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Arithmétique et Topologie des Nœuds Modulaires

Arithmetic and Topology of Modular knots

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Chapter 0

Introduction

Contents

This introductory chapter to the thesis is divided in four sections. The appetiser is meant to provide a direction and a possible frame of mind for reading the thesis. It is by no means exhaustive as it omits some of its main ideas and results. The second section provides some background explaining the relationships between the various interpretations of modular conjugacy classes. The third section explains most of the main ideas and results, and it is divided in two parts reflecting the structure of the thesis as a whole. After that the thesis is divided in five chapters which, besides a few detours, are meant to be read in this order, as each one contains sections building on the previous. Every chapter is preceded by a specific introduction indicating its relation to the others, its internal structure, and containing its relevant bibliography.

0.1 Appetiser

This thesis is dedicated to the investigation of various structures underlying the set of conjugacy classes in the *modular group* $\mathrm{PSL}_2(\mathbb{Z})$, arising from arithmetic or geometry and topology or combinatorics, such as equivalence relations and bilinear pairings.

The geometrical structures arise from the fact that $\mathrm{PSL}_2(\mathbb{Z})$ is a Fuchsian group, that is a finite type discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Thus $\mathrm{PSL}_2(\mathbb{Z})$ acts properly discontinuously on the upper half-plane $\mathbb{HP}$ with quotient the modular orbifold $\mathbb{M}$, a hyperbolic surface with conical singularities i & j of order 2 & 3, and a cusp ∞. The orbifold homotopy classes of loops in $\mathbb{M}$ correspond to the conjugacy classes in its orbifold fundamental group $\pi_1(\mathbb{M}) = \mathrm{PSL}_2(\mathbb{Z})$.

A primitive hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$ acts by translation along an axis $\gamma_A \subset \mathbb{HP}$ which projects to a closed oriented primitive geodesic $[\gamma_A] \subset \mathbb{M}$. Conversely, a closed oriented primitive geodesic in $\mathbb{M}$ lifts to a set of bi-infinite oriented geodesics in $\mathbb{HP}$; each one is the translation axis of a unique primitive hyperbolic element in $\mathrm{PSL}_2(\mathbb{Z})$, and all these elements are conjugate. Consequently, primitive hyperbolic conjugacy classes of $\mathrm{PSL}_2(\mathbb{Z})$ correspond to primitive closed oriented geodesics in $\mathbb{M}$.

The complexity of a modular geodesic $[\gamma_A]$ can be measured by its length λ_A, equal to the logarithm of the ratio between the eigenvalues of $\pm A$, in other terms:

$$\tfrac{1}{2}|\mathrm{Tr}(A)| = \cosh \tfrac{1}{2}\lambda_A \qquad \text{or} \qquad \mathrm{disc}(A) := \mathrm{Tr}(A)^2 - 4 = 4\left(\sinh \tfrac{1}{2}\lambda_A\right)^2$$

Observe that modular geodesics have the same length if and only if the corresponding conjugacy classes of $\mathrm{PSL}_2(\mathbb{Z})$ are conjugate in $\mathrm{PSL}_2(\mathbb{Q}(\sqrt{\mathbb{Z}}))$ where $\mathbb{Q}(\sqrt{\mathbb{Z}})$ is the field extension of $\mathbb{Q}$ obtained by adjoining the square roots of all integers. This equivalence relation, admitting both geometric and arithmetic interpretations, turns out to be non-trivial and groups the modular geodesics into finite classes.

We are thus led to consider a field $\mathbb{K}$ containing $\mathbb{Q}$, and ask: *when are conjugacy classes of* $\mathrm{PSL}_2(\mathbb{Z})$ *conjugate in* $\mathrm{PSL}_2(\mathbb{K})$*, and how to measure this* $\mathbb{K}$*-equivalence geometrically*? To address these questions, we shall study the adjoint action of the arithmetic group $\mathrm{PSL}_2(\mathbb{K})$ on its Lie algebra $\mathfrak{sl}_2(\mathbb{K})$, and derive the following.

Proposition (Arithmetic equivalence of modular geodesics)**.** *Two conjugacy classes of* $\mathrm{PSL}_2(\mathbb{Z})$ *with discriminant* $\Delta > 0$ *are* $\mathbb{K}$*-equivalent if and only if the corresponding modular geodesics satisfy the following equivalent conditions:*

- θ *There exists one intersection point with angle* $\theta \in]0, \pi[$ *such that* $\left(\cos \frac{\theta}{2}\right)^2 = \frac{1}{(2x)^2 - \Delta y^2}$ *for* $x, y \in \mathbb{K}$*, in which case all intersections have this property.*

- λ *There exists one co-oriented ortho-geodesic of length* λ *such that* $\left(\cosh \frac{\lambda}{2}\right)^2 = \frac{1}{(2x)^2 - \Delta y^2}$ *for* $x, y \in \mathbb{K}$*, in which case all such ortho-geodesics have this property.*

We now recall a combinatorial parametrization of infinite order conjugacy classes of $\mathrm{PSL}_2(\mathbb{Z})$. The euclidean algorithm shows that the group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the transvections $L\&R$, and more precisely that its submonoid $\mathrm{SL}_2(\mathbb{N})$ of matrices with non-negative entries is freely generated by $L\&R$. This submonoid can be identified with its image $\mathrm{PSL}_2(\mathbb{N}) \subset \mathrm{PSL}_2(\mathbb{Z})$.

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

In $\mathrm{PSL}_2(\mathbb{Z})$, the conjugacy class of an infinite order element intersects $\mathrm{PSL}_2(\mathbb{N})$ along all cyclic permutations of a non-empty $L\&R$-word. The primitivity of the conjugacy class is equivalent to the primitivity of those cyclic words, and the conjugacy class is hyperbolic exactly when both letters L and R appear.

We may wonder how to decipher the precise geometry of modular geodesics and in particular their isotopy classes from the combinatorics of these cyclic binary words. The most immediate measures of complexity of a binary word are given by the numbers of letters of each sort. For a primitive hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$ we call $\mathrm{Rad}([A]) = \#R - \#L$ the *Rademacher number* of its conjugacy class. In his paper [Ati87] on the Logarithm of the Dedekind eta function, M. Atiyah identified the Rademacher function with no less than six other important functions appearing in diverse areas of mathematics, showing how omnipresent it is. In [Ghy07] É. Ghys added a topological interpretation in terms of modular knots which we now explain.

The unit tangent bundle $\mathbb{U} = \mathrm{PSL}_2(\mathbb{Z})\backslash \mathrm{PSL}_2(\mathbb{R})$ of $\mathbb{M} = \mathrm{PSL}_2(\mathbb{Z})\backslash \mathbb{HP}$ is a three-manifold, and the structure of the Seifert fibration $\mathbb{U} \to \mathbb{M}$ reveals that it is homeomorphic to the complement of a trefoil knot in the sphere. The primitive closed oriented geodesics in $\mathbb{M}$ lift to the primitive periodic orbits for the geodesic flow in $\mathbb{U}$ which are the so called *modular knots*. Hence the primitive hyperbolic conjugacy classes in the modular group index the components of a link in the complement of a trefoil, and one may ask about their linking numbers.

In [Ghy07], É. Ghys showed that the linking number of a modular knot with the trefoil is equal to its Rademacher invariant, and concluded by asking for *an arithmetic interpretation of the linking pairing between modular knots.*

For this, we introduce for any pair of modular geodesics $[\gamma_A], [\gamma_B]$, the sum over their oriented intersection angles $\theta \in]0, \pi[$ of the quantities which appeared in our previous arithmetic interpretation:

$$\mathrm{L}_1([A],[B]) := \sum \left(\cos \tfrac{\theta}{2}\right)^2$$

and study its variations as we deform the metric on $\mathbb{M}$ by opening the cusp.

The complete hyperbolic metrics on the orbifold $\mathbb{M}$ correspond to the faithful and discrete representations $\rho\colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R})$ up to conjugacy. They form a 1-dimensional real algebraic set parametrized by $q \in \mathbb{R}^*$ and the matrix $A_q = \rho_q(A)$ is obtained from any $L\&R$-factorisation of A by replacing $L \mapsto L_q$ and $R \mapsto R_q$, where:

$$L_q = \begin{pmatrix} q & 0 \\ 1 & q^{-1} \end{pmatrix} \qquad \text{and} \qquad R_q = \begin{pmatrix} q & 1 \\ 0 & q^{-1} \end{pmatrix}.$$

The primitive hyperbolic conjugacy classes of $\mathrm{PSL}_2(\mathbb{Z})$ still index the hyperbolic geodesics in the quotient $\mathbb{M}_q = \rho_q(\mathrm{PSL}_2(\mathbb{Z}))\backslash\mathbb{HP}$ which do not surround the cusp. We may thus define the analogous sum $\mathrm{L}_q([A],[B])$ over the intersection angles $\theta_q \in]0,\pi[$ between the q-modular geodesics $[\gamma_{A_q}], [\gamma_{B_q}] \subset \mathbb{M}_q^*$ of the $\left(\cos\frac{1}{2}\theta_q\right)^2$.

As $q \to \infty$, the hyperbolic orbifold $\mathbb{M}_q$ has a convex core which retracts onto a thin neighbourhood of the long geodesic arc connecting its conical singularities, whose preimage in the universal cover $\mathbb{HP}$ is a trivalent tree. In the limit we recover the action of $\mathrm{PSL}_2(\mathbb{Z})$ on the infinite planar trivalent tree, and by studying its combinatorics we shall prove the following.

Theorem (Linking numbers from boundary evaluations)**.** *For primitive hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$, the limit of the function $\mathrm{L}_q([A],[B])$ at the boundary point of the $\mathrm{PSL}_2(\mathbb{R})$-character variety of $\mathrm{PSL}_2(\mathbb{Z})$ recovers their linking number:*

$$\mathrm{L}_q([A],[B]) \xrightarrow[q\to\infty]{} 2\,\mathrm{lk}([A],[B]).$$

This deformation of the hyperbolic metric provides another way to refine the partition of modular geodesics $[\gamma_A]$ according to their geometric length λ_A. It amounts to asking for the equality of the trace functions $\mathrm{Tr}(A_q) \in \mathbb{Z}[q,q^{-1}]$ on the character variety. This algebraic equivalence implies the equality of geometric lengths λ_A and of combinatorial lengths $\#R + \#L$ (which count the geometric intersection number of modular geodesics with the infinite geodesic arc $[i,\infty) \subset \mathbb{M}$). However it is not trivial, and to understand it better we propose a topological interpretation.

The modular knot associated to a (primitive) hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$ defines a conjugacy class in the fundamental group $\pi_1(\mathbb{U})$ of the complement of the trefoil. This group is isomorphic to the braid group on three strands $\mathcal{B}_3$, and to the conjugacy class of a braid is associated the link obtained as its closure. We shall relate ρ_q to the reduced Burau-Squier representation $\mathcal{B}_3 \to \mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$ to deduce the following.

Proposition. *For a (primitive) hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$, the Alexander polynomial of the link $[\sigma_A]$ is equal to $\Delta([\sigma_A]) = \frac{q^{\mathrm{Rad}(A)} - \mathrm{Tr}(A_q) + q^{-\mathrm{Rad}(A)}}{(q-q^{-1})^2}$, where $q = \sqrt{-t}$.*

This suggests that the arithmetic $\mathbb{Q}$-equivalence of $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ implies the "quantum equivalence" given by $\mathrm{Tr}(A_q) = \mathrm{Tr}(B_q)$ & $\mathrm{Rad}(A) = \mathrm{Rad}(B)$.

0.2 Aims of the thesis

This thesis is dedicated to the investigation of topological and arithmetical structures on the set of conjugacy classes in the modular group $\mathrm{PSL}_2(\mathbb{Z})$. These structures will consist in various equivalence relations or functions on pairs of conjugacy classes.

Several questions arise. What is the modular group, and why is it interesting? How to study this group, and which role is played by its conjugacy classes?

0.2.1 The modular quadratic dictionary

To fix some notations and a common ground, let us briefly retrace the emergence of the modular group in the history of mathematics, without shying the introduction of the matrix formalism even when it is anachronistic. We refer to [Wei84, Kle79] and [KLLP07] for much more depth and accuracy concerning historical matters.

Euclidean monoid & continued fractions

Denote $\mathbb{N} = \{0, 1, \dots\}$ the monoid of non-negative integers, $\mathbb{Z}$ the ring of integers and $\mathbb{Q}$ its field of fractions. The rational field was the realm of Pythagorean arithmetic: all numbers were given by the ratio of two commensurable lengths.

The common measure of two commensurable lengths can be computed from a repeated application of the transformations $J \colon z \mapsto z^{-1}$ and $R^{-1} \colon z \mapsto z - 1$ to their ratio $r \in \mathbb{Q}$ according to the Euclidean division algorithm. This leads to the sequence of partial quotients $q_i \in \mathbb{N}^*$ appearing in its continued fraction expansion:

$$r = \lfloor q_0, q_1, \dots, q_k \rfloor := q_0 + \frac{1}{q_1 + \frac{1}{\dots + \frac{1}{q_k}}} = (R^{q_0} J R^{q_1} J \dots R^{q_k} J) \cdot \infty$$

The transformations $J \& R$ generate the group $\mathrm{PGL}_2(\mathbb{Z})$ which acts on the rational projective line $\mathbb{QP}^1$ by linear fractional transformations. Its index two subgroup which preserves the cyclic order of $\mathbb{QP}^1$ is the *modular group* $\mathrm{PSL}_2(\mathbb{Z})$, generated by $R \colon z \mapsto z + 1$ and $L = JRJ \colon z \mapsto (z^{-1} + 1)^{-1}$. Over any ring, the action of PGL_2 on $\mathbb{P}^1$ is given by linear fractional transformations as follows:

$$A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \quad A \cdot z = \tfrac{az+b}{cz+d} \qquad \text{so we lift} \quad J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \quad R = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \quad L = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right).$$

The Euclidean algorithm leads to considering only those elements in the *euclidean monoid* freely generated by $L \& R$, which gets identified both to $\mathrm{SL}_2(\mathbb{N})$ and to its projectivization $\mathrm{PSL}_2(\mathbb{N})$. Thus in Antiquity the modular group was not yet an object of study.

The next two thousand years saw no change in the use of the Euclidean algorithm, although it was extended to numbers with no common measure around the turn of the XVII$^{\text{th}}$ century (by Bombelli, Brouncker, Wallis, Huygens and Euler among others). This led to the decomposition of irrational numbers into infinite continued fractions:

$$\tfrac{1}{2}(1+\sqrt{5}) = \lfloor 1, 1, \dots \rceil \qquad \sqrt[3]{2} = \lfloor 1, 3, 1, 5, 1, 1, \dots \rceil \qquad \pi = \lfloor 3, 7, 15, 1, 292, \dots \rceil$$

Now instead of considering the common measure of two rationals, we focus on the successive tails $r_k = \lfloor q_k, q_{k+1}, \dots \rceil$ of the continued fraction expansion of a real number $r = r_0 \in \mathbb{R}$.

We thus consider equivalent those numbers whose continued fraction expansions eventually coincide, in other terms whose orbits under the action of the aforementioned monoid have a non-empty intersection. By doing so we are secretly considering the action of the group generated by the monoid.

$$\begin{array}{lll} \text{If} & r_0 = \lfloor n_0, n_1, \dots\dots\dots, n_{k-1}, n_k, \quad r_{k+1} \rceil & \\ \text{then} & r_0 = (R^{n_0} J\, R^{n_1} J\ \dots\ R^{n_{k-1}} J\, R^{n_k} J) \cdot r_{k+1} & \text{for all } k \in \mathbb{N} \\ \text{whence} & r_0 = (R^{n_0} L^{n_1}) \dots\dots\dots (R^{n_{k-1}} L^{n_k}) \cdot r_{k+1} & \text{when } k \text{ is odd} \quad \text{as } JR^nJ = L^n. \end{array}$$

Hence we find that $x, y \in \mathbb{RP}^1$ belong to the same $\mathrm{PGL}_2(\mathbb{Z})$-orbit when some tails x_i and y_j of their continued fractions coincide, and they belong to the same $\mathrm{PSL}_2(\mathbb{Z})$-orbit when there exist $i, j \in 2\mathbb{N}$ such that the tails x_i and y_j coincide.

The rational numbers are those for which this tail is eventually empty. After that, the simplest numbers are those whose continued fraction expansions have periodic tails. Their study reached an effervescence in the works of Euler, Lagrange, Legendre, Galois and Gauss, in connection with the arithmetic of quadratic forms. Indeed, these preperiodic irrational numbers correspond to the roots of irreducible quadratic polynomials with integral coefficients and positive discriminant. They also correspond to primitive hyperbolic elements in the modular group.

One may learn more about continued fractions and diophantine approximation in [HW38, Khr08], and about their relation to quadratic forms in [Ser85a, Hat22].

Arithmetic dictionary: primitive hyperbolic matrices in $\mathrm{PSL}_2(\mathbb{Z})$

The continued fraction expansion of an irrational preperiodic number α admits a unique factorisation into a mantissa of length $2i$ and a period of length $2j$ with minimal $i, j \in \mathbb{N}$:

$$\alpha = \lfloor c_1, \dots, c_{2i}, \overline{b_1, \dots, b_{2j}} \rceil.$$

Since α is irrational we have $j > 0$, and α is called *purely periodic* when $i = 0$.

We have $\alpha = C \cdot \beta$ for $C = R^{c_1}L^{c_2} \dots R^{c_{2i-1}}L^{c_{2i}}$ and $\beta = B \cdot \beta$ is purely periodic with monodromy $B = R^{b_1} \dots L^{b_{2j}} \in \mathrm{PSL}_2(\mathbb{N})$, so α is fixed under

$$A = CBC^{-1} \in \mathrm{PSL}_2(\mathbb{Z}).$$

Since j is positive and minimal, the element B is hyperbolic and primitive: it satisfies $\mathrm{disc}(B) := \mathrm{Tr}(B)^2 - 4 > 0$ and is not the power of another matrix in $\mathrm{PSL}_2(\mathbb{Z})$. Exactly the same goes for A, which satisfies $\mathrm{disc}(A) = \mathrm{disc}(B)$.

The fixed point α of A satisfies the quadratic relation $c\alpha^2 + (d-a)\alpha - b = 0$ where a, b, c, d are the entries for a lift of A in $\mathrm{SL}_2(\mathbb{Z})$. Dividing this by $u = \gcd(c, d-a, -b)$ and by $\mathrm{sign}\,\mathrm{Tr}(A)$ yields a primitive integral binary quadratic form

$$Q(x, y) = lx^2 + mxy + ry^2$$

which is indefinite, meaning that its *discriminant* $\mathrm{disc}(Q) = m^2 - 4lr = \mathrm{disc}(A)/u^2$ is positive and non-square.

This quadratic form can be polarised with respect to the determinant as follows. Denote by $\mathfrak{sl}_2(\mathbb{Q})$ the Lie algebra of rational 2×2 matrices with trace 0. There exists a unique $\mathfrak{a} \in \mathfrak{sl}_2(\mathbb{Q})$ such that $Q(v) = \det(v, \mathfrak{a}v)$ and $\mathrm{disc}(\mathfrak{a}) := -4\det(\mathfrak{a}) = \mathrm{disc}(Q)$. It is given by the formula:

$$\mathfrak{a} = \frac{1}{2}\begin{pmatrix} -m & -2r \\ 2l & m \end{pmatrix}.$$

The $\mathbb{Q}$-vector space $\mathfrak{sl}_2(\mathbb{Q})$ is endowed with the non-degenerate quadratic form det of real signature $(1, 2)$. The dual lattice $\mathfrak{sl}_2(\mathbb{Z})^\vee = \frac{1}{2}\mathbb{Z}\mathbb{1} + \mathfrak{sl}_2(\mathbb{Z})$ to $\mathfrak{sl}_2(\mathbb{Z})$ with respect to det contains $\mathfrak{a}$ as a primitive vector. Moreover $\mathfrak{a}$ is space-like as $\det(\mathfrak{a}) < 0$.

Dictionary: objects. We have introduced the entries of the arithmetic dictionary:

- α Real quadratic irrationalities $\alpha = \lfloor c_1, \dots, c_{2i}, \overline{b_1, \dots, b_{2j}} \rfloor = \frac{-m+\sqrt{\mathrm{disc}(Q)}}{2l}$
- A Primitive hyperbolic matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$
- Q Primitive indefinite integral binary quadratic forms $Q(x, y) = lx^2 + mxy + ry^2$
- $\mathfrak{a}$ Primitive space-like vectors $\mathfrak{a} = \begin{pmatrix} -m/2 & -r \\ l & m/2 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{Z})^\vee$

The *extended modular group* $\mathrm{PGL}_2(\mathbb{Z})$ acts on each entry: by linear fractional transformation $\alpha \mapsto C \cdot \alpha$, by conjugacy $A \mapsto CAC^{-1}$, by change of variables $Q \mapsto Q \circ C^{-1}$ and by the *adjoint action* $\mathfrak{a} \mapsto C\mathfrak{a}C^{-1}$.

We also have an action of $\mathbb{Z}/2$ on these entries given by Galois conjugacy $\alpha \mapsto \alpha'$, by inversion $A \mapsto A^{-1}$, by change of sign $Q \mapsto -Q$ and $\mathfrak{a} \mapsto -\mathfrak{a}$. Note that it coincides with the operation of taking the transpose comatrix $A \mapsto A^{\#}$ and $\mathfrak{a} \mapsto \mathfrak{a}^{\#}$.

Dictionary: morphisms. We have an explicit correspondence between the entries $Q, \mathfrak{a}, \alpha, A$ of the arithmetic dictionary, that is a commutative diagram of $4^2 = 16$ bijective maps between any two of these families, which is equivariant under the extended modular group $\mathrm{PGL}_2(\mathbb{Z})$ as well as under the $\mathbb{Z}/2$-involution.

Some of them can be abbreviated by the following formulae, where we denote $v = {}^t(x, y)$ and $v_\alpha = {}^t(x_\alpha, y_\alpha)$ such that $x_\alpha/y_\alpha = \alpha$, as well as $t = \frac{1}{2}\mathrm{Tr}(A) > 0$, $u = \gcd(c, d-a, -b)$ and $\Delta = \mathrm{disc}(Q) = -\mathrm{disc}(\mathfrak{a}) = \mathrm{disc}(A)/u^2$:

$$Q \leftrightarrow \mathfrak{a}\colon \qquad Q = lx^2 + mxy + ry^2 \qquad Q(v) = \det(v, \mathfrak{a}v) \qquad \mathfrak{a} = \begin{pmatrix} -m/2 & -r \\ l & m/2 \end{pmatrix}$$

$$\{Q, \mathfrak{a}\} \leftrightarrow \alpha\colon \quad Q(x, y) = l(x - \alpha y)(x - \alpha' y) \qquad \alpha = \frac{-m + \sqrt{\Delta}}{2l} \qquad \mathfrak{a}v_\alpha = \frac{\sqrt{\Delta}}{2} v_\alpha$$

$$\{Q, \mathfrak{a}, \alpha\} \leftarrow A\colon \quad Q(v) = \tfrac{1}{u}\det(v, Av) \qquad A = t.\mathbb{1} + u.\mathfrak{a} \qquad Av_\alpha = \frac{\mathrm{Tr}(A) + u\sqrt{\Delta}}{2} v_\alpha$$

From now on we shall voyage fluently between the various entries of the dictionary. Out of all these, we prefer to work with matrices $\mathfrak{a}$ in the lattice $\mathfrak{sl}_2(\mathbb{Z})^\vee$ of $\mathfrak{sl}_2(\mathbb{Q})$. Indeed, the space $\mathfrak{sl}_2(\mathbb{Q})$ has the structure of a Lie algebra, which is preserved by the adjoint action of $\mathrm{PSL}_2(\mathbb{Q})$: the tools available for such a study are classical, allying geometrical insight with efficient linear algebra.

