the one skeleton of the Farey complex pictured below. Every pair of once intersecting essential simple loops $\alpha \perp_1 \beta$ gives rise to a positively oriented triangle $(\alpha, \beta, \alpha\beta)$ and conversely. It also gives rise to a negatively oriented triangle $(\alpha, \beta, \beta\alpha)$ adjacent to the previous one along the edge (α, β): together they form a *quadrilateral*. The triangles make up a homogeneous set under the action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ by Möbius transformations.

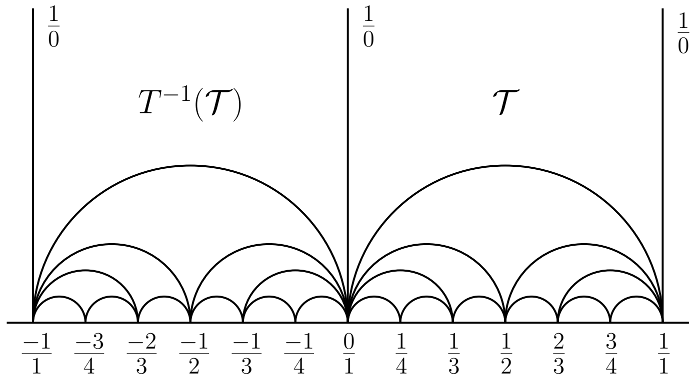

Figure 3: The Farey complex as a tiling of the hyperbolic plane.

The approximation of a rational number p/q by the partial series of its continued fraction expansion:

$$\frac{1}{0} \quad , \quad \frac{0}{1} \quad , \quad \frac{p_1}{q_1} \quad , \quad \ldots \quad , \quad \frac{p_n}{q_n} = \frac{p}{q}$$

decomposes any element of Γ representing the simple loop $p/q \in \mathbb{QP}^1$ as a non trivial product of two elements in Γ representing $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_{n-2}}{q_{n-2}}$ which intersect once. This implies that any simple loop can be written as a product of loops $\alpha = \alpha_1 \ldots \alpha_n$ such that every initial segment $\alpha_1 \ldots \alpha_j$ is a simple loop intersecting the simple loop α_j once. Thus the ideal I_Σ is generated by the **??** relation and the skein relations for pairs of elements in Γ corresponding to simple loops intersecting at most once. A more general phenomenon was mentioned in section 2.2, we shall come back to it in a moment.

In the four holed sphere $F_{0,4}$, two distinct simple loops α and β always intersect twice with opposite signs. This is denoted $\alpha \perp_2 \beta$. The hyperelliptic involution quotients $F_{0,4}$ to $F_{1,1}$ (after erasing conical singular points), and establishes a bijective correspondence between essential simple loops (a simple loop in an orbifold cannot loop around a singularity). Moreover this correspondance sends $\perp_2$ to $\perp_1$, so the preceding modular structure over the space of essential simple loops applies in the exact same way provided one multiplies all intersection numbers by two.

Modular structure on simple loops. We have seen that the set of essential simple loops on level 1 surfaces carry modular structures, and so does the level 0 surface (although it is empty, we shall comment on this just in a moment). Following Luo [**?**, section 3], each level 1 subsurface F' defines a charts for the modular structure on $\Sigma'(F)$ by considering the restriction of $\Sigma'(F)$ to the set of loops contained in F'. Since every simple loop is contained as an essential loop in level 1 subsurface, the union of all charts covers Σ'. Moreover, the automorphism group of the modular structure is [**?**] the orientation preserving modular group of the surface, and it acts with two orbits on Σ' so the structure is compact. There is a somewhat uneasy feeling about the transition maps: since any two charts intersecting in at least two elements are equal, there is nothing to check. To observe non-trivial overlappings, one has to consider level 2 sub-surfaces, and that is where all the richness of the modular structure on the set of simple loops appears.

Modular structure on Fenchel-Nielsen systems. Hatcher and Thurston showed in [HT80] (see [Luo98a, Lemma 5.2]) that any two pants decompositions are related by a finite number of simple moves which we define now. Denote $\alpha \top \alpha'$ when either $\alpha \perp_1 \alpha'$ or $\alpha \perp_2 \alpha'$. A *simple move* consists in replacing a component α_j of a pants decomposition α with another simple loop α'_j such that $\alpha'_j \top \alpha_j$ (of course α'_j must not intersect any of the other α_k to maintain a pants decomposition). Now following [Luo98b, appendix B], we define a charts $\mathcal{F}(F)$ around $\alpha \in \mathcal{F}$ as follows. Fix a component α_j and denote F' the only level 1 subsurface containing α_j and disjoint from the others; consider the set $U_j(\alpha) = \{\alpha' \mid \forall k\,,\, \alpha'_k = \alpha_k, \alpha'_j \top \alpha_j\}$, map $\mathcal{F}(F)$ to $\mathcal{F}(F')$. Then the natural map $\alpha' \in U_j(\alpha) \mapsto \alpha'_j \in \mathcal{F}(F')$ is a chart to $\mathbb{QP}^1$. Clearly, the charts satisfy the covering property, and again two charts intersect in at most one element trivially. Hatcher and Thurston's lemma, says that given two pants decompositions, there is a sequence of overlapping charts going from one to the another. This SHOULD? imply that that the full automorphism group of the structure is the oriented modular group and thus by the classification of surfaces that the structure is compact.

Generalized continued fraction approximation. The continued fraction expansion for simple loops in the holed torus may be generalized to a surface $F_{g,n}$ of general type. The relations $\perp_1$, $\perp_2$, and their union $\top$ as well as the notions of (oriented) triangles and quadrilaterals for simple loops readily extend to the space $\Sigma_{g,n}$ of simple loops in F. If $\alpha \top \beta$, denote $\partial(\alpha, \beta)$ the set of boundary components of a tubular neighborhood for $\alpha \cup \beta$. It has one either one or four elements according to whether $\alpha \perp_1 \beta$ or $\alpha \perp_2 \beta$. Given a subset $G_0 \subset \Sigma'$, we may construct a sequence $G_{n+1} = G_n \cup \{\gamma \mid \gamma = \alpha\beta \text{ where } \alpha, \beta, \beta\alpha \in G_n,\ \partial(\alpha, \beta) \subset G_n\}$, and define their union G_∞. Then say that G_0 *generates* Σ' if $G_\infty = \Sigma'$. In terms of the modular structure, the sequence is obtained by completing triangles and quadrilaterals.

Luo showed [Luo10, Proposition 1] (see also [**?**, Lemma 3.2]) that if we define $G_0(\gamma)$ for a simple multiloop γ as the set of simple loops γ' such that $\gamma' \top \gamma_i$ or $\gamma' \cap \gamma_i = \emptyset$ for every component γ_i; then $G_0(\gamma)$ generates Σ'. In particular, any function f defined on Σ whose value on $\alpha\beta$ is determined by those on α, β, $\beta\alpha$ and the elements of $\partial(\alpha \cup \beta)$ is uniquely determined by its restriction to $G_0(\gamma)$.

Remark (approximation of measured laminations)**.** *We may also generalize continued fraction approximation of measured laminations by their integral points, from the classical case of the (holed) torus to the general type surface. Since Dehn twists generalize naturally the two dimensional transvections (parabolic Möbius transformations), the analog for quadratic surds (periodic continued fractions by Lagrange), correspond to the stable and unstable foliations of a pseudo-Anosov diffeomorphisms.*

A.2 Luo-Grothendieck's reconstruction principle

In his Esquisse d'un programme [SL97], Grothendieck suggests a reconstruction principle for the geometry of the Teichmüller spaces $\mathcal{T}_{g,n}$ from that of the first two levels; the *level* being given by the dimension $3g - 3 + n$, which corresponds to the number of interior loops in a pants decomposition of the underlying topological surface. In level 0 we have the sphere with three holes, in level 1 the sphere with four holes and the once holed torus, and in level 2 the sphere with five holes and the twice holed torus. Mirzakhani's topolgical recursion formula can be thought as a quantitative reconstruction principle for the volumes of strata in moduli spaces induced by Weil-Petersen symplectic structures.