We refer to [Lac88] and [Coh78, Chapter 14 & Appendix B] for the correspondence between preperiodic continued fractions, real quadratic irrationals, quadratic forms and matrices in the modular group. The originality in our approach, inspired by [Thu97, Section 2.6] and [Bha04], is to consider matrices in lattices of the quadratic space $(\mathfrak{sl}_2(\mathbb{Q}), \det)$ in order to focus on their geometric study.

Remark (Algorithmic)**.** *It is straightforward to voyage between any two of* $\{Q, \mathfrak{a}, \alpha\}$*, and to deduce those from* A*; but computing* A *from* α *is rather delicate as it involves extracting the mantissa and period of a continued fraction. The mantissa could be long and contain periodic subwords beluding us into thinking we found the period (unless we keep an eye on the periodicity of the "quadratic left overs").*

However, we may express $A = t\mathbb{1} + u\mathfrak{a}$ *by computing the smallest positive element* $t + u\sqrt{\delta} \in \mathbb{Z}[\sqrt{\delta}]$ *of norm* $t^2 - \delta u^2 = 1$ *where* $\delta = \frac{1}{4}\Delta$ *(defining the* fundamental solution (t, u) *to the Pell-Fermat equation* $t^2 - \delta u^2 = 1$*), and replacing* $\sqrt{\delta}$ *by the square root* $\mathfrak{a} \in \mathfrak{sl}_2(\mathbb{Z})^\vee$ *of* $\delta\mathbb{1}$*.*

Remark (Negative discriminants)**.** *This dictionary can be extended to objects with negative discriminants:*

α *imaginary quadratic irrationals* $\alpha = \frac{-m+\sqrt{\Delta}}{2l}$

Q *primitive definite integral binary quadratic forms* $Q(x,y) = lx^2 + mxy + ry^2$

$\mathfrak{a}$ *primitive integral time-like vectors* $\mathfrak{a} = \frac{1}{2}\left(\begin{smallmatrix} -m & -2r \\ 2l & m \end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathbb{Z})^\vee$

Another reference about integral binary quadratic forms which eventually focuses on the case of negative discriminants is [Cox97].

Class groups and genera

Notice that two matrices $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ have the same discriminant Δ if and only if they are conjugate in $\mathrm{PSL}_2(\mathbb{C})$, or even in $\mathrm{PSL}_2(\mathbb{Q}(\sqrt{\mathbb{Z}}))$. Fix a positive non-square discriminant Δ and consider the set $\mathrm{Cl}(\Delta)$ of $\mathrm{PSL}_2(\mathbb{Z})$-equivalence classes of primitive integral binary quadratic forms with that discriminant. This set is finite because one may represent a class by a purely periodic number α, implying that $-1/\alpha'$ is purely periodic (with reverse period), so both of these real numbers are > 1. In particular $\alpha' < 0 < \alpha$ so the corresponding quadratic form Q satisfies $-lr > 0$ and there is a finite number of solutions to $\Delta = m^2 - 4lr$.

A class $[Q] \in \mathrm{Cl}(\Delta)$ is uniquely determined by its set of values $[Q](\mathbb{Z}^2)$. Indeed [Hat22, Proposition 6.7] says that if two forms of the same discriminant represent the same prime number or both represent 1 then they are equivalent, and by [Mey88] every primitive form represents at least one prime. In [Gau07] C.-F. Gauss showed that there is a well defined composition of classes corresponding to the multiplication of the associated sets of values, endowing $\mathrm{Cl}(\Delta)$ with the structure of a finite abelian group. It was later reformulated by Dirichlet as follows [Wei84]. One may represent two elements in $\mathrm{Cl}(\Delta)$ by forms Q_1 and Q_2 whose first coefficients l_1 and l_2 are coprime, and with the same middle coefficient m. Then their composition Q_3 of the same discriminant is determined by its first coefficient $l_3 = l_1 l_2$ and middle coefficient m.

Two classes *have same genus* when they represent the same values in $(\mathbb{Z}/\Delta)^\times$. By [Cox97, Theorem 3.21] this is equivalent to saying that they are conjugate by a matrix $C \in \mathrm{GL}_2(\mathbb{Q})$ with denominators coprime to 2Δ. The genera form a group $\mathrm{Gen}(\Delta)$ given by the multiplication of their sets of values in $(\mathbb{Z}/\Delta)^\times$. Gauss identified it with the quotient of his class group by the subgroup of squares. Moreover the kernel of the squaring map consists in the subgroup $\mathrm{Sym}(\Delta)$ of classes invariant by the Galois involution. In other terms we have a short exact sequence of abelian groups:

$$1 \to \mathrm{Sym}(\Delta) \to \mathrm{Cl}(\Delta) \xrightarrow{square} \mathrm{Cl}(\Delta) \to \mathrm{Gen}(\Delta) \to 1.$$

0.2.2 Topological dictionary for conjugacy classes

The modular dictionary descends to classes under the action of the modular group. The dictionary of $\mathrm{PSL}_2(\mathbb{Z})$-classes can be extended to include topological entries, in particular modular geodesics and modular knots, which we now briefly describe.

In this subsection, we refer to [Thu97, Sco83, Mon87] for more about hyperbolic geometry and geometric structures on orbifolds or their unit tangent bundles; to [Ser77, Hat22] for the action of the modular group on the trivalent tree.

The modular orbifold and the associated trivalent tree

The automorphism group $\mathrm{PGL}_2(\mathbb{C})$ of the complex projective line $\mathbb{CP}^1$ contains $\mathrm{PGL}_2(\mathbb{R})$ as the stabiliser of the real projective line $\mathbb{RP}^1$. The index-two subgroup $\mathrm{PSL}_2(\mathbb{R})$ also preserves the upper half-plane $\mathbb{HP} = \{z \in \mathbb{C} \mid \Im(z) > 0\} \subset \mathbb{CP}^1$, or equivalently the orientation induced on its boundary $\partial\mathbb{HP} = \mathbb{RP}^1$.

The complex structure on $\mathbb{HP}$ is conformal to a unique hyperbolic metric. The hyperbolic distance λ between $w, z \in \mathbb{HP}$ can be deduced from the cross-ratio by:

$$\frac{1}{\mathrm{bir}(\bar{z}, z, \bar{w}, w)} = \left(\cosh \tfrac{\lambda}{2}\right)^2$$

This realizes $\mathrm{PSL}_2(\mathbb{R})$ as the positive isometry group of the hyperbolic plane: it preserves the previous cross-ratio and acts simply-transitively on positive triples of distinct points of $\mathbb{RP}^1$, thus it preserves the hyperbolic metric and acts simply transitively on the unit tangent bundle of $\mathbb{HP}$.

The subgroup $\mathrm{PSL}_2(\mathbb{Q})$ is the stabiliser of the rational projective line $\mathbb{QP}^1$. The discrete subgroup $\mathrm{PSL}_2(\mathbb{Z})$ is the stabiliser of the ideal triangulation of $\mathbb{HP}$ with vertex set $\mathbb{QP}^1$ and edges all geodesics whose endpoints $\frac{p}{q}, \frac{r}{s}$ satisfy $|ps - qr| = 1$.

Consider the action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on this ideal triangulation. It is transitive on the set of edges, which is in bijection with the orbit of $i \in (0, \infty)$. The stabiliser of i is the subgroup of order 2 generated by S. It is transitive on the set of triangles, which is in bijection with the orbit of $j = \exp(i\pi/3) \in (0, 1, \infty)$. The stabiliser of j is the subgroup of order 3 generated by T. Thus it is freely transitive on the flags, or equivalently on the oriented edges, and we deduce that $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$ is the free amalgam of its subgroups generated by S and T.

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

We also find that $\mathrm{PSL}_2(\mathbb{Z})$ acts properly discontinuously on $\mathbb{HP}$ with fundamental domain the triangle $(\infty, 0, j)$. We may cut it along the geodesic arc (i, j) to obtain a pair of isometric triangles (i, j, ∞) and $(i, j, 0)$. Identifying them along their isometric edges yields the quotient

$$\mathbb{M} = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{HP}.$$

It is a hyperbolic two-dimensional orbifold, with conical singularities $[i]$ & $[j]$ of order 2 & 3 respectively associated to the fixed points i & j of S & T, and a cusp associated to the fixed point ∞ of R acting on $\partial\mathbb{HP}$. We call $\mathbb{M}$ the *modular orbifold*.

The preimage of the segment $([i], [j]) \subset \mathbb{M}$ in $\mathbb{HP}$ forms a bipartite tree $\mathcal{T}'$, the first barycentric subdivision of a trivalent tree $\mathcal{T}$ which is dual to the ideal triangulation. The group $\mathrm{PSL}_2(\mathbb{Z})$ acts freely transitively on the set of edges of $\mathcal{T}'$, or the set of oriented edges of $\mathcal{T}$. The *base edge* (i, j) of $\mathcal{T}'$ defines the *oriented base edge* $\vec{e}_i$ of $\mathcal{T}$.

Lyndon cycles and modular geodesics

A primitive hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathcal{T}$ by translation along an oriented geodesic g_A called its *combinatorial axis*. Since g_A has endpoints $\alpha', \alpha \in \partial\mathcal{T} = \mathbb{RP}^1$, it uniquely determines A. The conjugacy class of A corresponds to the orbit of g_A under the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{T}$, denoted $[g_A] = g_A \bmod \mathrm{PSL}_2(\mathbb{Z})$.

Observe that g_A passes through the oriented base edge $\vec{e}_i$ of $\mathcal{T}$ exactly when its endpoints satisfy $\alpha' < 0 < \alpha$, that is when $A \in \mathrm{PSL}_2(\mathbb{N})$. In that case, it follows a periodic sequence of left and right turns given by the $L\&R$-factorisation of A, or the continued fraction expansion of the periodic number α.

Consequently, the conjugacy classes of primitive hyperbolic matrices in $\mathrm{PSL}_2(\mathbb{Z})$ correspond to the primitive cyclic words on the alphabet $\{L, R\}$ with at least one occurrence of each letter. The linear representatives of such an $L\&R$-cycle parametrize the intersection of the corresponding conjugacy class with $\mathrm{PSL}_2(\mathbb{N})$, whose elements are called its *Lyndon representatives*. Lyndon words have appeared in the study of free Lie algebras [Lyn54, CFL58, BP07] and [Reu93] remarks that they index conjugacy classes in the modular group.

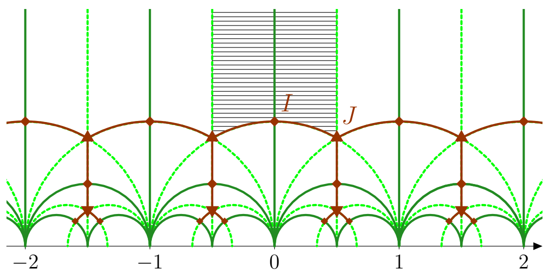
A hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathbb{HP}$ by translation along an oriented geodesic $\gamma_A \subset \mathbb{HP}$ with endpoints $\alpha', \alpha \in \mathbb{RP}^1$, called its *geometric axis*. Its projection in $\mathbb{M}$ is an oriented closed geodesic $[\gamma_A] = \gamma_A \bmod \mathrm{PSL}_2(\mathbb{Z})$ for the hyperbolic metric whose hyperbolic length λ_A is given by:

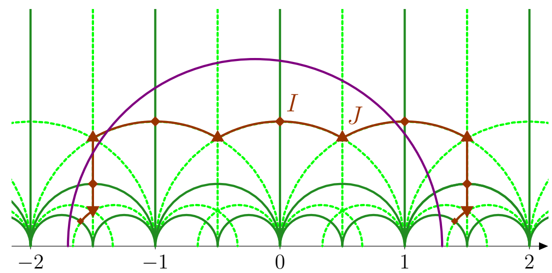
$$\tfrac{1}{2}\operatorname{Tr}(A) = \cosh \tfrac{1}{2}\lambda_A \quad \text{thus} \quad \operatorname{disc}(A) = 4\left(\sinh \tfrac{1}{2}\lambda_A\right)^2$$

Conversely, an oriented closed geodesic lifts to a set of bi-infinite oriented geodesics in $\mathbb{HP}$, each one being the translation axis of a unique primitive element in $\mathrm{PSL}_2(\mathbb{Z})$, and all these elements are conjugate. Consequently, primitive hyperbolic conjugacy classes of $\mathrm{PSL}_2(\mathbb{Z})$ correspond to primitive closed geodesics in $\mathbb{M}$.

Notice that the combinatorial and geodesic axes of $A \in \mathrm{PSL}_2(\mathbb{Z})$ have the same endpoints $\alpha', \alpha \in \mathbb{RP}^1 = \partial\mathbb{HP} = \partial\mathcal{T}$. More precisely, the geometric axis γ_A lies inside a $(\log\sqrt{\Delta})$-neighbourhood of the combinatorial axis g_A, where $\Delta = \mathrm{disc}(A)$.

In Section 3.1, we investigate the isotopy class of $[\gamma_A] \subset \mathbb{M}$ in terms of the corresponding $L\&R$-cycle.





Ideal triangulation $\triangle$ of $\mathbb{HP}$, its dual trivalent tree $\mathcal{T}$, and the modular tesselation. Geometric axis γ_A inside a $(\log\sqrt{\Delta})$-neighbourhood of the combinatorial axis g_A.

Modular knots and torus bundles

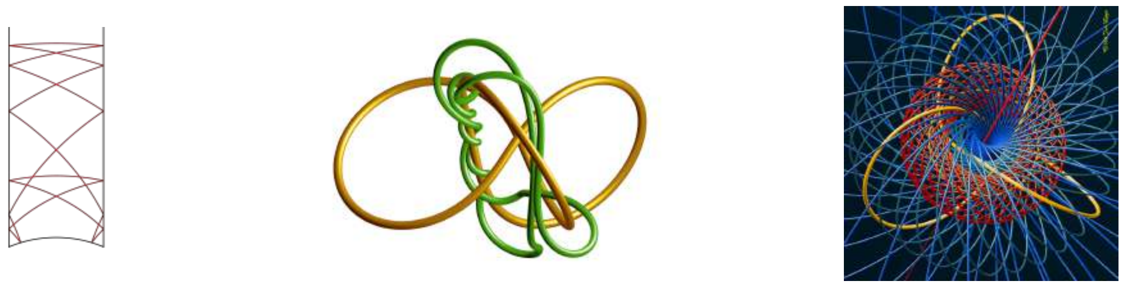
The Lie group $\mathrm{PSL}_2(\mathbb{R})$ identifies with the unit tangent bundle to the hyperbolic plane $\mathbb{HP}$. Its lattice $\mathrm{PSL}_2(\mathbb{Z})$ acts on the left with quotient $\mathbb{U} = \mathrm{PSL}_2(\mathbb{Z})\backslash \mathrm{PSL}_2(\mathbb{R})$ the unit tangent bundle to the modular orbifold $\mathbb{M} = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{HP}$.

The structure of the Seifert fibration $\mathbb{U} \to \mathbb{M}$ reveals that $\mathbb{U}$ is homeomorphic to the complement of a trefoil knot in the sphere. This was proved in [PH79, Klo16]. In particular, one may speak of linking numbers between disjoint loops in $\mathbb{U}$.

The closed hyperbolic geodesics $[\gamma_A]$ in $\mathbb{M}$ lift to the primitive periodic orbits for the geodesic flow in its unit tangent bundle $\mathbb{U}$. These *modular knots* form a second entry in our topological dictionary. The online article [GL16] proposes an animated introduction to the topology and dynamics of $\mathbb{U}$.

The fundamental group $\pi_1(\mathbb{U})$ is isomorphic to the braid group on three strands. We characterise in Chapter 4 which of its conjugacy classes correspond to homotopy classes of modular knots and describe the isotopy classes of these modular knots.

The conjugacy class of a braid β in $\mathcal{B}_3$ yields, by taking its closure, a link in the solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ with three strands which are transverse to the disc fibration. The ramified double cover of that solid torus over the link yields a punctured torus bundle over the circle whose monodromy is the projection $\beta \in \mathcal{B}_3 \mapsto A \in \mathrm{SL}_2(\mathbb{Z})$.



A modular geodesic in $\mathbb{M}$, lifted as a modular knot in $\mathbb{U}$, Seifert fibration $\mathbb{U} \to \mathbb{M}$. Images produced by Constantin Kogler and Jos Leys with Étienne Ghys.

The homeomorphism classes of oriented punctured torus bundles with an oriented base and primitive hyperbolic monodromy $A \in \mathrm{SL}_2(\mathbb{Z})$ form a third entry of the topological dictionary. However it will play a secondary role in this thesis.

The Lorenz template and the modular link

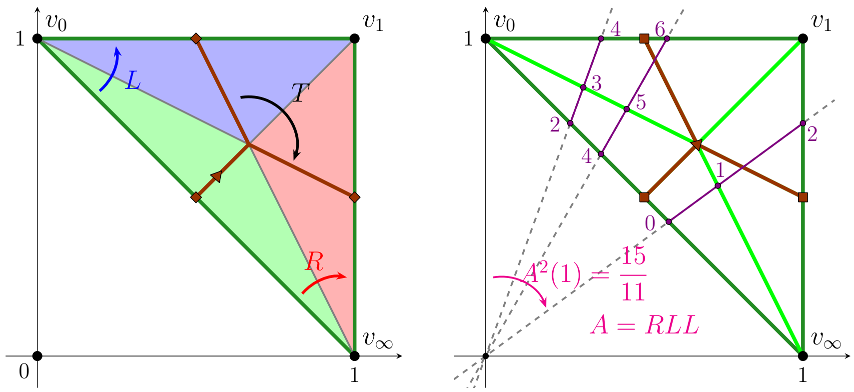
To describe the isotopy class of the *master modular link* consisting in all modular knots, we rely on the construction of the Lorenz template and its embedding in $\mathbb{U}$.

The *Lorenz template* $\mathbb{Y}$ is the branched surface obtained from the triangle $\nabla_1 \subset \mathbb{R}^2$ with vertices $R^{-1}v_1 = (0,1), v_1 = (1,1), L^{-1}v_1 = (1,0)$ by identifying its right side with the hypotenuse through R^{-1} and its top side with the hypotenuse through L^{-1}. The radial vector field on $\mathbb{R}^2$ intersected with ∇_1 descends on $\mathbb{Y}$ to define a semi-flow. The lines with rational inclination project to the orbit with finite future, forming the past of the vertex $(1,1)$.

Now consider a primitive hyperbolic conjugacy class in $\mathrm{PSL}_2(\mathbb{Z})$. The lifts of its Lyndon representatives in $\mathrm{SL}_2(\mathbb{N})$ act on $\mathbb{R}^2$ as hyperbolic transformations. Their stable eigen-directions intersect the triangle ∇_1 in a collection of disjoint segments which quotient to a closed loop in $\mathbb{Y}$. These are the primitive periodic orbits of the Lorenz semi-flow.

In [Ghy07], Ghys isotoped the *master modular link* formed by all modular knots, to the *master Lorenz link* formed by the primitive periodic orbits of the semi-flow on the Lorenz template, after its embedding in $\mathbb{U}$ as suggested in the figure below. The Lorenz template was introduced by Birman-Williams in [BW83] to study the periodic orbits for the dynamical system arising from Lorenz' equations.

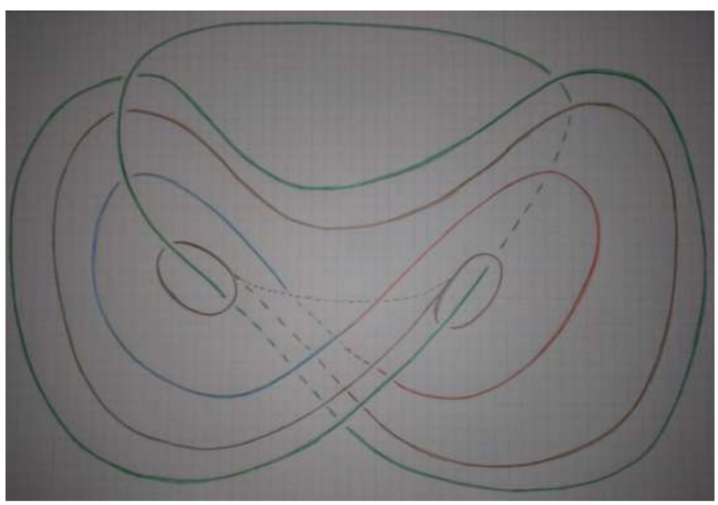
The *Rademacher number* of a primitive hyperbolic conjugacy class of $\mathrm{PSL}_2(\mathbb{Z})$ is computed from any Lyndon representative $A \in \mathrm{PSL}_2(\mathbb{N})$ as $\mathrm{Rad}([A]) = \#R - \#L$. In his paper [Ati87] on the Logarithm of the Dedekind eta function, M. Atiyah identified



Identifications of ∇_1 yielding the Lorenz template. From an $L\&R$-cycle to the cycle of segments in the base triangle of $\triangle_1$.

the Rademacher function with no less than six other important functions appearing in diverse areas of mathematics, showing how omnipresent it is.

In [Ghy07], É. Ghys showed that the linking number of a modular knot with the trefoil is equal to its Rademacher invariant, and concluded by asking for an arithmetic interpretation of the linking pairing between modular knots.



The Lorenz template $\mathbb{Y}$ and the trefoil. Modular knot in $\mathbb{Y}$ and a crossing.

Structures on conjugacy classes: equivalence & pairings

We have given our first motivation to study conjugacy classes in the modular group: they parametrize various kinds of mathematical objects.

The main questions about them depend on the objects under consideration and the frame of mind. One may ask about the numbers represented by quadratic forms, about the arithmetic of the quadratic rings generated by their roots, about the periods of their continued fraction expansions. We may care about the algebraic structures or the asymptotic distribution of their $\mathrm{PSL}_2(\mathbb{Z})$-classes.

The dictionary prompts us to translate structures from one entry to the other. In this thesis we focus on two such questions, namely we want to understand:

- arithmetic equivalence relations defined over the set of binary quadratic forms in terms of the geometry of the modular geodesics
- intersection and linking numbers of modular geodesics and modular knots in terms of the arithmetic of real quadratic irrationalities

We have presented those arithmetic equivalence relations and the linking numbers of modular knots in the introduction. In the next section we will say more about them as we present the main results of the thesis.

Before going on, let us collect here some of the equivalence relations we have already encountered on the set of hyperbolic conjugacy classes in the modular group. For this we consider Lyndon representatives $A, B \in \mathrm{PSL}_2(\mathbb{N})$ of the conjugacy classes and freely use the arithmetic dictionary. In particular we denote by $\alpha = \lfloor \overline{a_0, \dots, a_m} \rfloor$ and $\beta = \lfloor \overline{b_0, \dots, b_n} \rfloor$ the corresponding real quadratic irrationals.

$\mathrm{PSL}_2(\mathbb{Q})$ Arithmetic $\mathbb{Q}$-equivalence: $\exists C \in \mathrm{PSL}_2(\mathbb{Q})\colon\ CAC^{-1} = B$

$\mathrm{Cl}\,/\,\mathrm{Cl}^2$ Genus equivalence of Q_A and Q_B

$\mathrm{Tr}(A_q)$ Algebraic trace equivalence: $\mathrm{Tr}(A_q) = \mathrm{Tr}(B_q)$

λ_A Geometric length equivalence: $\lambda_A = \lambda_B$, that is $\mathrm{Tr}(A) = \mathrm{Tr}(B)$

$\mathrm{len}(A)$ Combinatorial length equivalence: $\mathrm{len}(A) = \mathrm{len}(B)$, that is $\sum a_k = \sum b_k$

$\mathrm{len}(\alpha)$ Arithmetic length equivalence: $\mathrm{len}(\alpha) = \mathrm{len}(\beta)$, that is $m = n$

$\mathrm{Rad}(A)$ Rademacher equivalence: $\mathrm{Rad}(A) = \mathrm{Rad}(B)$, that is $\sum(-1)^k a_k = \sum(-1)^k b_k$

Conjecture 0.1. *Our main conjectures are that arithmetic $\mathbb{Q}$-equivalence implies genus equivalence, as well as trace equivalence and Rademacher equivalence.*

0.3 Main results of the thesis

0.3.1 Arithmetic equivalence of modular geodesics

The adjoint action of $\mathrm{PSL}_2(\mathbb{K})$ on $\mathfrak{sl}_2(\mathbb{K})$

Let $\mathbb{K}$ be a field of characteristic different from 2. In the sequel, we specialize $\mathbb{K} = \mathbb{Q}$.

On the 3-dimensional associative $\mathbb{K}$-algebra $\mathfrak{sl}_2(\mathbb{K})$, the commutator

$$\{\mathfrak{a}, \mathfrak{b}\} = \tfrac{1}{2}(\mathfrak{a}\mathfrak{b} - \mathfrak{b}\mathfrak{a})$$

defines a Lie bracket. Its Killing form multiplied by $-\frac{1}{8}$ recovers the polarisation

$$\langle \mathfrak{a}, \mathfrak{b} \rangle = \tfrac{1}{2}\operatorname{Tr}(\mathfrak{a}\mathfrak{b})$$

of the non-degenerate quadratic form det.

The adjoint action of $\mathrm{GL}_2(\mathbb{K})$ on $\mathfrak{sl}_2(\mathbb{K})$ preserves all this structure and yields a representation $\mathrm{PGL}_2(\mathbb{K}) \to \mathrm{SO}(\mathfrak{sl}_2(\mathbb{K}), \det)$ which turns out to be an isomorphism, as we recall in Proposition 1.54. The main achievement of Chapter 1 is to describe the structure of the orbits for the action of $\mathrm{PSL}_2(\mathbb{K})$ on $\mathfrak{sl}_2(\mathbb{K})$.

Lemma (Parametrizing the cone)**.** *Denote by $\mathbb{X}$ the isotropic cone of $(\mathfrak{sl}_2(\mathbb{K}), \det)$. We define the quadratic map $\psi \colon \mathbb{K}^2 \to \mathfrak{sl}_2(\mathbb{K})$ by $\psi(v) = -v \cdot {}^t(Sv)$ where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.*

It has image $\mathbb{X}_S = \{p \in \mathbb{X} \mid \exists\, x, y \in \mathbb{K} : \langle p, S\rangle = x^2 + y^2\}$ and is two-to-one outside the origin. For all $u, v \in \mathbb{K}^2$ we have $2\langle\psi(u), \psi(v)\rangle = \det(u, v)^2$. The map ψ intertwines the tautological action of $\mathrm{SL}_2(\mathbb{K})$ on $\mathbb{K}^2$ with its adjoint action on $\mathbb{X}$.

Corollary (Actions on $\mathbb{KP}^1$)**.** *The projectivised map $\mathbb{P}(\psi) \colon \mathbb{P}(\mathbb{K}^2) \to \mathbb{P}(\mathbb{X})$ is an isomorphism of projective lines which intertwines the tautological action of $\mathrm{PSL}_2(\mathbb{K})$ on the projective line $\mathbb{P}(\mathbb{K}^2)$ to its adjoint action on the projective conic $\mathbb{P}(\mathbb{X})$.*

We deduce that the action of $\mathrm{PGL}_2(\mathbb{K})$ on $\mathbb{P}(\mathbb{X})$ is simply-transitive on triples of distinct lines. Moreover, after defining the Maslov index of three lines in $\mathbb{KP}^1$ as an element of $\{0\} \cup \mathbb{K}^\times/(\mathbb{K}^\times)^2$, we shall deduce that the action of $\mathrm{PSL}_2(\mathbb{K})$ on $\mathbb{P}(\mathbb{X})$ preserves the Maslov index and is simply-transitive on triples of distinct lines with a given Maslov index. This is Proposition 1.39 but we don't emphasise it here.

The map $\psi \colon \mathbb{Q}^2 \to \mathbb{X}$ and the isomorphism $\mathbb{P}\psi \colon \mathbb{P}(\mathbb{Q}^2) \to \mathbb{P}(\mathbb{X})$ of projective lines.

For $\mathfrak{a} \in \mathfrak{sl}_2(\mathbb{K}) \setminus \mathbb{X}$, choose a square root of $\Delta := \operatorname{disc}(\mathfrak{a}) = -4\det(\mathfrak{a})$ and extend the scalars to the field $\mathbb{K}' = \mathbb{K}[\sqrt{\Delta}]$. The tautological action of $\mathfrak{a}$ on the plane $\mathbb{K}'^2$

has two eigendirections for the eigenvalues $\pm\frac{1}{2}\sqrt{\Delta}$. These lines map under $\psi \otimes \mathbb{K}'$ to the intersection of the cone $\mathbb{X} \otimes \mathbb{K}'$ with the orthogonal plane $\mathfrak{a}^\perp$. We deduce an ordered pair of points $\alpha', \alpha \in \mathbb{K}'\mathbb{P}^1$.

We may now define the cosine $\cos(\mathfrak{a}, \mathfrak{b})$ and cross-ratio $\operatorname{bir}(\mathfrak{a}, \mathfrak{b})$ of $\mathfrak{a}, \mathfrak{b} \in \mathfrak{sl}_2(\mathbb{K}) \backslash \mathbb{X}$, and relate them. These quantities play important roles throughout the whole thesis.

Lemma 0.2. *For $\mathfrak{a}, \mathfrak{b} \in \mathfrak{sl}_2(\mathbb{K})$, if we choose a square root of $\det(\mathfrak{ab})$ then we may define their cosine $\cos(\mathfrak{a}, \mathfrak{b}) \in \mathbb{K}[\sqrt{\det(\mathfrak{ab})}]$:*

$$\cos(\mathfrak{a}, \mathfrak{b}) := \frac{\langle \mathfrak{a}, \mathfrak{b}\rangle}{\sqrt{\langle \mathfrak{a}, \mathfrak{a}\rangle\langle \mathfrak{b}, \mathfrak{b}\rangle}} = \frac{-\frac{1}{2}\operatorname{Tr}(\mathfrak{ab})}{\sqrt{\det(\mathfrak{ab})}}$$

and we may order their polar points $\mathbb{P}(\mathfrak{a}^\perp \cap \mathbb{X}) = \{\alpha', \alpha\}$ and $\mathbb{P}(\mathfrak{b}^\perp \cap \mathbb{X}) = \{\beta', \beta\}$ up to simultaneous inversion, so as to define their cross-ratio $\operatorname{bir}(\mathfrak{a}, \mathfrak{b}) \in \mathbb{K}[\sqrt{\det(\mathfrak{ab})}]$:

$$\operatorname{bir}(\mathfrak{a}, \mathfrak{b}) := \operatorname{bir}(\alpha', \alpha, \beta', \beta) = \frac{(\alpha - \alpha')(\beta - \beta')}{(\alpha - \beta')(\beta - \alpha')}$$

For a same choice of $\sqrt{\det(\mathfrak{ab})}$, these quantities are related by:

$$\frac{1}{\operatorname{bir}(\mathfrak{a}, \mathfrak{b})} = \frac{1 + \cos(\mathfrak{a}, \mathfrak{b})}{2} = \frac{\det(\mathfrak{a} + \mathfrak{b})}{4\sqrt{\det(\mathfrak{ab})}}$$

The following Proposition (which is 1.57 in Chapter 1) implies that $\mathrm{PGL}_2(\mathbb{K})$ acts transitively on each level set of the determinant.

Proposition 0.3. *Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{sl}_2(\mathbb{K})$ have determinant $d \neq 0$ and $\operatorname{bir}(\mathfrak{a}, \mathfrak{b}) \neq 0$.*

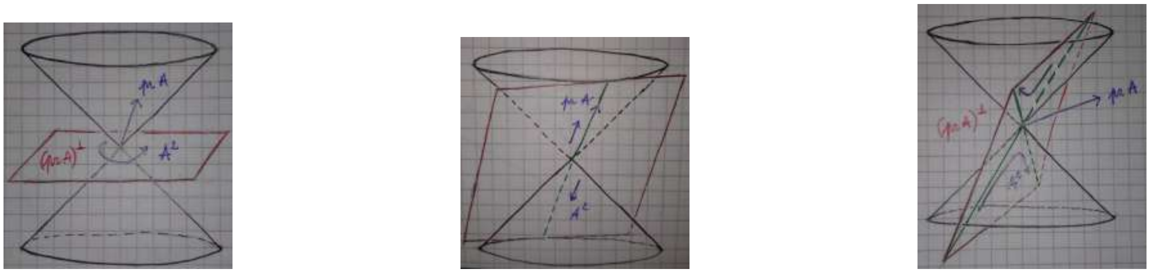
The quadratic subalgebra $\mathbb{K}[\{\mathfrak{a}, \mathfrak{b}\}]$ of $\mathfrak{gl}_2(\mathbb{K})$ contains a unique $M \in \mathrm{GL}_2(\mathbb{K})$ with $\operatorname{Tr}(M) = 2$ which conjugates $\mathfrak{a}$ to $\mathfrak{b}$. It is given by:

$$M = \mathbb{1} + \tfrac{\operatorname{bir}(\mathfrak{a},\mathfrak{b})}{2d} \cdot \{\mathfrak{a}, \mathfrak{b}\} = \frac{(d + \langle \mathfrak{a}, \mathfrak{b}\rangle)\mathbb{1} + \{\mathfrak{a}, \mathfrak{b}\}}{d + \langle \mathfrak{a}, \mathfrak{b}\rangle} \qquad \text{and} \qquad \det(M) = \operatorname{bir}(\mathfrak{a}, \mathfrak{b}).$$

We recall in Proposition 1.24 why the centralizer of $\mathfrak{a} \in \mathfrak{sl}_2(\mathbb{K}) \setminus \mathbb{X}$ in $\mathfrak{gl}_2(\mathbb{K})$ is reduced to the quadratic subalgebra $\mathbb{K}[\mathfrak{a}]$. We thus deduce the following Corollary which will have arithmetic applications bearing to the genus of quadratic forms.

Corollary 0.4. *Let $\mathfrak{a}, \mathfrak{b} \in \mathfrak{sl}_2(\mathbb{K})$ have discriminant $\Delta \neq 0$ and $\operatorname{bir}(\mathfrak{a}, \mathfrak{b}) \neq 0$.*

The matrices $M \in \mathrm{PGL}_2(\mathbb{K})$ conjugating $\mathfrak{a}$ to $\mathfrak{b}$ have a well defined determinant in the quotient $\mathbb{K}^\times / \operatorname{Norm}_{\mathbb{K}}(\mathbb{K}[\sqrt{\Delta}]^\times)$ and its is equal to the class of $\operatorname{bir}(\mathfrak{a}, \mathfrak{b})$.



Adjoint ation of A on $\mathbb{Q}\mathfrak{a} \oplus \mathfrak{a}^{\perp}$ for elliptic, parabolic and hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$.

Here is the main theorem of Chapter 1 describing the structure of the orbits for the adjoint action of $\mathrm{PSL}_2(\mathbb{K})$ on the non-zero level sets $\{\mathrm{disc} = \Delta\} \subset \mathfrak{sl}_2(\mathbb{K})$.

Theorem 0.5. *Let* $\mathfrak{a}, \mathfrak{b} \in \mathfrak{sl}_2(\mathbb{K})$ *have discriminant* $\Delta \neq 0$ *and* $\mathrm{bir}(\mathfrak{a}, \mathfrak{b}) \notin \{1, \infty\}$*. The* $C \in \mathrm{SL}_2(\mathbb{K})$ *such that* $C\mathfrak{a}C^{-1} = \mathfrak{b}$ *are parametrized by the Pell-Fermat conic:*

$$(x, y) \in \mathbb{K} \times \mathbb{K} : \quad (2x)^2 - \Delta y^2 = \mathrm{bir}(\mathfrak{a}, \mathfrak{b}) \qquad \text{by} \qquad C = x(\mathbb{1} + \mathfrak{b}\mathfrak{a}^{-1}) + y(\mathfrak{a} + \mathfrak{b})$$

In particular, $\mathfrak{a}$ *and* $\mathfrak{b}$ *are conjugate by an element of* $\mathrm{PSL}_2(\mathbb{K})$ *if and only if the Pell-Fermat equation* $(2x)^2 - \Delta y^2 = \mathrm{bir}(\mathfrak{a}, \mathfrak{b})$ *has a solution in* $\mathbb{K} \times \mathbb{K}$*.*

Remark 0.6. *For* $\mathfrak{a}, \mathfrak{b} \in \mathfrak{sl}_2(\mathbb{K}) \setminus \mathbb{X}$ *we have* $\mathrm{bir}(\mathfrak{a}, \mathfrak{b}) \notin \{1, \infty\} \iff \det\{\mathfrak{a}, \mathfrak{b}\} \neq 0$.

Scholium 0.7. *Of course, we may recast the previous discussion to characterise the orbits for the action of* $\mathrm{PSL}_2(\mathbb{K})$ *by conjugacy on itself, but we prefer to study the adjoint action of* $\mathrm{PSL}_2(\mathbb{K})$ *on its Lie algebra.*

Indeed, the linear algebra and geometry of the adjoint action are much more developed and fathomable than the corresponding features for the conjugacy action. Moreover, in the arithmetic dictionary, the elements of the lattice $\mathfrak{sl}_2(\mathbb{Z})$ *are the bridge between hyperbolic matrices in the modular group and binary quadratic forms. This also has the advantage of avoiding many confusions which may arise between the group which acts and the space which is acted upon.*

All this will become apparent inside the quaternion algebra $\mathfrak{gl}_2(\mathbb{K})$*: the group of units* $\mathrm{SL}_2(\mathbb{K})$*, kernel of the determinant morphism* $\det \colon \mathrm{GL}_2(\mathbb{K}) \to \mathbb{K}^{\times}$*, acts on the hyperplane of pure quaternions* $\mathfrak{sl}_2(\mathbb{K})$*, kernel of the trace form* $\mathrm{tr} \colon \mathfrak{gl}_2(\mathbb{K}) \to \mathbb{K}$*.*

Arithmetic implications for binary quadratic forms

The field $\mathbb{K}$ has characteristic different from 2, so we may consider the extension of scalars $\mathfrak{sl}_2(\mathbb{Z}[1/2]) \to \mathfrak{sl}_2(\mathbb{K})$, and its restriction to the lattice $\mathfrak{sl}_2(\mathbb{Z})^{\vee} = \frac{1}{2}\mathbb{Z} \oplus \mathfrak{sl}_2(\mathbb{Z})$.

We say that $\mathfrak{a}, \mathfrak{b} \in \mathfrak{sl}_2(\mathbb{Z})^\vee$ are $\mathbb{K}$-*equivalent* when their images in $\mathfrak{sl}_2(\mathbb{K})$ belong to the same orbit under the adjoint action of $\mathrm{PSL}_2(\mathbb{K})$. We may thus group the $\mathrm{PSL}_2(\mathbb{Z})$-classes of our dictionary into $\mathbb{K}$-classes and observe how this varies with $\mathbb{K}$.

When $\mathbb{K}$ is an extension of $\mathbb{Q}$, that is when it has characteristic zero, the extension of scalars $\mathfrak{sl}_2(\mathbb{Q}) \to \mathfrak{sl}_2(\mathbb{K})$ is injective so the $\mathbb{K}$-equivalence implies the equality of discriminants. When $\mathbb{K} = \mathbb{C}$, this groups the integral binary quadratic forms according to their discriminant, and we find the finite class groups. Geometrically, we are considering modular geodesics of the same length. When $\mathbb{K} = \mathbb{Q}$, this defines for each discriminant Δ a partition of the class group $\mathrm{Cl}(\Delta)$ into $\mathbb{Q}$-classes.

Let us reformulate the results of the previous section in terms of $\mathbb{Q}$-equivalence of binary quadratic forms using the quadratic dictionary $Q_a \in \mathcal{Q}(\mathbb{Z}) \leftrightarrow \mathfrak{a} \in \mathfrak{sl}_2(\mathbb{Z})^\vee$.

Corollary 0.8 (Reformulation)**.** *Consider primitive binary quadratic forms Q_a, Q_b with the same non-square discriminant $\Delta \in \mathbb{Z}$. Assume that they are not opposite (after conjugating one by $\mathrm{PSL}_2(\mathbb{Z})$ if necessary), so that $\mathrm{bir}(Q_a, Q_b)$ is finite.*

By Corollary 0.4 the class $\mathrm{bir}(Q_a, Q_b) \bmod \mathrm{Norm}_{\mathbb{Q}}(\mathbb{Q}(\sqrt{\Delta})^\times)$ only depends on the $\mathbb{Q}$-classes of Q_a and Q_b and by Theorem 0.5 it is trivial if and only if those coincide.

The matrices $C \in \mathrm{PSL}_2(\mathbb{Q})$ conjugating Q_a to Q_b are parametrized by the rational points of the Pell-Fermat conic $(2x)^2 - \Delta y^2 = \mathrm{bir}(Q_a, Q_b)$ with an explicit formula.

We also provide a computable criterion for determining the partition of $\mathrm{Cl}(\Delta)$ into $\mathbb{Q}$-equivalence classes.

Denote $\mathcal{P} = \{-1, 2\} \cup \{3, 5, 7, \dots\}$ the set of rational primes, and $\mathbb{Q}_p$ the p-adic completion of $\mathbb{Q}$. The prime -1 refers (following Conway [CF97]) to the place at which the completion of $\mathbb{Q}$ is the Archimedian field $\mathbb{Q}_{-1} = \mathbb{R}$.

For $\delta, \chi \in \mathbb{Q}_p^\times$ the *Hilbert symbol* $(\delta, \chi)_p$ takes the value 1 or -1 according to whether the Pell-Fermat equation $X^2 - \delta Y^2 = \chi Z^2$ admits a solution in $\mathbb{Q}_p\mathbb{P}^2$ or not. Thus we have $(\delta, \chi)_p = 1$ if and only if χ is the norm of an element in $\mathbb{Q}_p(\sqrt{\delta})$.

Let us define the set of prime obstructions to solving the Pell-Fermat equation $(2x)^2 - \Delta y^2 = \mathrm{bir}(Q_a, Q_b)$ by $\mathcal{P}(Q_a, Q_b) = \{p \in \mathcal{P} \mid (\Delta, \mathrm{bir}(Q_a, Q_b))_p = -1\}$.

Proposition 0.9. *Consider primitive integral binary quadratic forms Q_a, Q_b, Q_0 of discriminant Δ a non-square integer. Then:*

- *$\mathcal{P}(Q_a, Q_b) \setminus \{2\}$ is contained in the set of primes dividing Δ to an odd power.*
- *Q_a is $\mathbb{Q}$-equivalent to Q_b $\iff$ $\mathcal{P}(Q_a, Q_b) = \emptyset$ $\iff$ $\mathcal{P}(Q_a, Q_0) = \mathcal{P}(Q_0, Q_b)$.*

We apply Proposition 0.9 to determine the partition of $\mathrm{Cl}(\Delta)$ into $\mathbb{Q}$-classes in a few relevant examples involving positive discriminants (which are all fundamental). Observing the following tables, we are led to discover counter examples and formulate a conjecture, for which we provide further evidence in Section 1.5.

$\mathrm{Cl}(\Delta) = \mathbb{Z}/4$ for $\Delta = 4 \times 2022$. Since $\delta = 2022 = 2 \times 3 \times 337$ is square-free and $\equiv 2 \bmod 4$, the ring of integers of the field $\mathbb{Q}(\sqrt{2022})$ has discriminant $\Delta = 4 \times \delta$. The fundamental solution to the Pell-Fermat equation $t^2 - \delta u^2 = 1$ is $(t, u) = (1349, 30)$.

The ideal class group $\mathrm{Cl}(\Delta)$ is isomorphic to $\mathbb{Z}/4$. Its partition into genera is $\{\alpha_0, \alpha_2\}, \{\alpha_1, \alpha_3\}$ and this coincides with its partition into $\mathbb{Q}$-classes as shown by the following table.

$Q_j = (l_j, m_j, r_j)$	Period of α_j	$\mathcal{P}(Q_0, Q_j)$
$(1, -88, -86)$	$[88, 1, 28, 1]$	$\emptyset$
$(66, -72, -11)$	$[1, 4, 2, 2, 3, 1, 2, 7]$	$\{2, 337\}$
$(43, -84, -6)$	$[2, 44, 2, 14]$	$\emptyset$
$(34, -60, -33)$	$[2, 4, 1, 7, 2, 1, 3, 2]$	$\{2, 337\}$

$\mathrm{Cl}(\Delta) = \mathbb{Z}/5$ for $\Delta = 4 \times 439$. Since $\delta = 439$ is square-free and $\equiv 3 \bmod 4$ the ring of integers of the field $\mathbb{Q}(\sqrt{439})$ has discriminant $\Delta = 4 \times \delta$. The fundamental solution to the Pell-Fermat equation $t^2 - \delta u^2 = 1$ is $(t, u) = (440, 21)$.

The ideal class group $\mathrm{Cl}(\Delta)$ is isomorphic to $\mathbb{Z}/5$. Its partition into genera is trivial: there is only one genus since all elements of $\mathbb{Z}/5$ are squares. The partition into $\mathbb{Q}$-classes is $\{\alpha_0, \alpha_2, \alpha_4\}, \{\alpha_1, \alpha_3\}$ as shown by the following table.