Luo proved such reconstruction principles for hyperbolic length functions in [Luo98a] as well as for measured laminations in [Luo10], which we now explain since it will be essential to our discussion. A function $f\colon \Sigma \to \mathbb{R}$ is called <u>*geometric*</u> if it is the geometric intersection with an <u>*extended*</u> measured lamination: its leaves may have ends in the boundary of the surface. Call the <u>*restriction*</u> of f to an essential subsurface, its restriction to all simple loops which can be homotoped inside it (including its boundary components). Using Hatcher and Thurston's theorem saying that two pants-decompositions of a surface are related by a finite number of special moves, Luo reduces the geometric property to subsurfaces of level at most 2, and then proves the reconstruction principle for those to obtain the following.

Theorem (Luo, [Luo10])**.** *A function is geometric if and only if its restriction to (the simple loops contained in) any essential subsurface of level one is geometric.*

He then provides necessary and sufficient conditions on a function defined on the simple loops in a level one subsurface to be geometric. We denote δ the boundary component of a level one surface, and δ_i its components for $F_{0,4}$.

Theorem (Luo, [Luo10])**.** *A function $f\colon \Sigma(F_{1,1}) \to \mathbb{R}$ is geometric if and only if for all positive triangles $(\alpha_1, \alpha_2, \alpha_3)$ and all quadrilaterals, say made up of the previous one and the negatively oriented triangle $(\alpha_1, \alpha_2, \alpha_3')$, we have:*

$$\begin{aligned} f(\alpha_1) + f(\alpha_2) + f(\alpha_3) &= \max\{2f(\alpha_i), f(\delta)\} \\ f(\alpha_3) + f(\alpha_3') &= \max\{2f(\alpha_1), 2f(\alpha_2), f(\delta)\} \end{aligned}$$

A function $f\colon \Sigma(F_{0,4}) \to \mathbb{R}$ is geometric if and only if for all positive triangles $(\alpha_1, \alpha_2, \alpha_3)$ such that $(\alpha_j, \delta_s, \delta_r)$ bounds a holed torus and, and for all quadrilaterals $(\alpha_1, \alpha_3', \alpha_2, \alpha_3)$:

$$\begin{aligned} f(\alpha_1) + f(\alpha_2) + f(\alpha_3) &= \max_{\substack{1\leqslant i\leqslant 3\\ 1\leqslant p\leqslant 4}}\{2f(\alpha_i), 2f(\delta_p), f(\delta), f(\alpha_j) + f(\delta_r) + f(\delta_s)\} \\ f(\alpha_3) + f(\alpha_3') &= \max_{\substack{1\leqslant i\leqslant 2\\ 1\leqslant p\leqslant 4}}\{2f(\alpha_i), 2f(\delta_p), f(\delta), f(\alpha_j) + f(\delta_r) + f(\delta_s)\} \end{aligned}$$

Moreover in both cases, the corresponding extended measured lamination is integral if furthermore $f(\delta) \in 2\mathbb{Z}$.

Classical invariants for tropical geometric valuations Consider a measured lamination λ which is a weighted union of simple loops $\lambda_1 \cup \dots \lambda_r$ whose components are weighted by real coefficients. The associated geometric tropical valuation ν_λ has rank one and rational rank equal to the dimension of the $\mathbb{Q}$ vector space spanned by the weights. This is bounded by r.

Proposition A.1 (monomial geometric tropical valuations)**.** *For a real weighted multiloop λ as above, the valuation ν_λ is monomial with transcendance degree $3g-3-\mathtt{ratank}(\nu_\lambda)$.*

In particular, the ν_λ are Abyhankar

Geometric valuations of higher rank There are other ways to construct valuations from simple multiloops. Those generalize the previous case in that their rank r can take any value up to $3g-3+n$. Consider a multiloop λ which has been decomposed as a union $\lambda_1 \cup \cdots \cup \lambda_r$ over an ordered set of r non trivial simple multiloops. Equivalently, the components of λ (distinct non-trivial homotopy classes in Σ) have been partitioned into r non-empty sets, and those parts have been given an order. Such an element can be denoted $(\lambda_1, \dots, \lambda_r)$. Then minus the intersection numbers with the λ_i sends any monomial of the skein algebra into a subgroup of $\mathbb{R}^r$ endowed with the lexicographic order. The proof of the following proposition follows the same lines as the rank-one case.

Tangent space at a valuation. Fix ν a valuation on $\mathbb{K}(X)$. We wish to define a tangent space to the space of valuations at that point. If ν is a divisoral valuation with center W (in particular it has rank one) it is natural to expect that the tangent space at ν is the set of valuations ν' whose center is in the exceptional divisor $\widetilde{W}$ of the blowup of X at W. Algebraically, this implies that $\nu' \subset W = \nu$.