$Q_j = (l_j, m_j, r_j)$	Period of α_j	$\mathcal{P}(Q_0, Q_j)$
$(2, -38, -39)$	$[19, 1, 40, 1]$	$\emptyset$
$(15, -14, -26)$	$[1, 1, 6, 3, 13, 1]$	$\{2, 439\}$
$(18, -10, -23)$	$[1, 2, 3, 1, 3, 1, 7, 1]$	$\emptyset$
$(30, -34, -5)$	$[1, 3, 1, 3, 2, 1, 1, 7]$	$\{2, 439\}$
$(13, -40, -3)$	$[3, 6, 1, 1, 1, 1, 13]$	$\emptyset$

Remark 0.10 (Counter-examples)**.** *Genus equivalence does not imply $\mathbb{Q}$-equivalence: there exist forms of the same genus which are not $\mathbb{Q}$-equivalent.*

The $\mathbb{Q}$-equivalence does not control the period lengths: there exist $\mathbb{Q}$-equivalent forms whose roots have euclidean periods of different length.

Inverse elements in the class group can remain in different $\mathbb{Q}$-classes.

Question 0.11 (Conjecture)**.** *The $\mathbb{Q}$-equivalence implies genus-equivalence.*

More precisely, the $\mathbb{Q}$-equivalence classes seem to be described as follows. Decompose the class group into a product of primary cyclic groups:

$$\mathrm{Cl}(\Delta) = \prod_{p \in \mathcal{P}} \prod_{j \in \mathbb{N}} (\mathbb{Z}/p^e)^{n_{p,e}}$$

and denote $Q_{p,e,k} \in \mathbb{Z}/p^e$ *where* $1 \leq k \leq n_{p,e}$ *the coordinates of* Q. *Then the* $Q_{p,e,k} \bmod 2$ *provide a complete set of invariants for the* $\mathbb{Q}$*-classes.*

Remark 0.12. *Our initial motivation only involved positive discriminants, but Corollary 0.8 and Proposition 0.9 remain true for negative discriminants.*

Geometric interpretation in terms of modular geodesics

Let us derive from Theorem 0.5 a geometric interpretation of $\mathbb{Q}$-equivalence in terms of the modular geodesics.

Consider two elements $(\alpha, A, Q_a, \mathfrak{a})$ and $(\beta, A, Q_b, \mathfrak{b})$ in our modular dictionary with the same discriminant $\mathrm{disc}(\mathfrak{a}) = \Delta = \mathrm{disc}(\mathfrak{b})$. They define oriented geodesics $(\alpha', \alpha), (\beta', \beta)$ in the hyperbolic plane which either intersect at a point with a well defined angle $\theta \in]0, \pi[$ or have a unique common perpendicular geodesic arc of length λ and whose co-orientations inherited by each axis may coincide or not. The quantities θ, λ are given in terms of the cross-ratio $\mathrm{bir}(\mathfrak{a}, \mathfrak{b}) = \mathrm{bir}(\alpha', \alpha, \beta', \beta)$ by:

$$\left(\cos \tfrac{\theta}{2}\right)^2 = \frac{1 + \cos(\theta)}{2} = \frac{1}{\mathrm{bir}(\mathfrak{a}, \mathfrak{b})} \qquad \left(\cosh \tfrac{\lambda}{2}\right)^2 = \frac{1 + \cosh(\lambda)}{2} = \frac{1}{\mathrm{bir}(\mathfrak{a}, \mathfrak{b})}$$

Corollary 0.13. *Two* $\mathrm{PSL}_2(\mathbb{Z})$*-classes of* $\mathrm{disc} = \Delta$ *are* $\mathbb{Q}$*-equivalent if and only if the corresponding modular geodesics satisfy the following equivalent conditions:*

θ *There exists one intersection point with angle* $\theta \in]0, \pi[$ *such that:*

$$\left(\cos \tfrac{\theta}{2}\right)^2 = \tfrac{1}{(2x)^2 - \Delta y^2} \qquad \text{for} \quad x, y \in \mathbb{Q}$$

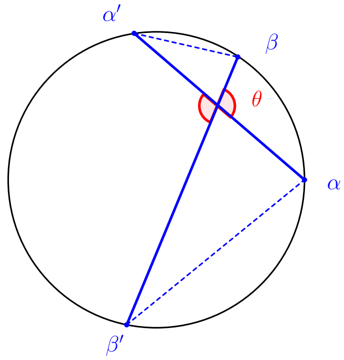
in which case all intersection points have this property.

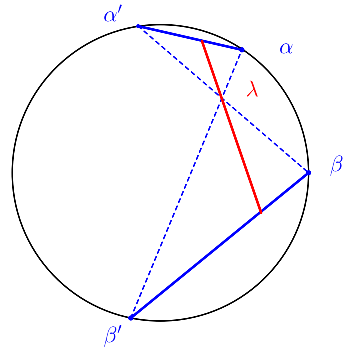
λ *There exists one co-oriented ortho-geodesic of length* λ *such that:*

$$\left(\cosh \tfrac{\lambda}{2}\right)^2 = \tfrac{1}{(2x)^2 - \Delta y^2} \qquad \text{for} \quad x, y \in \mathbb{Q}$$

in which case all co-oriented ortho-geodesics have this property.

In other terms, the geometric quantities on the left hand sides belong to the group of norms of the quadratic extension $\mathbb{Q}(\sqrt{\Delta})/\mathbb{Q}$ *(which is stable under inversion).*





Angle at intersection $\frac{1}{\text{bir}} = \left(\cos\frac{\theta}{2}\right)^2$. Length of ortho-geodesic $\frac{1}{\text{bir}} = \left(\cosh\frac{\lambda}{2}\right)^2$.

0.3.2 Linking numbers of modular knots

Our starting point for computing linking numbers is an algorithmic formula, which was used by Pierre Dehornoy in [Deh11]. It relies on the description of the modular link in terms of the Lorenz template explained at the end of Section 0.2.

We endow the submonoid $\mathrm{PSL}_2(\mathbb{N})$ of $\mathrm{PSL}_2(\mathbb{Z})$, which is freely generated by $L\&R$, with the lexicographic order extending $L < R$.

In the group $\mathrm{PSL}_2(\mathbb{Z})$ the conjugacy class of an infinite order element intersects the monoid $\mathrm{PSL}_2(\mathbb{N})$ along its Lyndon representatives, which consist in all cyclic permutations of a non-empty $L\&R$-word. The primitivity of the conjugacy class is equivalent to the primitivity of the cyclic words, and the conjugacy class is hyperbolic when both letters L and R appear.

The set $\{L,R\}^{\mathbb{N}}$ of infinite binary sequences on the letters $L\&R$ is given the lexicographic order extending $L < R$. The monoid $\mathrm{PSL}_2(\mathbb{N})$ maps to $\{L,R\}^{\mathbb{N}}$ by sending a finite word A to its periodisation A^∞. This map is increasing, and injective in restriction to primitive elements.

We use σ to denote the Bernoulli shift on $\{L,R\}^{\mathbb{N}}$ which removes the first letter, as well as the cyclic shift on $\mathrm{PSL}_2(\mathbb{N})$ which moves the first letter at the end. These shifts are intertwined by the periodisation map $A \mapsto A^\infty$, namely for all $A \in \mathrm{PSL}_2(\mathbb{N})$ we have $(\sigma^j A)^\infty = \sigma^j(A^\infty)$.

In particular, the Lyndon representatives for the conjugacy class of $A \in \mathrm{PSL}_2(\mathbb{N})$ are the cyclic permutations $\sigma^i A$ for $1 \le i \le \mathrm{len}(A)$, and we shall consider them with multiplicity when A is not primitive.

Denote by $W[-1] \in \{L,R\}$ the last letter of a non-empty word $W \in \mathrm{PSL}_2(\mathbb{N})$. Thus for instance, $(\sigma^1 W)[-1]$ is the first letter of W.

Following Iverson [Knu92], denote $[\![P]\!] \in \{0,1\}$ the truth value of a property P.

Proposition 0.14. *For primitive hyperbolic matrices $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ we have:*

$$\mathrm{lk}(A,B) = \frac{1}{2}\sum_{i=1}^{\mathrm{len}(A)}\sum_{j=1}^{\mathrm{len}(B)} \begin{pmatrix} [\![(\sigma^i A)[-1] > (\sigma^j B)[-1]]\!]\,[\![\sigma^i A^\infty < \sigma^j B^\infty]\!] \\ + \\ [\![(\sigma^i A)[-1] < (\sigma^j B)[-1]]\!]\,[\![\sigma^i A^\infty > \sigma^j B^\infty]\!] \end{pmatrix} \quad \text{(Algo-Sum)}$$

This Algo-Sum counts the pairs of Lyndon representatives whose periodisations are ordered in the opposite way to their last letters.

Remark 0.15. *When A and B are conjugate, this Algo-Sum returns the self-linking number of the modular knot for the framing defined by the Lorenz template.*

Example 0.16. *The linking number between the knots associated to the primitive hyperbolic matrices $RLL, RRL \in \mathrm{PSL}_2(\mathbb{N})$ is 1. Indeed, the only pairs of Lyndon representatives which add 1 to the Algo-Sum are (RLL, LRR) and (LLR, RRL).*

In Section 4.2 we propose a variation on this summation formula, but we will not relate its exploration to the core of the thesis. It can be interpreted as a factorisation (or polarisation) of the quadratic linking form, thus opening a door onto its Hilbertian analysis, and the investigation of its binomial statistics.

For a pattern $P \in \mathrm{PSL}_2(\mathbb{N})$ and a hyperbolic $A \in \mathrm{PSL}_2(\mathbb{N})$, let $\mathrm{pref}(P, A^\infty) = [\![A^\infty \in P \cdot \mathrm{PSL}_2(\mathbb{N})]\!] \in \{0,1\}$ tell whether P is a prefix of A^∞, and $\mathrm{occ}(P,A) = \sum_{j=1}^{\mathrm{len}\,A} \mathrm{pref}\left(P, A_j^\infty\right)$ count the number of cyclic occurrences of P in $A \bmod \sigma$.

Proposition 0.17 (Sum of linked patterns)**.** *For coprime hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{N})$ the corresponding modular knots have linking number:*

$$\mathrm{lk}(A,B) = \frac{1}{2}\sum_{w} \begin{pmatrix} \mathrm{occ}(RwL, A)\cdot \mathrm{occ}(LwR, B) \\ + \\ \mathrm{occ}(RwL, B)\cdot \mathrm{occ}(LwR, A) \end{pmatrix}$$

where the summation extends over all words $w \in \mathrm{PSL}_2(\mathbb{N})$ including the empty one.

Consider the free $\mathbb{Z}$-module generated by the set $\mathrm{PSL}_2(\mathbb{N})/\sigma$ of all cyclic words, endowed with the symmetric bilinear form lk. Let us reformulate Proposition 4.34 as a factorisation of the corresponding symmetric matrix.

Definition 0.18 (Occurrence matrices)**.** *Denote $P(w,A)$ the infinite "rectangular matrix" with entries $\mathrm{occ}(RwL, A)$ indexed by $w \in \mathrm{PSL}_2(\mathbb{N})$ and $A \in \mathrm{PSL}_2(\mathbb{N})/\sigma$.*

Denote $P^\#(w,A) = P(w^\#, A^\#)$ where $w^\#\&A^\#$ are the mirror images of $w\&A$. Its entries are given by $\mathrm{occ}(LwR, A) = \mathrm{occ}(Lw^\# R, A^\#)$.

Finally we define $Z = P + iP^\#$ over the ring $\mathbb{Z}[i]$ of Gaussian integers.

Corollary 0.19 (Factorising the linking matrix)**.** *The matrix of the bilinear form $\mathrm{lk}(A,B)$ is the imaginary part of the product ${}^{t}ZZ^\#$.*

Invariants on pairs of conjugacy classes

We wish to compute functions of pairs of conjugacy classes in $\mathrm{PSL}_2(\mathbb{Z})$, such as linking numbers of modular knots. For this we now explain how to average conjugacy invariants for pairs of matrices to obtain functions of pairs of conjugacy classes.

Consider a group Γ acting on a space Σ and a function f defined on $\Sigma \times \Sigma$ with values in a commutative group Λ which is invariant under the diagonal action of Γ:

$$f \colon \Sigma \times \Sigma \to \Lambda \qquad \forall W \in \Gamma,\ \forall a, b \in \Sigma:\ f(a,b) = f(W \cdot a, W \cdot b)$$

We define an invariant F for pairs of Γ-orbits $[a], [b]$ by summing f over all pairs of representatives of the orbits considered modulo the diagonal action of Γ.

The pairs of representatives for the orbits are parametrized by the $(U \cdot a, V \cdot b)$ for $(U,V) \in \Gamma/(\mathrm{Stab}\, a) \times \Gamma/(\mathrm{Stab}\, b)$, and the quotient of this set by the diagonal action of Γ by left translations is denoted $\Gamma/(\mathrm{Stab}\, a) \times_\Gamma \Gamma/(\mathrm{Stab}\, b)$.

Consequently, the sum indexed by $(U,V) \in (\Gamma/\operatorname{Stab} a) \times_\Gamma (\Gamma/\operatorname{Stab} b)$ defines our desired invariant:

$$F([a],[b]) = \sum_{(U,V)} f(U \cdot A, V \cdot b)$$

This can also be written as the sum over double cosets $W \in (\mathrm{Stab}\, a)\backslash\Gamma/(\mathrm{Stab}\, b)$:

$$F([a],[b]) = \sum_{W} f(a, W \cdot b)$$

because the map $(\Gamma/\operatorname{Stab} a) \times (\Gamma/\operatorname{Stab} b) \to (\mathrm{Stab}\, a)\backslash\Gamma/(\mathrm{Stab}\, b)$ sending (U,V) to $W = U^{-1}V$ is surjective, and its fibers are the orbits under the diagonal action of Γ by left translations.

We shall apply this discussion to the action of $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ on the group $\mathrm{PSL}_2(\mathbb{R})$ or the lattice $\mathrm{PSL}_2(\mathbb{Z})$, on its Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ or the lattice $\mathfrak{sl}_2(\mathbb{Z})$, on the symmetric space $\mathbb{H} \cup \mathbb{H}'$ or the trivalent tree $\mathcal{T}$ and thus on its bi-infinite oriented geodesics $\mathcal{G}$. These actions are closely related, for instance we may associate to $A \in \mathrm{PSL}_2(\mathbb{R})$ its projection in $\mathbb{P}(\mathfrak{sl}_2(\mathbb{R}))$, and to $A \in \mathrm{PSL}_2(\mathbb{Z})$ a point in $\mathcal{T} \cup \partial\mathcal{T} \cup \mathcal{G}$.

Of course when $\mathrm{PSL}_2(\mathbb{Z})$ acts on itself by conjugacy, we obtain invariants for pairs of conjugacy classes. We will explain how to compute them on pairs of hyperbolic conjugacy classes in Section 2.3, after recalling why the centraliser of a hyperbolic matrix in $\mathrm{PSL}_2(\mathbb{Z})$ is the subgroup generated by its primitive root.

The function $f(a,b)$ could be obtained from geometrical invariants such as the scalar product $\langle a, b \rangle$, like the cross-ratio $\mathrm{bir}(\alpha', \alpha, \beta', \beta)$, as well as combinatorial invariants like $\mathrm{cross}(g_a, g_b)$ or $\mathrm{cosign}(g_a, g_b)$ introduced in the next paragraph.

Linking and intersection numbers from the action on $(\mathcal{T}, \mathrm{cord})$

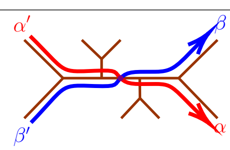
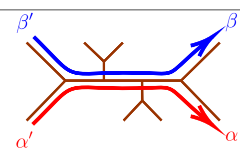
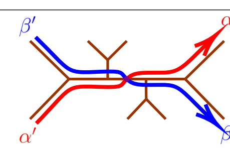
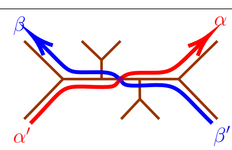
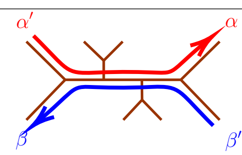
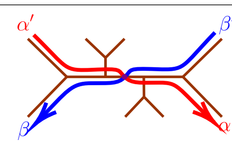
The group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ acts on $\Sigma = \mathcal{T}$ preserving its cyclic order structure, defined on the set of edges incident to each vertex (given by the surface embedding $\mathcal{T} \subset \mathbb{HP}$). This is equivalent to the cyclic order function $\mathrm{cord}(x, y, z) \in \{-1, 1\}$ of three distinct points $x, y, z \in \mathcal{T} \cup \partial\mathcal{T}$, or to the crossing function $\mathrm{cross}(u, v, x, y) \in \{-1, 0, 1\}$ of four distinct points $u, v, x, y \in \mathcal{T} \cup \partial\mathcal{T}$ defined by:

$$\mathrm{cross}(u, v, x, y) = \tfrac{1}{2}\left(\mathrm{cord}(u, x, v) - \mathrm{cord}(u, y, v)\right)$$

that is the algebraic intersection number of the oriented geodesics (u, v) and (x, y). We denote $|\mathrm{cross}|(u, v, x, y) \in \{0, 1\}$ the absolute value of $\mathrm{cross}(u, v, x, y)$ which is the linking number of the cycles (u, v), (x, y) in the cyclically ordered boundary $\partial\mathcal{T}$.

The intersection of two oriented bi-infinite geodesics $g_a = (\alpha', \alpha)$ and $g_b = (\beta', \beta)$ of $\mathcal{T}$ is either empty in which case we define $\mathrm{cosign}(g_A, g_b) = 0$, or else it consists in a geodesic containing at least one edge along which we may thus compare their orientations by $\mathrm{cosign}(g_a, g_b) \in \{-1, +1\}$.

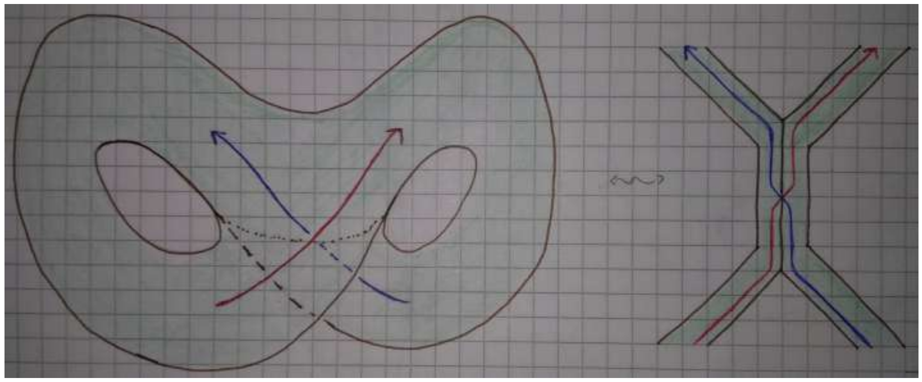
The functions cross and cosign are $\mathrm{PSL}_2(\mathbb{Z})$-invariant, symmetric, and inverting the orientation of one argument results in a change of sign.

cosign \ cross	**+1**	**0**	**−1**
+1			
−1			

Configurations of axes: cross and cosign. Note that $\mathrm{cross} \neq 0 \implies \mathrm{cosign} = \pm 1$.

For coprime hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ with axes $g_A = (\alpha', \alpha)$ and $g_B = (\beta', \beta)$ in $\mathcal{T}$, we write $\mathrm{cross}(A, B) = \mathrm{cross}(\alpha', \alpha, \beta', \beta)$ and $\mathrm{cosign}(A, B) = \mathrm{cosign}(g_A, g_B)$.

Lemma 0.20 (cosign = 1)**.** *Consider infinite order* $A, B \in \mathrm{PSL}_2(\mathbb{Z})$. *There exists* $C \in \mathrm{PSL}_2(\mathbb{Z})$ *such that* $CAC^{-1}, CBC^{-1} \in \mathrm{PSL}_2(\mathbb{N})$ *if and only if the combinatorial axes* $g_A, g_B \subset \mathcal{T}$ *share an oriented edge, that is when* $\mathrm{cosign}(A, B) = 1$.



Template crossings $\leftrightarrow \{(g_A, g_B)\colon\ |\mathrm{cross}|(g_A, g_B) = 1 = \mathrm{cosign}(g_A, g_B)\} \bmod \Gamma \times \Gamma$.

We shall use this important observation to recast the algorithmic formula (Algo-Sum) according to the general framework introduced in the previous paragraph.

Theorem 0.21. *For coprime hyperbolic matrices $A, B \in \Gamma = \mathrm{PSL}_2(\mathbb{Z})$ we have:*

$$\mathrm{lk}(A,B) = \frac{1}{2}\sum \left(|\mathrm{cross}| \times \frac{1+\mathrm{cosign}}{2}\right)(A_u, B_v)$$

where the sum extends over pairs of representatives $A_u = UAU^{-1}$ and $B_v = VBV^{-1}$ for the conjugacy classes with $(U,V) \in \Gamma/\operatorname{Stab}(A) \times_\Gamma \Gamma/\operatorname{Stab}(B)$.

In particular, we recover the intersection number between modular geodesics as:

$$\mathrm{lk}(A,B) + \mathrm{lk}(A,B^{-1}) = \frac{1}{2}\sum |\mathrm{cross}|(A_u, B_v) = \tfrac{1}{2}\cdot I(A,B)$$

whereas the sum of the cosign over pairs of intersecting axes yields:

$$\mathrm{lk}(A,B) - \mathrm{lk}(A,B^{-1}) = \frac{1}{2}\sum \left(|\mathrm{cross}| \times \mathrm{cosign}\right)(A_u, B_v).$$

We deduce an efficient algorithm computing the intersection number $I(A,B)$ from the $L\&R$-factorisation of A, B by applying Algo-Sum formula to the linking numbers $\mathrm{lk}(A,B)$ and $\mathrm{lk}(A,B^{-1})$.

Remark 0.22. *Note that if A is conjugate to B, then $I(A,B)$ is the intersection number between two parallel copies of the corresponding modular geodesic, which is twice its self-intersection number (counted as the number of double points).*

For instance, the modular geodesic corresponding to RLL has self-intersection

$$\tfrac{1}{2}I([RLL],[RLL]) = \mathrm{lk}([RLL],[RLL]) + \mathrm{lk}([RLL],[LLR]) = \tfrac{1}{2}4 + \tfrac{1}{2}2 = 3.$$

Deforming the $\mathrm{PSL}_2(\mathbb{Z})$-action on $\mathbb{HP}$ to the $\mathrm{PSL}_2(\mathbb{Z})$-action on $\mathcal{T}$

Let us finally show how to recover the linking number of two modular knots as the limiting value of a function defined on the character variety of the modular group.

We first define a one parameter family of representations $\rho_q \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{R})$ depending algebraically on the parameter $q \in \mathbb{R}^*$ and with integral coefficients. Fix $S_q = S$ and let T_q be the conjugate of T by $\exp \frac{1}{2}\log(q)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Given $A \in \mathrm{PSL}_2(\mathbb{Z})$, we deduce $A_q = \rho_q(A)$ from any $S\&T$-factorisation by replacing $T \mapsto T_q$, for instance:

$$R_q = \begin{pmatrix} q & 1 \\ 0 & q^{-1} \end{pmatrix} \quad \text{and} \quad L_q = \begin{pmatrix} q & 0 \\ 1 & q^{-1} \end{pmatrix}.$$

This descends to a representation $\bar{\rho}_q \colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R})$ which is faithful and discrete (this follows from the positivity of $\mathrm{disc}(R_q) = (q - q^{-1})^2$), and positive in the sense that T_q is a $2\pi/3$-rotation of $\mathbb{HP}$ in the positive direction. Conversely, every such representation is conjugate to $\bar{\rho}_q$ for a unique $q > 0$, so we have parametrized the Teichmüller space of $\mathrm{PSL}_2(\mathbb{Z})$ by the real algebraic set $\mathbb{R}_+^*$.

As $q \to \infty$, the hyperbolic orbifold $\mathbb{M}_q = \rho_q(\Gamma)\backslash\mathbb{HP}$ has a convex core which retracts onto the long geodesic arc (i, j_q) connecting the conical singularities. The hyperbolic geodesics of $\mathbb{M}_q$ remain in this convex core, so their angles tend to 0 mod π.

Proposition 0.23. *Consider hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ such that $|\mathrm{cross}|(A, B) = 1$.*

For all $q > 0$ the elements $A_q, B_q \in \mathrm{PSL}_2(\mathbb{R})$ are hyperbolic, and their oriented geometric axes intersect at an angle whose cosine is given by:

$$\cos(A_q, B_q) = \frac{\mathrm{Tr}(A_q B_q) - \mathrm{Tr}(A_q B_q^{-1})}{\sqrt{\mathrm{disc}(A_q)\,\mathrm{disc}(B_q)}}$$

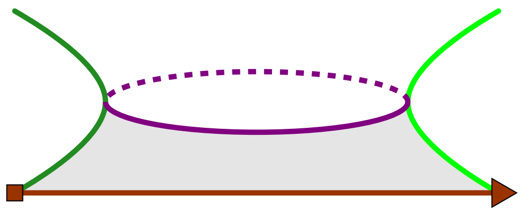
which is an algebraic function of q with limit $\cos(A_q, B_q) \xrightarrow[q\to\infty]{} \mathrm{cosign}(A, B)$.

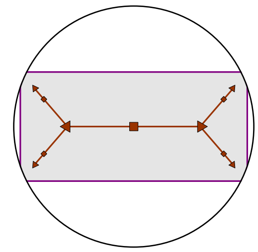
The formula for the cosine is derived from Lemma 0.2 and some trace identities. To find the limit as $q \to \infty$ we compute the degrees and dominant terms of the polynomials involved in this expression as follows.

For $C \in \mathrm{PSL}_2(\mathbb{Z})$, denote $\mathrm{len}(C) \in \mathbb{N}$ the minimum displacement length $d(e, C \cdot e)$ of an edge $e \in \mathcal{T}$. When C has infinite order, it is the $L\&R$-length of a Lyndon representative, and when C has finite order it is zero.

For all $C \in \mathrm{PSL}_2(\mathbb{Z})$ the Laurent polynomial $\mathrm{Tr}(C_q)$ is reciprocal of degree $\mathrm{len}(C)$. To identify the limit, we show that for infinite order elements $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ whose axes intersect, we have $\mathrm{cosign}(A, B) = \mathrm{sign}\left(\mathrm{len}(AB) - \mathrm{len}(AB^{-1})\right)$.

This should not surprise someone acquainted with compactifications of Teichmüller space by actions on trees or by valuations [Ota15, MS21]. Here the unique boundary point $q = \infty$ corresponds to the action on $\mathcal{T}$ or to the valuation $-\deg_q$.





The convex core of $\mathbb{M}_q$ lifts in $\mathbb{HP}$ to an ϵ-neighbourhood of $\mathcal{T}_q$ with $\epsilon = \Theta(1/q^2)$.

Definition 0.24. *For conjugacy classes $[A], [B]$ of hyperbolic elements in $\mathrm{PSL}_2(\mathbb{Z})$, consider the algebraic functions of q defined by:*

$$\mathrm{L}_q([A],[B]) = \sum \left(\frac{[\![\mathrm{bir} > 1]\!]}{\mathrm{bir}}\right)\left(\tilde{A}_q, \tilde{B}_q\right) \tag{L_q}$$

$$\mathrm{C}_q([A],[B]) = \sum (|\mathrm{cross}| \times \cos)\left(\tilde{A}_q, \tilde{B}_q\right) \tag{C_q}$$

where the sums extend over pairs of representatives $\tilde{A} = UAU^{-1}$ and $\tilde{B} = VBV^{-1}$ for the conjugacy classes with $(U, V) \in \Gamma/\operatorname{Stab}(A) \times_\Gamma \Gamma/\operatorname{Stab}(B)$.

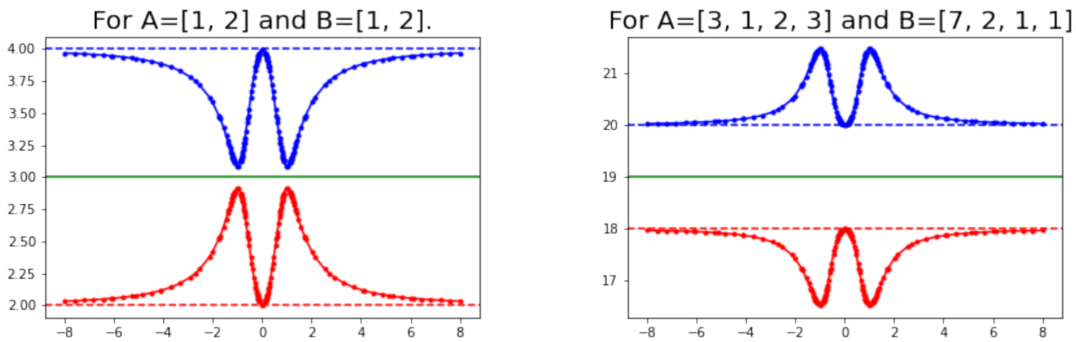
The appearance of $[\![\mathrm{bir} > 1]\!] = |\mathrm{cross}|$ as a factor in the terms of L_q and C_q amounts to restricting the summations over pairs of matrices whose axes intersect. Hence the support of the sums corresponds to the intersection points of the modular geodesics $[\gamma_A]$ and $[\gamma_B]$ associated to the conjugacy classes, which must be counted with appropriate multiplicity when A or B is not primitive, and we have:

$$\mathrm{L}_q([A],[B]) = \sum \left(\cos\tfrac{\theta}{2}\right)^2 \qquad \text{and} \qquad \mathrm{C}_q([A],[B]) = \sum (\cos\theta)\,.$$

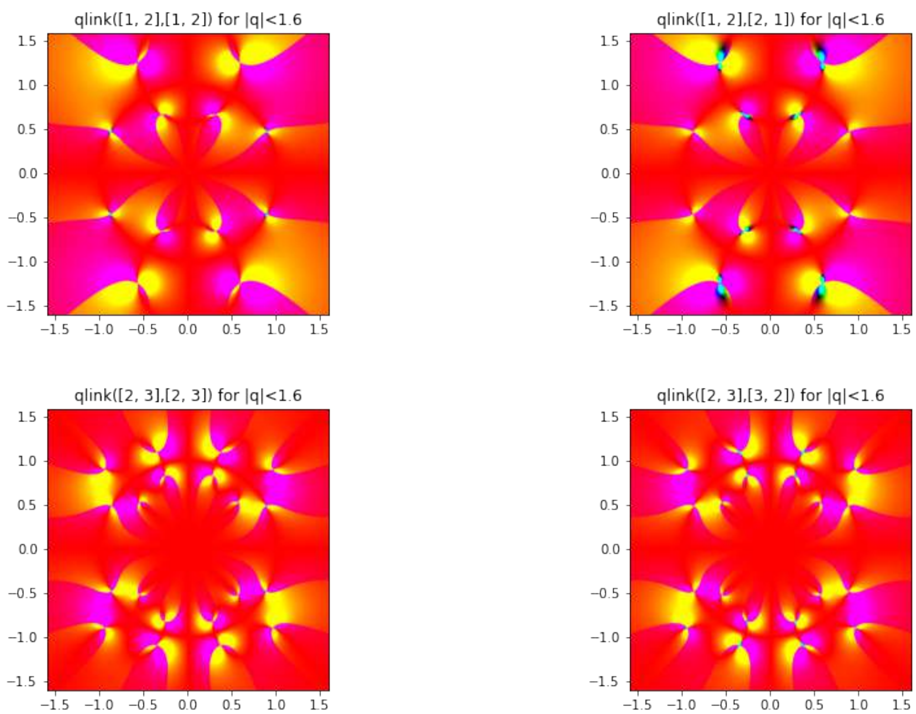
Theorem 0.25. *For conjugacy classes $[A], [B]$ of hyperbolic elements in $\mathrm{PSL}_2(\mathbb{Z})$, the functions $\mathrm{L}_q([A],[B])$ and $\mathrm{C}_q([A],[B])$ have limits at the boundary point of the $\mathrm{PSL}_2(\mathbb{R})$-character variety of $\mathrm{PSL}_2(\mathbb{Z})$, which recover the linking and intersection numbers of the corresponding modular knots and modular geodesics:*

$$\tfrac{1}{2}\,\mathrm{L}_q([A],[B]) \xrightarrow[q\to\infty]{} \mathrm{lk}(A,B)$$
$$\tfrac{1}{2}\,\mathrm{C}_q([A],[B]) \xrightarrow[q\to\infty]{} 2\,\mathrm{lk}(A,B) - \tfrac{1}{2}I(A,B)$$

Let us display here some graphs of $q \mapsto \mathrm{L}_q(A,B)$ for a few pairs (A,B), and refer to the end of Section 5.3 for more.



The graphs of $\mathrm{L}_q(A,B)$ & $\mathrm{L}_q(A,B^{-1})$ and their average $I(A,B)=I(A,B^{-1})$.



Graphs of $\mathrm{L}_q(A,B)$ for $q\in\mathbb{C}$ and $|q|<1.6$.

0.4 Further directions of research

0.4.1 Linking forms of Fuchsian groups

To begin with, we compare the definitions of the functions L_q and C_q and their limiting behaviour at $q = \infty$ with similar considerations which have been made for non-oriented loops in a closed surface S of genus $g \geq 2$. Such loops, corresponding to the conjugacy classes of $\alpha, \beta \in \pi_1(S)$ up to inversion, define trace functions $\mathrm{Tr}(\alpha), \mathrm{Tr}(\beta)$ on the $\mathrm{SL}_2(\mathbb{C})$-character variety of $\pi_1(S)$ (whose real locus contains the Teichmüller space of S as a Zariski dense open set). This character variety carries a natural symplectic structure [Gol84], given by the Weil-Petersson symplectic form.

The sum $\mathrm{C}_q(A, B)$ looks very much like Wolpert's cosine formula [Wol82, Wol81] computing the Poisson bracket $\{\mathrm{Tr}(\alpha), \mathrm{Tr}(\beta)\}$ of the trace functions. The major difference is that Wolpert's formula is a skew-symmetric expression in two non-oriented loops. In fact, we are able to define an analog of Wolpert's formula by summing the product $\mathrm{cross}(A, B) \times \cos(A, B)$. However, the Teichmüller space of $\mathbb{M}$ is reduced to a point so any Poisson structure in the usual sense would be trivial, and we expect this function to be zero (as corroborated by our computer experimentation).

Moreover, the Weil-Petersson symplectic form has been extended to several compactifications of the character variety [PP91, SB01, MS]. The limits of the Poisson bracket $\{\mathrm{Tr}(\alpha), \mathrm{Tr}(\beta)\}$ at the respective boundary points have been interpreted in [Bon92, Proposition 6] and [MS]. Thus, we may generalise the definitions of our functions L_q & C_q to oriented geodesics in hyperbolic surfaces and ask for an interpretation of their limits at boundary points of the Teichmüller space.

Pursuing this direction, one may ask for extension of L_q & C_q to pairs A, B of oriented geodesic currents. This should be analogous to the extension of the intersection form described in Bonahon [Bon88]. One may also wish to replace ρ with a semi-conjugacy class of representations $\Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ or a generalised cross-ratio [Ota92]. The aim would be to think of L_q & C_q as differential forms on the "tangent bundle" to these spaces of representations or generalized cross-ratios. The semi-conjugacy class of representations $\Gamma \to \mathrm{Homeo}(\mathbb{S}^1)$ form a cone in the first bounded cohomology group $H^1_b(\Gamma; \mathbb{R})$, and we suspect that something similar is true for some spaces of general cross-ratios. Thus we ask

Question 0.26. *How to interpret* $\mathrm{L}_\rho(A, B)$ *as a "differential form" on (an appropriate subspace in) the first bounded cohomology group* $H^1_b(\Gamma; \mathbb{R})$ *?*

Besides, we believe that the functions L_q would yield some kind of Killing form on Goldman's Lie algebra of oriented loops [Gol86].

0.4.2 Arithmetic and Geometric deformations

Let us mention another general context in which our definitions L_q & C_q seem to apply with almost no changes. Recall that our definitions of the cross-ratios and cosine in Lemma 0.2 hold for pairs of semi-simple elements in $\mathrm{PGL}_2(\mathbb{K})$. Thus for any faithful representation of a group $\rho\colon \Gamma \to \mathrm{PSL}_2(\mathbb{K})$ sending $A, B \in \Gamma$ to semi-simple elements, one may try define the following invariants for the pair of conjugacy classes:

$$\mathrm{L}_\rho(A,B) = \sum \mathrm{bir}(\rho\tilde{A}, \rho\tilde{B})^{-1} \qquad \mathrm{C}_\rho(A,B) = \sum \cos(\rho\tilde{A}, \rho\tilde{B})$$

where the sum is indexed by the double-coset space $\mathrm{Stab}\, A\backslash\Gamma/\,\mathrm{Stab}\, B$ with some restrictions analog to $[\![\mathrm{bir} > 1]\!]$ and $[\![|\mathrm{cross}| > 1]\!]$ ensuring that it has finite support, which we shall comment later on. These define functions on (a subset in) the space of representations $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{K}))$ considered up to $\mathrm{PSL}_2(\mathbb{K})$-conjugacy at the target. One may ask for interpretations of their limiting values at special points in its appropriate compactifications.

As suggested above, this construction works in particular for discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ called Fuchsian groups. In general, we may want to specify that $\rho(\Gamma)$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{K})$ after $\mathbb{K}$ has been given a topology, or furthermore that $\rho(\Gamma)$ has finite covolume for the Haar measure on $\mathrm{PSL}_2(\mathbb{K})$ with respect to a measure on $\mathbb{K}$. In that case, one may consider the quotient of the symmetric space $\mathbb{P}(\mathfrak{sl}_2(\mathbb{K}) \setminus \mathbb{X})$ of $\mathrm{PSL}_2(\mathbb{K})$ by $\rho(\Gamma)$, and observe the relative position between the "cycles" corresponding to A, B in that quotient.

We may now suggest some tantalising connections between arithmetic and topology. For this, we should compare our summations (L_q) and (C_q) with the modular cocycles introduced in [DIT17] and the products appearing in [DV22].

Let us note however that [DIT17] considers the linking numbers $\mathrm{lk}(A + A^{-1}, B + B^{-1})$ between cycles obtained by lifting a geodesic and its inverse: this number amounts to the geometric intersection $I(A, B)$ of the modular geodesics. Furthermore [DV22] considers deformations of an arithmetic nature for these intersection numbers.

None of these address the actual linking numbers, and their approach is motivated by the arithmetic of modular forms, while ours will be inspired by the geometry of the character variety. Thus it would be interesting on the one hand to understand the arithmetic of linking numbers in terms of the modular forms appearing in [Kat84] or the modular cocycles in [DIT17], and on the other hand to relate the p-arithmetic intersections numbers considered in [DV22] to the special values of functions L_ρ & C_ρ defined for representations $\rho\colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Q}_p)$ as suggested above.

0.4.3 Special values of Poincaré Series

We may apply the general averaging procedure explained at the beginning of subsection 0.3.2 to other conjugacy invariants $f_q(A, B)$ and define new functions $F_q(A, B)$ on the character variety of $\mathrm{PSL}_2(\mathbb{Z})$. Their limit at the boundary point $q = \infty$ will be expressed in terms of the linking number $\mathrm{lk}(A, B)$ as soon as $f_q(A, B)$ converges to an expression of $\mathrm{cosign}(A, B)$ as $q \to \infty$.

Various motivations (including special values for Poincaré series [Sie65, Dir42], and McShane's identity [Bow96]) suggest to choose $f_q(A, B) = (x + \sqrt{x^2 - 1})^{-s}$ for some complexe variable $s \in \mathbb{C}$ where $x = -\frac{1}{2}\,\mathrm{tr}(\mathfrak{a}_q \mathfrak{b}_q)$ is the numerator of $\cos(A_q, B_q)$ in the formula of Proposition 0.23.

This summand $f_q(A, B)$ can also be written $e^{-si\theta}$ where θ is the angle between the oriented geometric axes of A_q and B_q when they intersect and e^{-sl} where l is the length of the ortho-geodesic arc γ connecting the geometric axes of A_q and B_q when they are disjoint. In formula:

$$F_q(A, B) = \sum (x + \sqrt{x^2 - 1})^{-s} = \sum_{[\gamma_A] \perp \gamma \perp [\gamma_B]} \exp(-s l_\gamma) - \sum_{p \in [\gamma_A] \cap [\gamma_B]} \exp(-s i \theta_p).$$

So the sum over all double cosets splits as a finite sum computable in a similar way to the Algo-Sum, and an infinite series which converges for $\Re(s) > 1$ (the topological entropy for the action of $\mathrm{PSL}_2(\mathbb{Z})$ on the hyperbolic plane). The infinite sum is a bivariate analog (in (A, B)) of the univariate Poincaré "theta-series" which appeared in the works of Eisenstein: those admit meromorphic continuation to $s \in \mathbb{C}$ and their special values in the variable s have been of interest for arithmetics and dynamics. Similar Poincaré series associated to one modular geodesic are also defined in [Kat84]. The earliest appearance we found for such bivariate series is in [For23, Section 50], and the only other one is [Pau13].

When $q = \infty$ and $s = 1$, the real part of the finite sum evaluates to $2\,\mathrm{lk}(A, B) - I(A, B)$, but one may wonder about the infinite series (now the order in which we take limits in s and q may import). More generally, one strategy to relate modular topology and quadratic arithmetic is to choose f with appropriate symmetries and analyticity properties so that the sum over all double cosets can be understood: then one deduces a relationship between a topologically meaningful finite sum, and the infinite series whose special values may be of interest in arithmetic.

Part I

Conjugacy classes in PSL_2

Chapter 1

Geometric algebra of $\mathfrak{gl}_2$

Outline of the chapter

In this chapter $\mathbb{K}$ is a field of characteristic different from 2. A good example to keep in mind is the field of rational numbers, as one can both visualise the geometry and wonder about the arithmetic. Many statements will remain true over a subring containing $1/2$, which can be any commutative integral ring in which 2 is invertible. Let $\sqrt{\mathbb{K}}$ be a universal quadratic closure. For instance $\sqrt{\mathbb{Q}}$ is the venerable field of numbers constructible by ruler and compas.

We consider a $\mathbb{K}$-vector space $\mathbb{V}$ of dimension 2 with no additional structure, and denote $\mathfrak{gl}(\mathbb{V})$ its $\mathbb{K}$-algebra of linear endomorphisms, whose invertible elements form the group $\mathrm{GL}(\mathbb{V})$. Only after choosing a basis of $\mathbb{V}$ do we have the identifications $\mathbb{V} = \mathbb{K}^2$ as well as $\mathfrak{gl}(\mathbb{V}) = \mathfrak{gl}_2(\mathbb{K})$ and $\mathrm{GL}(\mathbb{V}) = \mathrm{GL}_2(\mathbb{K})$.

Involutive algebra : Quaternion algebra and Lie algebra

In the first two sections, we describe from a synthetic viewpoint the algebra and geometry underlying the space $\mathfrak{gl}(\mathbb{V})$, emphasizing the role played by the canonical involution $M \mapsto M^{\#}$ given by the transpose comatrix. One guiding thread is the interplay of two algebraic structures characterising $\mathfrak{gl}_2(\mathbb{V})$ over its underlying four-dimensional vector space: a quaternion algebra and a Lie algebra. Both yield the non degenerate quadratic form det with which they bare tight relations. In short, commutativity rhymes with colinearity whereas anti-commutativity rhymes with orthogonality. The orthogonal decomposition $\mathfrak{gl}(\mathbb{V}) = \mathbb{K}\mathbb{1} \oplus \mathfrak{sl}(\mathbb{V})$ with respect to det will also play an important role. The orthogonal projections are given by the half-trace $\mathrm{tr}\colon \mathfrak{gl}(\mathbb{V}) \to \mathbb{K}$, and the quotient $\mathrm{pr}\colon \mathfrak{gl}(\mathbb{V}) \to \mathfrak{sl}(\mathbb{V})$.

Along the way we derive various identities involving tr & pr which serve as stepping stones in the current chapter as well as the future ones. Most of them are probably well known, but let us highlight the presumably new identities in Lemma 1.30, for which lots of effort was spent in devising an ingenious proof. It relates the projection of the commutator $\operatorname{pr}[M, N]$ of elements $M, N \in \mathrm{GL}(\mathbb{V})$ to the commutator of their projection $[\operatorname{pr} M, \operatorname{pr} N] \in \mathfrak{sl}(\mathbb{V})$. The idea behind those identities is to use the orthogonal projections $\operatorname{tr}\colon \mathrm{GL}(\mathbb{V}) \to \mathbb{K}$ and $\operatorname{pr}\colon \mathrm{SL}(\mathbb{V}) \to \mathfrak{sl}(\mathbb{V})$ which satisfy $\operatorname{tr}^2 - \operatorname{pr}^2 = \det$ by the Pythagorean theorem, and invert them by solving such Pell-Fermat equations. This "quadratic algebra" replaces the "infinitesimal analysis" involved in the use of the transcendental function $\exp\colon \mathfrak{gl}(\mathbb{V}) \to \mathrm{GL}(\mathbb{V})$.

Let us mention that a basis of $\mathbb{V}$ yields a canonical way to choose a basis $(\mathbb{1}, S, J, K)$ for $\mathfrak{gl}(\mathbb{V})$. This relies on the representation of the dihedral group $\mathcal{D}_4 \subset \mathrm{GL}(\mathbb{V})$ acting by symmetries of the square. We shall often use this basis, once it will be clear what depends on this coordinate system.

This forms the content of the first two sections, which is mostly basic. The core of the material can be found in [Car92, Art57, Die71] for the geometry of Lie groups and Lie algebras, and [Vig80, Sha90, MR03] for the arithmetic of quaternion algebras. The trace identities also have a long history, for which we refer to [Mag81].

Geometry of the isotropic cone

The structures of quaternion algebra and Lie algebra on $\mathfrak{gl}_2(\mathbb{V})$ yield complementary insights on the geometry of its isotropic cone for the quadratic form det, which we investigate in a third section. We shall provide a parametrization for the projectivized isotropic cone of $(\mathfrak{gl}(\mathbb{V}), \det)$ in 1.11 and 1.37.

More importantly, we also define a parametrization $\psi\colon \mathbb{K}^2 \to \mathbb{X}$ for the isotropic cone of $(\mathfrak{sl}_2(\mathbb{K}), \det)$ in Lemma 1.33. This map ψ is responsible for some of the originality and coherence in our presentation: it will serve repeatedly, including in the future chapters. As suggested by its notation, it is expressed in our favourite coordinate system (S, J, K) obtained after choosing a basis of $\mathbb{V}$. Yet one only needs to fix a symplectic form on $\mathbb{V}$, so we also explain its (not so obvious) intrinsic counterpart in Lemma 1.36.