See appendix B of Favre Jonsson for tangent space of divisoral valuations.

Poisson structure on the space of valuations

B Idées

Kauffman décomposition des courants sur $\mathbb{P}\mathcal{ML}$ Si on parvient a mettre tructure convexe sur Teichmüller plus PML dans courrants géodésiques, alors en s'inspirant du théorème de Choquet, on aimerait désintégrer la mesure (courrant) associée à une métrique comm combinaison linéaire convexe des points extrêmaux c'est à dire certainement de PML. Cela donnerait une formule du genre

$$i(\rho, \cdot) = \int_{\mathbb{P}\mathcal{ML}} i(\lambda, \cdot) \times i(\rho, \lambda) \, d \operatorname{vol}_{Th}(\lambda)$$

La preuve pourrait passer le théorème de Bowen Margulis ou des choses analogues à ce qui est fait en termes de convergence des mesures dans Rafi-Souto.

On pourrait peut être même montrer (mais alors ce serait en soi un fait remarquable) que si l'on prend une suite de courbes fermées aléatoires parmis toutes les courbes de longueur N, alors leur Kauffman-décomposition dans le module de skein converge (au sens de courants géodésiques) vers la métrique. Pour cela la preuve s'appuyerait sur "la courbe alétoire de Thurston" présentée par Bonahon [Bon88]. On peut également au lieu de prendre des courbes aléatoires, moyenner sur toutes les courbes de longeur N; la preuve passerait alors plutôt par un argument style Bowen Margulis.

A quoi ça sert ? C'est vrai qu'on a déjà le pendant algébrique de cette formule et il est bien meilleur: si l'on connait la longueur, c'est à dire la trace, pour ρ de $2^m - 1$ courbes simples (de Horowitz) alors la longueur de toute courbe (même pas simple) s'exprime comme l'évaluation en ρ de son polynôme de Fricke relatif à ces $2^n - 1$ courbes.

L'avantage de la précédente formule, réside surtout dans le fait de faire correspondre (grâce à la structure convexe) les points de vus courants et skein. En effet on peut la penser comme un analogue de "la fonction de partition" d'une courbe: on décompose un courant géodésique comme barycentre de multicourbes simples.

Série L d'une valuation Formule de McShane.

Isotopy and regular homotopy loop invariants. We define the *state sum* of an i-multiloop and show it is a regular homotopy invariant. This notion will make it easy to deduce the promised linear basis, and will be of crucial importance as we revisit compactifications in the sequel.

Recall the topological procedure employed as we reduced the number of intersection points of a loop γ in order to derive our simpler presentation: this was an instance of *splicing* [Kau01] also called *smoothing* [Thu09] an i-multiloop γ at the intersection point x. This consists in modifying the multiloop γ in a sufficiently small neighborhood of the intersection point as in Figure 4 to produce either one of the two resolutions γ^+ or γ^-. In general, the crossing may involve one or two strands of the multiloop, and in both cases the total number of strands may stay the same, or else respectively increase or decrease by one.

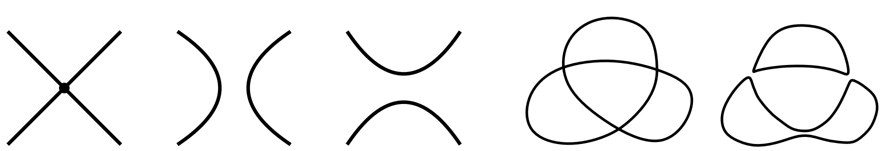

Figure 4: Smoothing an intersection point : $\gamma, \gamma^+, \gamma^-$. A state of an i-multiloop in $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$.

The simultanenous choice of a smoothing for each self intersection of γ yields a one-dimensional submanifold $\varphi \in \Phi$, which is the disjoint union $\psi \sqcup \phi$ of a state $\psi \in \Psi$, and trivial or torsion components ϕ. The collection (multiset) of those $2^{\mathrm{si}(\gamma)}$ submanifolds is denoted $\Phi(\gamma)$. Note that $\mathrm{si}(\gamma)$ and even moreso $\Phi(\gamma)$, highly depend on the *isotopy* class of γ, not just on its homotopy class.