Then we recall the definition and properties of the cross-ratio between four elements in $\mathbb{KP}^1$. As an amusing curiosity, we apply the parametrization ψ to derive an analog of Ptolemy's theorem for quadruples of lines in the cone. When $\mathbb{K} = \mathbb{R}$, such a quadruple of lines corresponds to an ideal quadrilateral in the hyperbolic plane. This Proposition 1.40 relies on the identity (CRS) relating the cross-ratio between lines in $\mathbb{X}$ to the scalar product of vectors $\mathfrak{sl}_2(\mathbb{K})$, which may be of independent interest.

Finally, we define and relate the ubiquitous notions of cross-ratio and cosine between two semi-simple elements of $\mathrm{PGL}(\mathbb{V})$ (e.g. hyperbolic matrices in $\mathrm{SL}_2(\mathbb{R})$). The important statements are summed up in Lemma 0.2. Those relations are crucial ingredients in our Theorem 0.25 expressing linking numbers of modular knots as limits of functions on the character variety of the modular group. In fact, the cosine identity in Remark 1.51 was used in [Wol81, Wol82] to express the Weil-Petersson scalar product of certain functions on character varieties of closed hyperbolic surfaces. We believe that our presentation sheds light on the geometric nature of this identity.

Adjoint action and equivalence of binary quadratic forms

This brings us to the culminating point of the chapter: in the fourth section we study the adjoint action of $\mathrm{PGL}(\mathbb{V})$ on its symmetric space, defined as its subset of elements having order two, which is in correspondence with $\mathbb{P}(\mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X})$. The aim is to describe the conjugacy classes in $\mathrm{PGL}(\mathbb{V})$ and in $\mathrm{PSL}(\mathbb{V})$. More precisely, for $a, b \in \mathfrak{sl}(\mathbb{V})$ of the same determinant such that $\det\{a, b\}$, we parametrize the set of $\mathbb{C} \in \mathrm{PSL}(\mathbb{V})$ which conjugate a to b. This is the content of Theorem 0.5 = 1.64 which can be considered as the main result in this chapter.

In a fifth section, which is independent from the sequel, we apply this main theorem over the fields $\mathbb{Q}$ and its completions $\mathbb{Q}_p$. The goal is to describe certain "arithmetic $\mathbb{K}$-equivalence relations" on the set of integral binary quadratic forms, which arise naturally in this context, and seem to be new. For this we will need to solve Pell-Fermat equations using the notion of Hilbert symbol which we briefly recall from [Ser70]. After displaying some experimental data concerning this $\mathbb{Q}$-equivalence, we comment on its relation to genus equivalence. One may find background concerning binary quadratic forms and their genera in [Ser70, Coh78, Cox97, Hat22].

Hyperbolic geometry

In a last section we focus on the case $\mathbb{K} = \mathbb{R}$ and investigate the geometry of the Lorentz space $\mathfrak{sl}_2(\mathbb{R})$ with a special emphasis on orientation matters. We made our best to make this last section accessible for the geometer who has not assimilated the algebraic results accumulated until then, by recalling the basic facts in this context, and proposing alternative arguments or geometric intuition instead of systematically referring to the first two sections. One may benefit from reading this section first to gain some geometric intuition before starting afresh. This projective perspective on hyperbolic geometry is akin to [Thu97, Chapter 2] and [Mon87, Appendix B].

We end with the geometric interpretation 0.13 = 0.13 of our main theorem.

1.1 Quaternion algebra of $\mathfrak{gl}(\mathbb{V})$

Central algebra with involution

On a finite dimensional vector space, the set of alternate multilinear forms of top degree constitutes a one-dimensional vector space, which is generated by the determinant in some basis. Thus in dimension two we have, up to scalar multiplication, a unique anti-symmetric non-degenerate bilinear form, also called a *symplectic form*.

The adjoint of an endomorphism $M \in \mathfrak{gl}(\mathbb{V})$ for such a form ω on $\mathbb{V}$ is the unique endomorphism $M^\# \in \mathfrak{gl}(\mathbb{V})$ satisfying $\omega(u, Mv) = \omega(M^\# u, v)$ for all $u, v \in \mathbb{V}$. The adjoint map $M \mapsto M^\#$ is invariant under scaling of the symplectic form ω.

To insist, if we consider a two-dimensional $\mathbb{K}$-vector space $\mathbb{V}$ with no additional structure, then its algebra of endomorphisms $\mathfrak{gl}(\mathbb{V})$ admits a canonically defined adjoint map, which we use as a starting point to investigate its algebra and geometry.

Proposition 1.1. *The adjoint map $M \mapsto M^\#$ is an* involution *of the algebra $\mathfrak{gl}(\mathbb{V})$, meaning that for all $M, N \in \mathfrak{gl}(\mathbb{V})$ and $\lambda \in \mathbb{K}$ we have:*

Linear map: $(\lambda M + N)^\# = \lambda M^\# + N^\#$

Anti-multiplicative: $(MN)^\# = N^\# M^\#$

Order two: $(M^\#)^\# = M$

The fixed points of $\#$ consist in $\mathbb{K}\mathbb{1}$, that is the center of $\mathfrak{gl}(\mathbb{V})$.

Proof. Suppose $M = M^\#$. Then for all $v \in \mathbb{V}$ we have $\det(v, Mv) = \det(M^\# v, v)$ equal to $\det(Mv, v) = -\det(v, Mv)$ thus $2\det(v, Mv) = 0$. Since the characterstic is different from 2 we deduce that $Mv = \lambda_v \cdot v$ for some $\lambda_v \in \mathbb{K}$. If there exist $u, v \in \mathbb{V}$ for which $\lambda_u \neq \lambda_v$ then $\lambda_u u + \lambda_v v = M(u + v) = \lambda_{u+v}(u + v)$ yields a linear relation between them so $\lambda_u = \lambda_v$ which is a contradiction. Hence $M \in \mathbb{K}\mathbb{1}$.

The center of $\mathfrak{gl}(\mathbb{V})$ contains $\mathbb{K}\mathbb{1}$. Conversely, the previous argument shows that if $M \notin \mathbb{K}\mathbb{1}$ then it sends a line $L_0 \subset \mathbb{V}$ to a distinct line $L_1 \subset \mathbb{V}$. Then the projection P on L_0 parallel to L_1 does not commute with M. □

Remark 1.2. *Notice that the determinant $\det(M)$ of an endomorphism is invariant by rescaling the symplectic form ω on $\mathbb{V}$.*

Composing $(M, M^\#)$ with addition or multiplication yields the central elements:

$$\mathrm{Tr}(M)\mathbb{1} := M + M^\# \qquad \det(M)\mathbb{1} := MM^\#$$

which define the linear map $\mathrm{Tr}\colon \mathfrak{gl}(\mathbb{V}) \to \mathbb{K}$ called the *trace*, and the multiplicative map $\det\colon \mathfrak{gl}(\mathbb{V}) \to \mathbb{K}$ called the *determinant*.

Remark 1.3. *The maps* Tr *and* det *can be defined intrinsically in terms of tensor algebra as follows.*

Using the canonical isomorphism $\mathbb{V} \otimes \mathbb{V}^* \mapsto \mathfrak{gl}(\mathbb{V})$ *sending a pure tensor* $f \otimes v$ *to the projector* $p\colon u \mapsto f(u)v$ *we have* $\mathrm{Tr}(f \otimes v) = f(v)$.

The action of $m \in \mathfrak{gl}(\mathbb{V})$ *on the* 1*-dimensional vector space* $\Lambda^2\mathbb{V}$ *is the multiplication by a scalar* $\det(m)$.

We leave it as an exercise to define the adjoint action in terms of tensor powers.

Developing the relation $M^2 - (M + M^{\#})M + (MM^{\#}) = 0$ yields the Cayley-Hamilton identity $\chi_M(M) = 0$ for $\chi_M(X) = X^2 - \mathrm{Tr}(M)X + \det(M) \in \mathbb{K}[X]$ the *characteristic polynomial* of M. Hence an element $M \in \mathfrak{gl}_2(\mathbb{K})$ generates a commutative subalgebra $\mathbb{K}[M]$ which has dimension at most 2, and equals $\mathrm{Span}(\mathbb{1}, M)$.

Moreover, if $M \notin \mathbb{K}\mathbb{1}$ then this subalgebra $\mathbb{K}[M]$ has dimension at least 2 and is thus isomorphic to the quadratic extension $\mathbb{K}[X]/(\chi_M)$ of $\mathbb{K}$. Hence when M is not central, its minimal polynomial equals its characteristic polynomial. In any case, the involution # restricts on $\mathbb{K}[M]$ to the Galois involution of this $\mathbb{K}$-extension.

Scholium 1.4. *We shall see, by studying the adjoint action that* $M, N \in \mathfrak{gl}(\mathbb{V})$ *are conjugate by* $\mathrm{GL}(\mathbb{V})$ *if and only if they have the same characteristic polynomial.*

The discriminant of $M \in \mathfrak{gl}(\mathbb{V})$ is defined as that of its characteristic polynomial, equal to $\mathrm{disc}(M) = \mathrm{Tr}(M)^2 - 4\det(M)$.

We call M *semi-simple* when $\mathrm{disc}(M) \neq 0$, that is when χ_M has simple roots in $\sqrt{\mathbb{K}}$. If these roots belong to $\mathbb{K}$ the algebra $\mathbb{K}[M]$ is isomorphic to the direct product $\mathbb{K} \times \mathbb{K}$, otherwise $\mathbb{K}[M]$ is a simple $\mathbb{K}$-algebra (no proper ideals). In both cases $\mathbb{K}[M]$ is a semi-simple $\mathbb{K}$-algebra (a product of simple algebras).

When $\mathrm{disc}(M) = 0$ we have $\chi_M(X) = (X - \lambda)^2$ for $\lambda \in \mathbb{K}$ so the algebra $\mathbb{K}[M]$ is not integral (it has zero divisors). Either $\lambda = 0$ in which case M is nipotent, otherwize M/λ is idempotent.

Remark 1.5. *In general for* $F \in \mathbb{K}[X]$ *the condition* $\mathrm{disc}(P) \neq 0$ *is equivalent to saying that* F *has simple roots in an algebraic closure of* $\mathbb{K}$. *Hence the* $\mathbb{K}$*-algebra* $\mathbb{K}[X]/(F)$ *is semi-simple if and only if* $\mathrm{disc}(F) \neq 0$.

Indeed, it splits as the direct product of the $\mathbb{K}[X]/(P^k)$ *where* $P \in \mathbb{K}[X]$ *range over the irreducible factors of* F. *If* $\mathrm{disc}(P) \neq 0$ *then* $\mathbb{K}[X]/(P^k)$ *defines a simple algebra (with no proper ideals) if and only if* $k = 1$.

Note that the irreducibly of P *does not imply that* $\mathrm{disc}(P) \neq 0$. *However it does provided that* $\deg(P)$ *is prime to the characteristic of the field (which is always the case when* $\deg P = 2$ *and* $\mathbb{K}$ *has characteristic different from* 2*) or when the field* $\mathbb{K}$ *is assumed to be perfect (which we do not need to assume).*

The involution # preserves the group $\mathrm{GL}(\mathbb{V})$ of invertible elements. It consists in those $A \in \mathfrak{gl}(\mathbb{V})$ such that $\det(A) \in \mathbb{K}^\times$, in which case $A^{-1} = \det(A)^{-1} A^\#$.

For $A \in \mathrm{GL}(\mathbb{V})$ and $M \in \mathfrak{gl}(\mathbb{V})$ we have $(AMA^{-1})^\# = AM^\# A^{-1}$, so the left adjoint linear action of $\mathrm{GL}(\mathbb{V})$ on $\mathfrak{gl}(\mathbb{V})$ preserves the involution, whence the trace and determinant.

Dihedral basis for the quaternion algebra

A choice of basis for $\mathbb{V}$ amounts to an isomorphism with the numerical space $\mathbb{K}^2$, this identifies its endomorphism algebra $\mathfrak{gl}(\mathbb{V})$ with the algebra $\mathfrak{gl}_2(\mathbb{K})$ of 2×2 matrices.

In such coordinates, the adjoint corresponds to the transpose comatrix, while the determinant and trace have their usual expressions:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad M^\# = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \mathrm{Tr}(M) = a + d \qquad \det(M) = ad - bd$$

Remark 1.6. *Note that choosing a basis (u_0, v_0) of $\mathbb{V}$ yields a preferred normalisation of the area form so that $\omega(u_0, v_0) = 1$. Then the area $\omega(u, v)$ of the parallelogram spanned by two vectors is the determinant of the endomorphism sending the basis (u_0, v_0) to (u, v) and one may use $\det(u, v)$ to denote both of these.*

More importantly a basis defines a unique euclidean structure, that is a symmetric non-degenerate bilinear form β, for which this basis is orthonormal. In $\mathfrak{gl}_2(\mathbb{K})$, the adjoint with respect to that symmetric bilinear form is the transposition $M \mapsto {}^tM$.

If we have a Euclidean scalar product β, hence a symplectic form ω one may polarise the latter with respect to the former and find the unique element $S \in \mathfrak{gl}(\mathbb{V})$ such that $\beta(u, v) = \omega(u, Sv)$ for all $u, v \in \mathbb{V}$. This is nothing else than the rotation $S \in \mathrm{SO}(\mathbb{V}, \beta)$ of order 4 in the positive direction.

In fact, given a symplectic form, it is equivalent to fix an element of order 4 up to inversion and a euclidean metric inducing the same area. (But one must be careful about the notion of orientation over general fields.)

Let us explain why choosing a basis of $\mathbb{V}$ leads to a preferred basis for $\mathfrak{gl}_2(\mathbb{V})$, in other terms we exhibit a favourite basis for $\mathfrak{gl}_2(\mathbb{K})$. The following elements form, together with their opposites, the dihedral group $\mathcal{D}_4$ of order 8 which acts faithfully on the square in $\mathbb{K}^2$ whose vertices have coordinates ± 1, as in Figure 1.1.

$$\imath = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Having coordinates in $\{-1, 0, 1\}$, these elements are defined over any field of characteristic different from 2, and they form a basis for $\mathfrak{gl}_2(\mathbb{K})$. In particular, we can present $\mathfrak{gl}_2(\mathbb{K})$ as the group algebra $\mathbb{K}[\mathcal{D}_4]$ quotiented by the identification between the units $\{\pm 1\} \in \mathbb{K}^\times$ and the central elements $\{\pm \mathbf{1}\} \subset \mathcal{D}_4$.

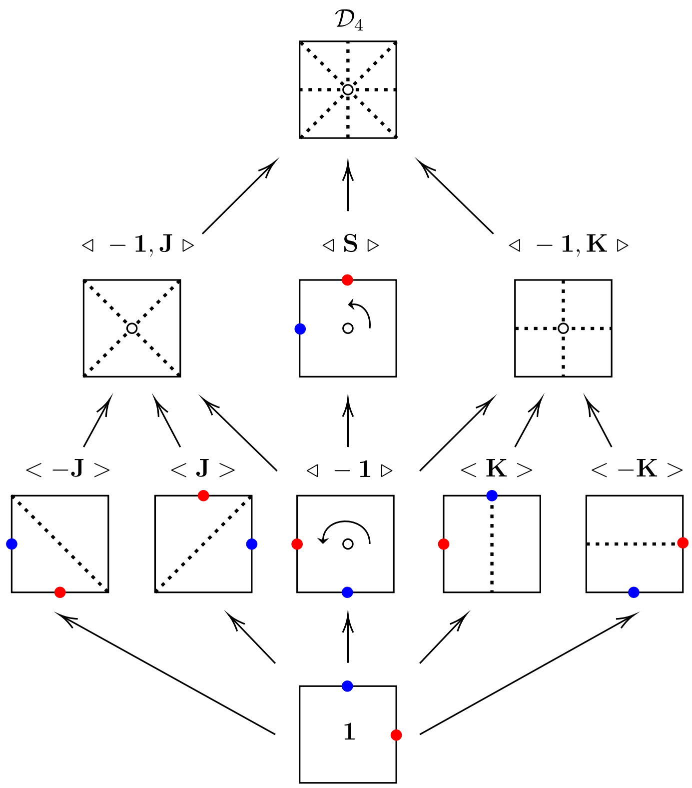


Figure 1.1: Poset of subgroups in $\mathcal{D}_4$ and their action on the square.

For future reference, we display some of the relations satisfied between S, J, K. Geometrically, the dihedral group $\mathcal{D}_4$ is generated by two reflections J, K whose product $KJ = S$ is a rotation of order 4. In particular we have the squaring relations:

$$J^2 = \mathbf{1} \qquad K^2 = \mathbf{1} \qquad S^2 = -\mathbf{1}$$

Algebraically, the trace relations $\mathrm{Tr}(S) = \mathrm{Tr}(J) = \mathrm{Tr}(K) = 0$ imply that $\#$ changes S, J, K in their opposite, and then $\det(S) = -\det(J) = -\det(K) = 1$ recover the squaring relations.

Combining them with $KJ = S$ yields the cyclic permutation relations:

$$SJ = K \qquad JK = S^{-1} \qquad KS = J$$

Combining those with $\mathrm{Tr}(K) = \mathrm{Tr}(S) = \mathrm{Tr}(J) = 0$ yields the anti-commutation relations:

$$SJ = -JS \qquad JK = -KJ \qquad KS = -SK$$

which imply that every element S, J, K conjugates any other in its opposite.

We may recombine those anti-commutation relations with the cyclic permutation relations to yield the following commutator relations, which only make sense in the group algebra:

$$SJ - JK = 2K \qquad JK - KJ = -2S \qquad KS - SK = 2J.$$

Scholium 1.7. *A quaternion algebra over* $\mathbb{K}$ *is a* 4*-dimensional* $\mathbb{K}$*-algebra spanned by elements* $(\mathtt{1}, jk, j, k)$ *satisfying* $j^2 = x\mathtt{1}$*,* $k^2 = y\mathtt{1}$ *and* $kj = -jk$ *for* $x, y \in \mathbb{K}^\times$*. For* $x = y = 1$ *we recover* $\mathfrak{gl}_2(\mathbb{K})$ *whereas for* $x = y = -1$ *and* $\mathbb{K} = \mathbb{R}$ *we find Hamilton's algebra of quaternions.*

Over $\mathbb{K}$*, the only quaternion algebra which is not a division algebra is* $\mathfrak{gl}_2(\mathbb{K})$*. On may consult [Sha90, Vig80, MR03] for much more about quaternion algebras.*

We can also present the four-dimensional $\mathbb{K}$*-algebra* $\mathfrak{gl}_2(\mathbb{K})$ *as a non-commutative multiquadratic extension of* $\mathbb{K}$*, or as a hermitian extension of a quadratic extension:*

$$\mathfrak{gl}_2(\mathbb{K}) \simeq \frac{\mathbb{K}[J,K]}{(J^2 = K^2 = 1,\ JK = -KJ)} \simeq \frac{\mathbb{K}[J][K]}{(K^2 = 1, KJ = J^\# K)}$$

Such a definition extends to quaternion algebras over fields of characteristic 2*.*

Structure of the dihedral group

Let us investigate the structure of $\mathcal{D}_4$, starting with its poset of subgroups, depicted in 1.1. Recall that a subgroup is normal if it is invariant by inner-automorphism, and characteristic if it is invariant by every automorphism.

Since every element S, J, K conjugates any other in its opposite, we deduce on the one hand that its center equals $\{\pm 1\}$, and on the other hand that its normal subgroups must contain $\{\pm 1\}$. Hence the normal subgroups are given by:

$$\{\pm 1\} \qquad \{\pm 1, \pm S\} \approx \mathbb{Z}/4 \qquad \{\pm 1, \pm J\} \approx \mathbb{Z}/2 \times \mathbb{Z}/2 \qquad \{\pm 1, \pm K\} \approx \mathbb{Z}/2 \times \mathbb{Z}/2$$

The center $\{\pm 1\}$ is characteristic (as always), and so is $\{\pm 1, \pm S\}$ since $\pm S$ are the only elements of order 4. The automorphism Φ swapping the generators $J \leftrightarrow K$ (and thus changing S into S^{-1}) exchanges the two other normal subgroups, which are thus not characteristic.

Since an automorphism must preserve the non-central elements of order two $\{\pm J, \pm K\}$, it must equal Φ up to composition with an inner-automorphism (corresponding to the quotient $\mathcal{D}_4/\{\pm 1\}$).

Remark 1.8. *We have just seen that J and K are not conjugate in $\mathcal{D}_4$. Still, as elements of $\mathrm{GL}_2(\mathbb{K})$ they have the same trace and determinant, so Corollary 1.52 should imply that they are conjugate.*

A straightforward computation shows that $C \in \mathrm{GL}_2(\mathbb{K})$ satisfies $CJ = KC$ if and only if it has the form:

$$C = \begin{pmatrix} x & -x \\ y & y \end{pmatrix} \qquad \text{for} \quad x, y \in \mathbb{K} \quad \text{with} \quad \det(C) = 2xy \neq 0.$$

Thus by taking $x = 1$ and $y = 1$ we have an element in $\mathrm{GL}_2(\mathbb{K})$ defined over any field $\mathbb{K}$ of characteristic different from 2 which conjugates J to K as desired.

They are also conjugate by the rotation of $\pi/4$ which belongs to $\mathrm{SO}_2(\mathbb{K}) \subset \mathrm{SL}_2(\mathbb{K})$ provided that $\mathbb{K}$ contains $\cos(\pi/4) = \sqrt{2}/2$, that is a root to the equation $z^2 = 1/2$.

In fact, the matrix C with $x = y = 1$ is nothing else than the rotation of $\pi/4$ multiplied by $\sqrt{2}$. We are lucky enough that its multiples in $\mathfrak{gl}_2(\mathbb{R})$ contains such a primitive vector C of the lattice $\mathfrak{gl}_2(\mathbb{Z})$ whose entries are defined over any field $\mathbb{K}$ of characteristic different from 2.

Notice that J and K are also conjugate by an element $\mathrm{SL}_2(\mathbb{K})$ by choosing $2x = y = 1$. We shall come back to this example in Remark 1.63 and Corollary 1.62.

Now let us describe the quotients of $\mathcal{D}_4$ by its normal subgroups and analyse the corresponding short exact sequences

The quotient of $\mathcal{D}_4$ by its center $\{\pm 1\}$ is the Klein four group $\mathbb{Z}/2 \times \mathbb{Z}/2$, generated by the classes of J, K, S. Thus we have a central extension:

$$1 \to \{\pm 1\} \to \mathcal{D}_2 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1$$

We may compute the classifying cohomology class $H^2(\mathbb{Z}/2 \times \mathbb{Z}/2; \{\pm 1\})$ from the squaring relations and the anti-commutation relations. A representative is given by:

$$c(J, J) = c(K, K) = 1 \qquad c(S, S) = -1$$

$$c(S, J) = -c(J, S) = 1 \quad - c(J, K) = c(K, J) = 1 \quad c(K, S) = -c(S, K) = 1$$

We may further quotient by S to find an extension whose quotient $\mathbb{Z}/2$ is generated by the class of J or K.

$$1 \to \mathbb{Z}/4 \to \mathcal{D}_4 \to \mathbb{Z}/2 \to 1$$

The quotient acts on the kernel by its only non trivial automorphism (equal to inversion, which is trivial only in restriction to the $\{\pm 1\}$ central subgroup). Since the sequence is split, it gives rise to a semi-direct product $\mathcal{D}_4 = \mathbb{Z}/4 \rtimes \mathbb{Z}/2$.

If we quotient by the center and either J or K we find two extensions, which are isomorphic by Φ:

$$1 \to \{\pm 1\} \times \mathbb{Z}/2 \to \mathcal{D}_4 \to \mathbb{Z}/2 \to 1 \tag{1.1}$$

The quotient $\mathbb{Z}/2$ is generated by the class of S, which has order 4 in $\mathcal{D}_4$, so the sequence cannot be split. The quotient acts on the kernel by its only non trivial automorphism (equal to inversion, which is trivial only in restriction to the $\{\pm 1\}$ central subgroup). Denoting $\phi\colon \mathbb{Z}/2 \to \mathrm{Out}(\mathbb{Z}/2 \times \mathbb{Z}/2) = \mathrm{Aut}(\mathbb{Z}/2 \times \mathbb{Z}/2)$ the corresponding representation, we may compute classifying twisted cohomology class in $H^2(\mathbb{Z}/2; (\mathbb{Z}/2 \times \mathbb{Z}/2)_\phi)$, but we already know it is a non trivial element (because the sequence is not split), and its explicit computation follows from that for the central extension: $c(S, S) = -1$.

Finally, we described the inner-automorphisms $\mathrm{Int}(\mathcal{D}_4) = \mathcal{D}_4/\{\pm 1\} \approx \mathbb{Z}/2 \times \mathbb{Z}/2$ and the outer-automorphisms $\mathrm{Out}(\mathcal{D}_4) = \mathrm{Aut}(\mathcal{D}_D)/\mathrm{Int}(\mathcal{D}_4) \approx \mathbb{Z}/2$ generated by Φ. They fit into the following extension:

$$1 \to \mathrm{Int}(\mathcal{D}_4) \to \mathrm{Aut}(\mathcal{D}_4) \to \mathrm{Out}(\mathcal{D}_4) \to 1$$

Since $\Phi \circ \mathrm{Int}_J = \mathrm{Int}_K \circ \Phi$ and Φ commutes with Int_S we find that the corresponding representation is the same $\phi\colon \mathbb{Z}/2 \to \mathrm{Out}(\mathbb{Z}/2 \times \mathbb{Z}/2)$ as the previous extension 1.1. But this time the sequence is split, in other terms $\mathrm{Aut}(\mathcal{D}_4) \approx (\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes_\phi \mathbb{Z}/2$.

Remark 1.9. *The transcomatrix $M \mapsto M^\#$ and the tranpose $M \mapsto {}^tM$ restrict to anti-automorphisms of $\mathcal{D}_4$. The former changes S, J, K in their opposite, while the latter changes S in its opposite. (In particular the restriction of the transpose is equal to the inversion map $M \mapsto M^{-1}$.)*

Their composition yields the comatrix, which is thus an automorphism of $\mathcal{D}_4$. It changes K and J in their opposites while leaving S invariant, that is Int_S.

In Chapter 2, we will often use the fact that $S^tMS^{-1} = M^\#$ for all $M \in \mathfrak{gl}(\mathbb{V})$.

Scholium 1.10. *For a group G denote $\mathrm{Aut}^\pm(G)$ be its group of automorphisms and anti-automorphisms. If G is abelian then $\mathrm{Aut}^\pm(G) = \mathrm{Aut}(G)$. If G is not abelian then $\mathrm{Aut}^\pm(G) = \mathrm{Aut}(G) \times \mathbb{Z}/2$ where the second factor is generated by the inversion.*

Indeed, in the non-abelian case, the inversion is an anti-automorphism, and by definition every automorphism commutes with it. Moreover, since the product of two anti-automorphisms is an automorphism, there is a unique anti-automorphism up to multiplication by an automorphism.

Quadratic space

On the vector space $\mathfrak{gl}(\mathbb{V})$ the determinant is a non degenerate quadratic form, and as $\det(M+N)\mathbf{1} = (M+N)(M+N)^{\#} = \big(\det(M) + \mathrm{Tr}(MN^{\#}) + \det(N)\big)\,\mathbf{1}$, its polar symmetric bilinear form is:

$$\langle M, N\rangle = \mathrm{tr}(MN^{\#}) \quad \text{where} \quad \mathrm{tr}(P) := \tfrac{1}{2}\,\mathrm{Tr}(P) \tag{1.2}$$

The isotropic cone of $\mathfrak{gl}(\mathbb{V})$ is the set $\mathfrak{gl}(\mathbb{V})\setminus\mathrm{GL}(\mathbb{V})$ of non invertible elements, and it is a cone over non-degenerate projective quadric in the projective space $\mathbb{P}(\mathfrak{gl}(\mathbb{V}))$.

The units $\mathrm{SL}(\mathbb{V}) = \{A \in \mathfrak{gl}(\mathbb{V}) \mid \det(A) = 1\}$ form a subgroup of $\mathrm{GL}(\mathbb{V})$, kernel of the determinant morphism $\det\colon \mathrm{GL}(\mathbb{V}) \to \mathbb{K}^{\times}$, thus $A \in \mathrm{SL}(\mathbb{V}) \iff A^{\#} = A^{-1}$.

Denote $\mathfrak{sl}(\mathbb{V})$ the kernel of the trace form, or the anti-symmetric part for the involution, thus $a \in \mathfrak{sl}(\mathbb{V}) \iff \mathrm{tr}(a) = 0 \iff a^{\#} = -a$.

The involution # has eigenvalues ± 1 and its eigenspaces provide a decomposition $\mathfrak{gl}(\mathbb{V}) = \mathbb{K}\mathbf{1} \oplus \mathfrak{sl}(\mathbb{V})$ which is orthogonal with respect to the determinant form, thus every element $M \in \mathfrak{gl}(\mathbb{V})$ splits as the sum of its symmetric and anti-symmetric parts with respect to the involution:

$$M = \mathrm{tr}(M)\mathbf{1} + \mathrm{pr}(M) \qquad \text{where} \quad \mathrm{tr}(M)\mathbf{1} = \frac{M + M^{\#}}{2} \quad \text{and} \quad \mathrm{pr}(M) := \frac{M - M^{\#}}{2}.$$

In particular $\det(M)\mathbf{1} = \mathrm{tr}(M)^2\mathbf{1} - \mathrm{pr}(M)^2$ which we may write $\det = \mathrm{tr}^2 - \mathrm{pr}^2$.

The 4-dimensional space $\mathfrak{gl}(\mathbb{V})$, which contains the isotropic cone $\mathfrak{gl}(\mathbb{V}) \setminus \mathrm{GL}(\mathbb{V})$ defined by $\det(M) = \langle M, M\rangle = 0$, decomposes as the direct sum of the anistropic line $\mathbb{K}\mathbf{1}$ and its orthogonal hyperplane $\mathfrak{sl}(\mathbb{V})$ defined by $\mathrm{tr}(M) = \langle \mathbf{1}, M\rangle = 0$. Denote $\mathbb{X} = \{M \in \mathfrak{gl}(\mathbb{V}) \mid \langle M, M\rangle = 0 = \langle \mathbf{1}, M\rangle\}$ the isotropic cone for the determinant restricted to the kernel of the trace.

In the projective 3-space $\mathbb{P}(\mathfrak{gl}(\mathbb{V}))$, the point $\mathbb{P}(\mathbb{K}\mathbf{1})$ and the plane $\mathbb{P}(\mathfrak{sl}(\mathbb{V}))$ are mutually polar with respect the non-degenerate quadric $\mathbb{P}(\mathfrak{gl}(\mathbb{V})\setminus\mathrm{GL}(\mathbb{V}))$. The point does not lie on the quadric and its polar plane intersects the quadric transversely, in the non-degenerate conic $\mathbb{P}(\mathbb{X})$.

In projective coordinates, this conic consists in the set of points $p \in \mathfrak{gl}(\mathbb{V})$ such that $\langle p, p\rangle = 0 = \langle \mathbf{1}, p\rangle$, or equivalently $\det(x\mathbf{1} + yp) = x^2$ for all $[x : y] \in \mathbb{KP}^1$, which means that the restriction of the quadratic form on the line $(\mathbf{1}, p)$ vanishes to the second order at p.

Hence the conic $\mathbb{P}(\mathbb{X})$ consists in the set of tangency points between the quadric $\mathbb{P}(\{\det = 0\})$ and the pencil of lines through $\mathbb{P}(\mathbf{1})$. Figure 1.2 provides a schematic picture (recall that this is valid over any field of characteristic different from 2).

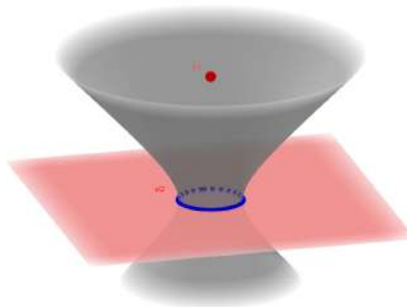


Figure 1.2: A non-degenerate projective quadric in $\mathbb{P}^3$. A point $\imath$ off the quadric and its polar plane $\mathbb{P}^2$, which intersects the quadric in a non degenerate conic.

The isomorphism types of quadrics over $\mathbb{K}$ are given, in terms of the diagonalisation of the quadratic form, by the classes of the diagonal elements in $\mathbb{K}^\times/(\mathbb{K}^\times)^2$.

Proposition 1.11. *The map* $\Psi\colon M \mapsto (\ker M, \operatorname{im} M)$ *defines a bijective algebraic correspondence between the projective quadric* $\mathbb{P}(\{\det = 0\})$ *and* $\mathbb{P}(\mathbb{V}) \times \mathbb{P}(\mathbb{V})$ *sending the conic projective conic* $\mathbb{P}(\mathbb{X})$ *to the diagonal* $\mathbb{P}(\mathbb{V})$*.*

It conjugates the adjoint action of $\mathrm{PGL}(\mathbb{V})$ *restricted to* $\mathbb{P}(\{\det = 0\})$ *with its tautological diagonal action on* $\mathbb{P}(\mathbb{V}) \times \mathbb{P}(\mathbb{V})$*.*

Proof. Notice that in terms of endomorphisms, an element $M \in \mathfrak{gl}(\mathbb{V})$ in the isotropic cone if and only if its rank is either 0 or rank 1.

If M has rank 0 then it is nilpotent, and the line $\operatorname{im}(M) = \ker(M) \subset \mathbb{V}$ uniquely determines $\mathbb{P}(M)$. In this case we have $\operatorname{tr}(M) = 0$. We recover the fact that the cone $\mathbb{P}(\mathbb{X})$ is parametrized by $\mathbb{P}(\mathbb{V})$.

Otherwize M has rank 1, so it is the multiple of an idempotent element, that is the projector on one line $\operatorname{im}(M)$ parallel to another $\ker(M)$. In this case it has $\operatorname{tr}(M) \neq 0$. Thus $\mathbb{P}(\{\det = 0\} \setminus \mathbb{X})$ corresponds to pairs of distinct lines in $\mathbb{V}$.

If $A \in \mathrm{GL}(\mathbb{V})$ then $\operatorname{im}(AMA^{-1}) = A(\operatorname{im} M)$ and $\ker(AMA^{-1}) = A(\ker M)$. □

Scholium 1.12. *In Section 1.3 we shall parametrize the istropic quadric and cone providing an inverse to the map* Ψ*.*

Quadratic identities: trace, projection and discriminant

Recall that for all $M \in \mathfrak{gl}(\mathbb{V}) = \mathbb{K}\imath \oplus \mathfrak{sl}(\mathbb{V})$ we have $\det(M)\imath = \operatorname{tr}(M)^2\imath - \operatorname{pr}(M)^2$ which we may write $\det = \operatorname{tr}^2 - \operatorname{pr}^2$.

Proposition 1.13. *For all $M, N \in \mathfrak{gl}(\mathbb{V})$ we have the following trace identity:*

$$\operatorname{tr}(MN) = \operatorname{tr}(M)\operatorname{tr}(N) - \langle \operatorname{pr} M, \operatorname{pr} N\rangle \tag{1.3}$$

This implies in particular:

$$\operatorname{tr}(MN) = \operatorname{tr}(NM) \tag{1.4}$$
$$\operatorname{tr}(MN) + \operatorname{tr}(MN^{\#}) = 2\operatorname{tr}(M)\operatorname{tr}(N) \tag{1.5}$$
$$\operatorname{tr}(MN) - \operatorname{tr}(MN^{\#}) = -2\langle \operatorname{pr}(M), \operatorname{pr}(N)\rangle \tag{1.6}$$

Finally $\operatorname{Tr}(MNM^{\#}N^{\#})$ *equals:*

$$\det(M)\operatorname{Tr}(N)^2 + \det(N)\operatorname{Tr}(M)^2 + \operatorname{Tr}(MN)^2 - \operatorname{Tr}(M)\operatorname{Tr}(N)\operatorname{Tr}(MN) - 2\det(MN).$$

Proof. We obtain identity (1.3) by multiplying the orthogonal decompositions of M, N into $\mathbb{K}\mathbb{1} \oplus \mathfrak{sl}(\mathbb{V})$ and projecting on $\mathbb{K}\mathbb{1}$. Then (1.4), (1.5), (1.6) follow easily. To show the last identity we apply the identities (1.4) and (1.5) several times:

$$\begin{aligned}
\operatorname{Tr}(MNM^{\#}N^{\#}) &= \operatorname{Tr}(MNM^{\#})\operatorname{Tr}(N^{\#}) - \operatorname{Tr}(MNM^{\#}N)\\
&= \det(M)\operatorname{Tr}(N)^2 - (\operatorname{Tr}(MN)\operatorname{Tr}(M^{\#}N) - \operatorname{Tr}(MNN^{\#}M))\\
&= \det(M)\operatorname{Tr}(N)^2 + \det(N)\operatorname{Tr}(M^2) - \operatorname{Tr}(MN)\operatorname{Tr}(M^{\#}N)
\end{aligned}$$

Substituting $\operatorname{Tr}(M^2) = \operatorname{Tr}(M)^2 - 2\det(M)$ and $\operatorname{Tr}(M^{\#}N) = \operatorname{Tr}(M)\operatorname{Tr}(N) - \operatorname{Tr}(MN)$ which are consequences of (1.4), yields the desired identity. □

Remark 1.14. *We recover the usual trace relation formulated for $A, B \in \mathrm{SL}(\mathbb{V})$ as* $\operatorname{Tr}(AB) + \operatorname{Tr}(AB^{-1}) = \operatorname{Tr}(A)\operatorname{Tr}(B)$, *which implies* $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ *and:*

$$\operatorname{Tr}[A, B] = \operatorname{Tr}(A)^2 + \operatorname{Tr}(B)^2 + \operatorname{Tr}(AB)^2 - \operatorname{Tr}(A)\operatorname{Tr}(B)\operatorname{Tr}(AB) - 2.$$

The discriminant of $M \in \mathfrak{gl}(\mathbb{V})$ was defined as that of its characteristic polynomial $X^2 - \operatorname{Tr}(M)X + \det(M)$, equal to $\operatorname{disc}(M) = 4\left(\operatorname{tr}(M)^2 - \det(M)\right)$. Thus $A \in \mathrm{SL}(\mathbb{V})$ has $\operatorname{disc}(A) = 4\operatorname{tr}(M)^2 - 4$ and $a \in \mathfrak{sl}(\mathbb{V})$ has $\operatorname{disc}(a) = -4\det(a)$.

The orthogonal projection $\operatorname{pr}\colon \mathfrak{gl}(\mathbb{V}) \to \mathfrak{sl}(\mathbb{V})$ preserves the discriminant since the characteristic polynomial of $t\mathbb{1} + u$ is that of u shifted by t.

Corollary 1.15. *For all $M, N \in \mathfrak{gl}(\mathbb{V})$ we have:*

$$-\langle \operatorname{pr} M, \operatorname{pr} N\rangle = \frac{\operatorname{Tr}(MN)^2 - \operatorname{Tr}(MN^{\#})^2}{\operatorname{Tr}(M)\operatorname{Tr}(N)} = \frac{\operatorname{disc}(MN) - \operatorname{disc}(MN^{\#})}{\operatorname{Tr}(M)\operatorname{Tr}(N)}$$

Proof. Multiply relations (1.5) & (1.6), and notice that $\det(MN) = \det(MN^{\#})$. □

Scholium 1.16. *We will apply Lemma 1.15 in Propositions 1.89 and 1.94 to hyperbolic elements $M, N \in \mathrm{SL}_2(\mathbb{R})$ acting on the hyperbolic plane.*

The left hand side will have a geometric interpretation in terms of the translation axes of M and N whereas the right hand side will have a dynamical interpretation in terms of the translation lengths of MN and MN^{-1}.

1.2 The Lie algebra $\mathfrak{sl}(\mathbb{V})$

Lie algebra and Killing form

The kernel $\mathfrak{sl}(\mathbb{V})$ of the trace form is a 3-dimensional Lie algebra for the *bracket* $\{a,b\} = \frac{1}{2}(ab - ba)$, that of the Lie group $\mathrm{SL}(\mathbb{V})$, kernel of the determinant morphism. Notice that if $M, N \in \mathfrak{gl}(\mathbb{V})$ have orthogonal projections $\operatorname{pr} M = a, \operatorname{pr} N = b \in \mathfrak{sl}(\mathbb{V})$ then $\frac{1}{2}(MN - NM) = \{a,b\}$, which explains why we restrict our attention to $\mathfrak{sl}(\mathbb{V})$. For $a, b \in \mathfrak{sl}(\mathbb{V})$, the decomposition $ab = \operatorname{tr}(ab)\mathbb{1} + \operatorname{pr}(ab)$ rewrites as

$$ab = -\langle a, b\rangle \mathbb{1} + \{a,b\} \tag{1.7}$$

hence a, b are orthogonal if and only if they anticommute, in which case $\{a,b\} = ab$. The Jacobi relation implies that $\{a,b\} \perp \operatorname{Span}(a,b)$ for all $a, b \in \mathfrak{sl}(\mathbb{V})$.

Remark 1.17. *The squaring and anti-commutation relations imply that* $(\mathbb{1}, S, J, K)$ *is an orthonormal basis of* $\mathfrak{gl}_2(\mathbb{K})$ *respecting the decomposition* $\mathbb{K}\mathbb{1} \oplus \mathfrak{sl}_2(\mathbb{K})$. *We have:*

$$\det(t\mathbb{1} + sS + jJ + kK) = t^2 + s^2 - j^2 - k^2$$

Moreover, we combined the anti-commutation and permutation relations to find:

$$\{S,J\} = K \qquad \{J,K\} = -S \qquad \{K,S\} = J$$

The Killing form associated to the bracket is proportional to the scalar product:

$$-\tfrac{1}{8}\operatorname{Tr}(c \mapsto 2\{a, 2\{b,c\}\}) = -\operatorname{tr}(c \mapsto \{a,\{b,c\}\}) = -\operatorname{tr}(ab) = \operatorname{tr}(ab^{\#}) = \langle a, b\rangle$$

The non degeneracy of the polar form $\langle \cdot, \cdot \rangle$, implies that of the Killing form, hence of the bracket $\{\cdot,\cdot\}$ (we thus recover the fact the center of $\mathfrak{gl}(\mathbb{V})$ is precisely $\mathbb{K}\mathbb{1}$).

Lemma 1.18. *Two elements* $a, b \in \mathfrak{sl}(\mathbb{V})$ *are colinear if and only if* $\{a,b\} = 0$.

Proof. Of course the bracket vanishes on colinear pairs. Let us prove the converse.

Suppose $\{a,b\} = 0$, so that $ab = -\langle a, b\rangle$.

If $\det(a) \neq 0$ then multiplying by $a^{\#} = -a$ we have $\det(a)b = \langle a,b\rangle a$ and dividing by $\det(a)$ yields the conclusion. If $\det(b) \neq 0$ the argument is similar.

Otherwize $\det(a) = 0 = \det(b)$, whence $0 = \det(ab) = \langle a,b\rangle^2$. Thus for all $x, y \in \mathbb{K}$ we have $\det(xa + yb) = \det(a)x^2 + 2\langle a,b\rangle xy + \det(b)y^2 = 0$. But the isotropic spaces of a non degenerate quadratic form cannot exceed half the dimension hence $\dim \operatorname{Span}(a,b) \leq 3/2$ and we are done. □

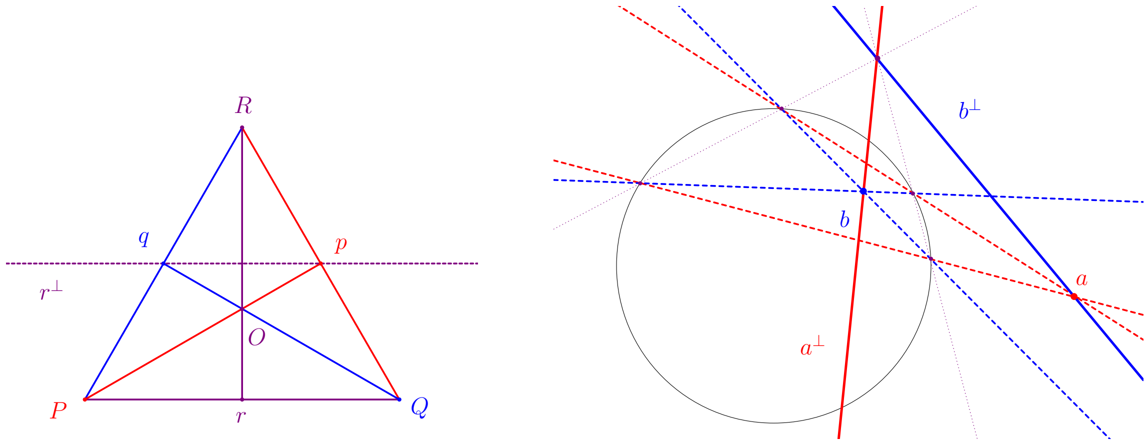


Figure 1.3: Constructing polarities with respect to a conic.It relies on choosing projective basis O, P, Q, R on the conic, to exploit the following property. The pencil of conics passing through such base points contains three singular members: the pairs of lines $(OP)\cup(QR), (OQ)\cup(PR), (OR)\cup(PQ)$, with intersection points p, q, r. The polarity relations $p^\perp = (qr)$, $q^\perp = (pr)$, $r^\perp = (pq)$ hold for any conic of the pencil.

When $a, b \in \mathfrak{sl}(\mathbb{V})$ are not colinear, they span a plane $\mathrm{Span}(a, b)$ which is orthogonal to $\{a, b\}$ and thus equal to $\{a, b\}^\perp$. In the projective plane $\mathbb{P}(\mathfrak{sl}(\mathbb{V}))$ the line through the distinct points $\mathbb{P}(a), \mathbb{P}(b)$ is polar to the point $\mathbb{P}(\{a, b\})$ with respect to the conic $\mathbb{P}(\mathbb{X})$.