Now let us define a weaker invariant $\Psi(\gamma)$ of the $\mathfrak{i}$-multiloop belonging to the free $\mathbb{Z}$-module over the base of states Ψ. For this, in every submanifold $\varphi = \psi \sqcup \phi$ in the collection $\Phi(\gamma)$, replace ϕ by $(-2)^{k}$ where k counts the number of (trivial) components, and sum the terms $(-2)^{k}\psi$ in this modified collection to get an element $\Psi(\gamma) \in \bigoplus_{\Psi} \mathbb{Z}\psi$. We call this the <u>*state sum*</u> of γ as we recognize an expression reminiscent of the state sum formula for the Kauffman bracket [Kau87, Kau01].

Proposition (Regular homotopy invariant)**.** *The state sum* $\Psi(\gamma) \in \bigoplus \mathbb{Z}\psi$ *is invariant under regular homotopy, that is under homotopy in the space of* smoothly differentiable immersions.

Proof. Note that unlike general states, a state without trivial components has a unique simple representative up to isotopy. We must check that such an expression is locally invariant under the second and third Reidemeiseter moves. This follows from an easy graphical computation using the skein relation to resolve the two or three intersection points before and after the local move. □

Remark. *Since by [HS94] the set of taut* $\mathfrak{i}$*-representatives of an* $\mathfrak{h}$*-loop are connected by the third Reidemeister move, this expression can actually be canonically defined for an* $\mathfrak{h}$*-multiloop.*

This whole discussion about isotopy and regular homotopy invariants will serve later on, as we consider the character variety of the unit tangent bundle $\mathrm{U\,F}$ to the orbifold. Indeed, isotopy classes and homotopy classes of knots in $\mathrm{U\,F}$ can be represented by immersed loops in F up to the third Reidemeister move and regular homotopy respectively. Moreover, a knot is the lift of a primitive geodesic for some riemannian metric on F if [NC01] and only if [MFS82] its projection is taut.

Question 5. *How to recognize the state sums of* $\mathfrak{i}$*-loops (one strand) ? And how about taut* $\mathfrak{i}$*-loops, in other terms what is the image* $\Psi(\overline{\pi}) \subset \bigoplus \mathbb{Z}\psi$ *?*

Define the <u>*skein module*</u> $\mathrm{Sk}(\mathrm{F})$ as the quotient of the free module over the set of all $\mathfrak{i}$-multiloops up to *regular homotopy*, by $\epsilon = -2\emptyset$ and all relations of the form $\gamma + \gamma^{+} + \gamma^{-} = 0$ obtained by smoothing an intersection point of an $\mathfrak{i}$-multiloop. The state sum decomposes any element of the skein module as linear combination of states, in other words $\bigoplus \mathbb{Z}\psi$ generates $\mathrm{Sk}(\mathrm{F})$ as a module.

Proposition (Skein algebra)**.** *The union of* $\mathfrak{i}$*-multiloops, extended by bilinearity, defines on the skein module* $\mathrm{Sk}(\mathrm{F})$ *the structure of a commutative algebra, whose unit is the empty multiloop.*

Proof. Note that the union of $\mathfrak{i}$-multiloops is well defined up to regular homotopy, denote it $\alpha \cup \beta$. In particular, if $\varphi = \psi \sqcup \phi$ and γ are in $\mathrm{Sk}(\mathrm{F})$ then $\gamma \cup \varphi = \gamma \cup \psi \cup \phi$ independently of the position for trivial loops. To ensure the well definition of the product, we must check that for all $\alpha, \beta \in \mathrm{Sk}(\mathrm{F})$ such that decomposes as a sum $\beta = \beta^{+} + \beta^{-}$ we have $\alpha \cup \beta$ equals $\alpha \cup \beta^{+} + \alpha \cup \beta^{-}$ in the module. In other words all identities obtained by forcing the equalities given by the product rule are already contained in those we have quotiented by to define the skein module, so the underlying module to the skein algebra is not smaller (not a non trivial quotient) of the skein module.

In particular this shows $\Psi(\Psi(\alpha) \cup \Psi(\beta)) = \Psi(\Psi(\alpha) \cup \Psi(\beta))$ so that one can move by isotopy □

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