When $\det\{a, b\} \neq 0$ the restriction of det to the plane $\mathrm{Span}(a, b)$ is a non degenerate quadratic form. In this case the point $\mathbb{P}(\{a, b\})$ lies off the conic and its polar line $\mathbb{P}\{a, b\}^\perp$ is transverse to the conic (but their intersection may be empty over $\mathbb{K}$). Notice that the points $\mathbb{P}(a)$ and $\mathbb{P}(b)$ could either be on or off the conic, and this independently of one another.

When $\det\{a, b\} = 0$ the restriction of det to the plane $\mathrm{Span}(a, b)$ is a degenerate quadratic form. In this case the point $\{a, b\}$ lies on the conic $\mathbb{P}(\mathbb{X})$ and its polar line $\mathbb{P}\{a, b\}^\perp$ is tangent to the conic. Notice that at most one among $\mathbb{P}(a)$ and $\mathbb{P}(b)$ can lie on the conic, in which case it equals $\mathbb{P}\{a, b\}$.

These configurations have been represented in Figure 1.4.

Remark 1.19. *As there may exist non-colinear $a, b \in \mathfrak{sl}(\mathbb{V})$ such that $\det\{a, b\} = 0$, one may thus think of $(a, b) \mapsto \det\{a, b\}$ as the square of a degenerate area form.*

Corollary 1.20. *The quantity $[a, b, c] := \langle\{a, b\}, c\rangle = \frac{1}{2}(\mathrm{tr}(bac) - \mathrm{tr}(abc))$ defines a volume form on $\mathfrak{sl}(\mathbb{V})$, that is an alternate non-degenerate trilinear form.*

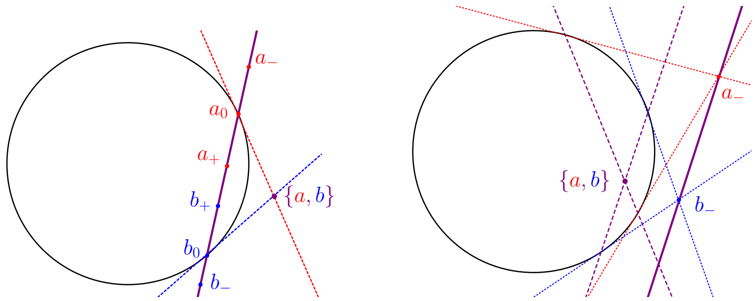


Figure 1.4: In the projective plane $\mathbb{P}(\mathfrak{sl}(\mathbb{V}))$ with the conic $\mathbb{P}(\mathbb{X})$: the line $(\mathbb{P}(a), \mathbb{P}(b))$ and its pole $\mathbb{P}\{a, b\}$.

Proof. It is clearly trilinear. It vanishes if and only if: either $\{a, b\} = 0$ in which case they are colinear; or else $c \in \{a, b\}^\perp = \mathrm{Span}(a, b)$. The alternate property can be read of the expression in terms of the trace. □

Example 1.21. *On $\mathfrak{sl}_2(\mathbb{K})$ we have $[K, J, S] = 1$ and $[S, J, K] = -1$.*

Subalgebras and their commutants

Elements $a, b \in \mathfrak{sl}(\mathbb{V})$ generate an associative subalgebra $(\mathbb{K}[a, b], \cdot)$ of $(\mathfrak{gl}(\mathbb{V}), \cdot)$ and generate a Lie subalgebra $(\mathfrak{L}(a, b), \{\cdot, \cdot\})$ of $(\mathfrak{sl}(\mathbb{V}), \{\cdot, \cdot\})$. Since the Lie bracket has been defined as half the commutator of the associative product we clearly have $\mathbb{K}[a, b] \supset \mathbb{K}\mathbf{1} \oplus \mathfrak{L}(a, b)$.

Proposition 1.22. *Let $a, b \in \mathfrak{sl}(\mathbb{V})$. In terms of the underlying vector spaces we have $\mathbb{K}[a, b] = \mathbb{K}\mathbf{1} \oplus \mathfrak{L}(a, b)$ and $\mathfrak{L}(a, b) = \mathrm{Span}(a, b, \{a, b\})$.*

There are four possibilities for the isomorphism type of $\mathfrak{L}(a, b)$ given by the relative position of $a, b, \{a, b\}$ with respect to the isotropic cone $\mathbb{X} \subset \mathfrak{sl}(\mathbb{V})$, as follows.

0 If $a = 0 = b$, then $\mathfrak{L}(a, b) = \{0\}$.

1 If $\{a, b\} = 0$ but $\det(a) \neq 0$ then $\mathfrak{L}(a, b)$ is the abelian Lie algebra of $\dim = 1$.

2 If $\{a, b\} \neq 0$ but $\det\{a, b\} = 0$ then $\mathfrak{L}(a, b)$ is the affine Lie algebra of $\dim = 2$.

3 If $\det\{a, b\} \neq 0$ then $\mathfrak{L}(a, b) = \mathfrak{sl}(\mathbb{V})$.

Over every field $\mathbb{K}$ of characteristic different from 2, each of these cases can be realised by choosing a, b appropriately.

Remark 1.23. *The previous corollary could have been formulated for* $M, N \in \mathfrak{gl}(\mathbb{V})$*.*

Indeed, the algebra generated by $M, N \in \mathfrak{gl}(\mathbb{V})$ *is the same as that generated by their projections* $a, b \in \mathfrak{sl}(\mathbb{V})$*, and we already saw that* $\{a, b\} = \frac{1}{2}(MN - NM)$*.*

Proof. After permuting a and b we are in one of the four cases, and these are mutually exclusive. The analysis of each case will show that $\mathbb{K}[a, b] = \mathbb{K} \oplus \mathfrak{L}(a, b)$.

The case 0 is obvious. The case 1 follows from Lemma 1.18; it includes the possibility $b = 0$, and we could have either $\det(a) = 0$ or $\det(a) \neq 0$.

To prove the others, suppose that a, b are not colinear, and recall that $\{a, b\}$ is a non zero vector orthogonal vector to the plane $\mathrm{Span}(a, b)$.

If $\det\{a, b\} \neq 0$ then $(a, b, \{a, b\})$ is a basis of $\mathfrak{sl}(\mathbb{V})$ and we have shown case 4.

Otherwize $\{a, b\}$ generates the unique isotropic line in $\mathrm{Span}(a, b)$, and we may decompose $\{a, b\} = xa + yb$ for $x, y \in \mathbb{K}$. At least one of the vectors a, b is not isotropic (as they are not colinear), say $\det(a) \neq 0$. In particular we have $a \neq \{a, b\}$ and $y \neq 0$. Thus $(a, \{a, b\})$ form a linear basis of $\mathfrak{L}(a, b)$ and $\{a, \{a, b\}\} = y\{a, b\}$. This proves the isomorphism with the affine Lie algebra of dimension 2.

Notice that in case 2, the pair (a, b) also forms a basis of $\mathfrak{L}(a, b)$. Besides, the relations $ab = -\langle a, b\rangle \mathbb{1} + xa + yb$ and $ba = -\langle a, b\rangle \mathbb{1} - xa - yb$ imply that $(\mathbb{1}, a, b)$ form a linear basis of the (non commutative) algebra $\mathbb{K}[a, b]$.

To realise each of these configurations, we explicit pairs of elements of $\mathfrak{sl}_2(\mathbb{K})$ with coefficients in $\{-1, 0, 1\}$ so that everything makes sense over all $\mathbb{K}$ characteristic different from 2. It is clear how one may realise cases $0, 1$. An instance of case 2 is obtained by choosing $a = K$ diagonal and $b = \frac{1}{2}(J - S)$ upper triangular such that $\{a, b\} = b$. An instance of 3 is given by choosing two elements in an orthogonal basis of $\mathfrak{sl}_2(\mathbb{K})$, such as the one appearing the next paragraph. □

In the next two corollaries the symbol $\perp$ refers to orthogonality in $(\mathfrak{gl}(\mathbb{V}), \det)$.

Proposition 1.24. *Consider the adjoint actions of the groups* $\mathrm{GL}(\mathbb{V})$ *and* $\mathrm{SL}(\mathbb{V})$ *on the space* $\mathfrak{sl}(\mathbb{V})$ *and its projectivisation* $\mathbb{P}(\mathfrak{sl}(\mathbb{V}))$*.*

The stabilizer of $p \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ *under* $\mathrm{GL}(\mathbb{V})$ *is* $(\mathbb{K}[p])^\times$*, which is the complement of the degenerate conic* $x^2 + y^2 \det(p) = 0$ *in the plane* $\mathbb{K}[p] = \{x\mathbb{1} + yp\}$*.*

The stabilizer of $p \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ *under* $\mathrm{SL}(\mathbb{V})$ *is* $(\mathbb{K}[p])^\times \cap \mathfrak{sl}(\mathbb{V})$*, which is the non-degenerate conic* $x^2 + y^2 \det(p) = 1$ *in the plane* $\mathbb{K}[p] = \{x\mathbb{1} + yp\}$*.*

The stabiliser of $\mathbb{P}(p) \in \mathbb{P}(\mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X})$ *under* $\mathrm{GL}(\mathbb{V})$ *is the* $\mathbb{Z}/2$*-graded subgroup* $(\mathbb{K}[p])^\times \sqcup (\mathbb{K}[p]^\perp)^\times$ *formed by the union of the complements of two degenerate conics.*

The stabiliser of $\mathbb{P}(p) \in \mathbb{P}(\mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X})$ *under* $\mathrm{SL}(\mathbb{V})$ *is the* $\mathbb{Z}/2$*-graded subgroup* $(\mathbb{K}[p] \cap \mathrm{SL}(\mathbb{V})) \sqcup (\mathbb{K}[p]^\perp \cap \mathfrak{sl}(\mathbb{V}))$ *formed by the union of two non-degenerate conics.*

Proof. Let $p \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ and $C \in \mathrm{GL}(\mathbb{V})$ and suppose $CpC^{-1} = \lambda p$ for some $\lambda \in \mathbb{K}$.

Then $0 \neq \det(p) = \det(CpC^{-1}) = \lambda^2 \det(p)$ so $\lambda^2 = 1$. The first two statements correspond to the case $\lambda = 1$, and the analysis for the last two statements is completed by considering the case $\lambda = -1$.

Denoting $z + c$ the decomposition of C according to $\mathbb{K}\mathbb{1} \oplus \mathfrak{sl}(\mathbb{V})$ we have:

$CpC^{-1} = p \iff (z + c)p = p(z + c) \iff \{c, p\} = 0$ and by Lemma 1.18 this is equivalent to $C \in \mathbb{K}[p]^\times$,

$CpC^{-1} = -p \iff (z + c)p = -p(z + c) \iff 2zp = 2\langle c, p\rangle \iff z = 0 = \langle c, p\rangle$ which is equivalent to $C \in \mathfrak{sl}(\mathbb{V}) \cap (\mathbb{K}[p]^\perp)^\times$. □

Remark 1.25. *The $\mathbb{Z}/2$-graded group $(\mathbb{K}[p])^\times \sqcup (\mathbb{K}[p]^\perp)^\times$ is a non trivial extension of $\mathbb{Z}/2$ by $\mathbb{K}[p]$. Hence the $\mathbb{Z}/2$-graded group $(\mathbb{K}[p] \cap \mathrm{SL}(\mathbb{V})) \sqcup (\mathbb{K}[p]^\perp \cap \mathfrak{sl}(\mathbb{V}))$ is a non trivial extension of $\mathbb{Z}/2$ by $\mathbb{K}[p] \cap \mathrm{SL}(\mathbb{V})$.*

Question 1.26 (Exercise)**.** *Describe the isomorphism types of the stabilisers in the previous proposition in terms of $\mathbb{K}$ and $\mathbb{K}^\times$ and $\mathrm{disc}(p) \in (\mathbb{K}^\times)/(\mathbb{K}^\times)^2$.*

Example 1.27. *The stabiliser of S by $\mathrm{GL}_2(\mathbb{K})$ is $\{t\mathbb{1} + sS \mid t^2 + s^2 \neq 0\}$ and by $\mathrm{SL}_2(\mathbb{K})$ is $\{t\mathbb{1} + sS \mid t^2 + s^2 = 1\}$.*

The latter is also equal to the stabiliser of $\mathbb{P}(S)$ by $\mathrm{SL}_2(\mathbb{K})$. However, the stabiliser of $\mathbb{P}(S)$ by $\mathrm{GL}_2(\mathbb{K})$ is $\{t\mathbb{1} + sS \mid t^2 + s^2 = 1\} \sqcup \{jJ + kK \mid j^2 + k^2 = 1\}$.

For completeness, and to compare with the previous proposition, we also describe the stabiliser of an isotropic vector $p \in \mathfrak{sl}(\mathbb{V})$ and a its projectivisation $\mathbb{P}(p) \in \mathbb{P}(\mathbb{X})$, but we will not make much use of this result.

Proposition 1.28. *Consider the adjoint actions of the groups $\mathrm{GL}(\mathbb{V})$ and $\mathrm{SL}(\mathbb{V})$ on the space $\mathfrak{sl}(\mathbb{V})$ and its projectivisation $\mathbb{P}(\mathfrak{sl}(\mathbb{V}))$, and assume $p \in \mathbb{X} \setminus \{0\}$.*

The stabilizer of $p \in \mathbb{X} \subset \mathfrak{sl}(\mathbb{V})$ under $\mathrm{GL}(\mathbb{V})$ is $(\mathbb{K}[p])^\times$, which is the complement of the very degenerate conic $x^2 = 0$ in the plane $\mathbb{K}[p] = \{x\mathbb{1} + yp\}$.

The stabilizer of $p \in \mathbb{X} \subset \mathfrak{sl}(\mathbb{V})$ under $\mathrm{SL}(\mathbb{V})$ is $(\mathbb{K}[p])^\times \cap \mathfrak{sl}(\mathbb{V})$, which is the degenerate conic $x^2 = 1$ in the plane $\mathbb{K}[p] = \{x\mathbb{1} + yp\}$.

The stabiliser of $\mathbb{P}(p) \in \mathbb{P}(\mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X})$ under $\mathrm{GL}(\mathbb{V})$ is $(p^\perp)^\times$ which is the complement of a degenerate quadric in the hyperplane $p^\perp \subset \mathfrak{gl}(\mathbb{V})$.

The stabiliser of $\mathbb{P}(p) \in \mathbb{P}(\mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X})$ under $\mathrm{SL}(\mathbb{V})$ is $\mathfrak{sl}(\mathbb{V}) \cap (p^\perp)^\times$ which is a degenerate quadric in the hyperplane $p^\perp \subset \mathfrak{gl}(\mathbb{V})$.

Proof. We use the same notations as in the previous proof. The first two statements follow from $Cp = pC \iff \{c, p\} = 0$. As for the last two statements, notice first that by Proposition 1.22 for $c \in \mathfrak{sl}(\mathbb{V})$ we have $\{c, p\} \in \mathbb{K}p \iff c \in p^\perp$.

Now $Cp = \lambda pC \iff (z-\lambda)p + \lambda\{c,p\} = (1-\lambda)\langle c,p\rangle\mathbb{1}$ which is equivalent to $(x-\lambda)p + \lambda\{c,p\} = 0 = \langle c,p\rangle$, whether $\lambda \neq 1$ or $\lambda = 1$ by the previous remark. But for every $c \in \mathfrak{sl}(\mathbb{V}) \cap p^\perp$ and $z \in \mathbb{K}\mathbb{1}$ there exists some $\lambda \in \mathbb{K}$ such that $\lambda\{c,p\} = (\lambda - z)p$ hence $Cp = \lambda pC \iff c \perp p \iff C \perp p$. □

Question 1.29 (Exercise)**.** *Describe the isomorphism types of the stabilisers in the previous proposition in terms of* $\mathbb{K}$ *and* $\mathbb{K}^\times$*.*

We may define the *one parameter subgroup* of $\mathrm{SL}(\mathbb{V})$ generated by $c \in \mathfrak{sl}(\mathbb{V})$ as the intersection $\mathbb{K}[c] \cap \mathfrak{sl}(\mathbb{V})$. It consists in the set of elements $C = x + yc$ satisfying $x^2 + y^2 \det(c) = 1$, a conic in $\mathrm{Span}(\mathbb{1}, c)$ defined by the Pell-Fermat equation of parameter $-\det(c) = c^2$.

We have not defined the exponential map because it would lead to considering transcendental elements over $\mathbb{K}$. Instead, one may think of the (restricted) projection $\mathrm{pr}\colon \mathrm{SL}(\mathbb{V}) \to \mathfrak{sl}(\mathbb{V})$ as a renormalised version of the logarithm.

Computing with the orthogonal decomposition $\mathfrak{gl}(\mathbb{V}) = \mathbb{K}\mathbb{1} \oplus \mathfrak{sl}(\mathbb{V})$ and the relation $\det = \mathrm{tr}^2 - \mathrm{pr}^2$ remains in the realm of quadratic algebra over $\mathbb{K}$.

Quadratic identities and commutators

Lemma 1.30. *Let* $A, B \in \mathrm{SL}(\mathbb{V})$ *and* $a = \mathrm{pr}\, A$*,* $b = \mathrm{pr}\, B$ *their projections in* $\mathfrak{sl}(\mathbb{V})$*. Then their commutator* $[A,B] = ABA^{-1}B^{-1}$ *has projection and discriminant:*

$$\mathrm{pr}[A,B] = 2\,(AB - \{a,b\})\,\{a,b\} \tag{1.8}$$

$$\mathrm{disc}[A,B] = 2\,\mathrm{tr}([A,B])\,\mathrm{disc}\{a,b\} \tag{1.9}$$

The second formula implies that $t = \mathrm{tr}[A,B] \neq 0$*, and can then be rewritten as:*

$$\det\{a,b\} = -\tfrac{1}{2}(t - t^{-1})$$

In particular $\det\{a,b\} = 0 \iff \mathrm{tr}[A,B] = \pm 1$.

Proof. If $A = x + a$, $B = y + b$, then $AB = z + (u + v)$ where $z = xy - \langle a,b\rangle$, $u = xb + ya$ and $v = \{a,b\}$, while $BA = z + u - v$ whence $(BA)^\# = z - u + v$.

Now $[A,B] = (AB)(BA)^\# = (z^2 + \langle v+u, v-u\rangle) + (2zv + \{v+u, v-u\})$, so $[A,B] = (z^2 + u^2 - v^2) + \mathrm{pr}[A,B]$ with $\mathrm{pr}[A,B] = 2zv + 2\{u,v\}$. But $u \in \mathrm{Span}(a,b)$ and $v = \{a,b\}$ imply $u \perp v$, whence $uv = \{u,v\}$. Consequently $\mathrm{pr}[A,B] = (z+u)2v$.

Now focus on $\mathrm{disc}[A,B] = \mathrm{disc}\,\mathrm{pr}[A,B] = -4\det\mathrm{pr}[A,B] = 4\det(z+u)\,\mathrm{disc}(v)$. Develop $\det(z+u) = \langle a,b\rangle^2 - (xy)^2 + x^2\det(b) + y^2\det(a)$ and replace $\det(a) = 1 - x^2$, $\det(b) = 1 - y^2$. Then notice that $\mathrm{tr}(AB)\,\mathrm{tr}(AB^\#) = (xy)^2 - \langle a,b\rangle^2$. Thus $4\det(z+u) = 4\,(\mathrm{tr}(A)^2 + \mathrm{tr}(B)^2 - \mathrm{tr}(AB)\,\mathrm{tr}(AB^{-1})) = 2\,\mathrm{tr}([A,B])$.

Finally for $t = \mathrm{tr}[A,B]$ we have $4(t^2-1) = \mathrm{disc}[A,B] = 2t\,\mathrm{disc}\{a,b\}$. □

Scholium 1.31 (Motivation)**.** *Let $A, B, \in \mathrm{SL}(\mathbb{V})$ and $C = AB$. Denote $x + a$, $y + b$ and $z + c$ their decomposition according to $\mathbb{K}\mathbb{1} \oplus \mathfrak{sl}(\mathbb{V})$.*

The lemma expresses $\operatorname{disc}\{a, b\} = 2(t - t^{-1})$ in terms of $2t = \operatorname{tr}[A, B]$ which is in turn given in terms of x, y, z by the trace formula $t/2 = x^2 + y^2 + z^2 - 2xyz$.

The motivation which led to proving such a lemma was to show that when $\mathbb{K} \subset \mathbb{R}$, $\operatorname{sign}\operatorname{disc}\{a, b\}$ is positive for $t \in]-1, 0[\cup]1, \infty[$ and negative for $t \in]-\infty, -1[\cup]0, 1[$.

Let us apply identity 1.8 to describe the pairs of elements $A, B \in \mathrm{SL}(\mathbb{V})$ whose commutator $[A, B]$ belongs to the center $\{\pm\mathbb{1}\}$ of $\mathrm{SL}(\mathbb{V})$.

Clearly we have $[A, B] = \mathbb{1} \iff \{a, b\} = 0$ and this situation has already been described in Lemma 1.18 and its Corollary 1.22

Corollary 1.32. *Let $A, B \in \mathrm{SL}(\mathbb{V})$. We have $[A, B] = -\mathbb{1}$ if and only if A and B belong to $\mathfrak{sl}(\mathbb{V})$ and are orthogonal. This can be written:*

$$[A, B] = -\mathbb{1} \iff \operatorname{tr}(A) = \operatorname{tr}(B) = \operatorname{tr}(AB) = 0$$

Proof. Denote $x + a$, $y + b$ the decompositions of A, B according to $\mathbb{K}\mathbb{1} \oplus \mathfrak{sl}(\mathbb{V})$.

Since A, B do not commute we have $\{a, b\} \neq 0$. Identity 1.8 implies $AB = \{a, b\}$. This yields $\operatorname{tr}(AB) = 0$ along with $xy = \langle a, b\rangle$ and $xb + ya = 0$.

Suppose by contradiction that $xy \neq 0$. Then the previous equalities imply that $xy = \langle a, -ay/x\rangle$ whence $x^2 + \langle a, a\rangle = 0$ which is $\det(A) = 0$, a contradiction.

The relation $xb + ya = 0$ reveals, together with the non commutativity of $a\&b$, that $x = 0 \iff y = 0$. We have thus shown that $x = y = 0$, as desired. $\square$

1.3 The projective conic $\mathbb{P}(\mathbb{X})$ and cross-ratios

Parametrizing the isotropic cone $\mathbb{X}$ of $\mathfrak{sl}(\mathbb{V})$

In this paragraph we parametrize the isotropic cone $\mathbb{X}$, and construct an explicit inverse to the map Ψ appearing in Proposition 1.11.

For sake of clarity we start with the case of $\mathfrak{sl}_2(\mathbb{K})$ using our preferred basis, before providing an intrinsic formulation in the case of $\mathfrak{sl}(\mathbb{V})$ which has the advantage of revealing the nature of the parametrization.

Thus we choose a basis of $\mathbb{V}$ identifying it with $\mathbb{K}^2$, which yields one for $\mathfrak{gl}(\mathbb{V})$ as we saw by exhibiting our favourite basis $\{\mathbb{1}, S, J, K\}$ of $\mathfrak{gl}_2(\mathbb{K})$.

Lemma 1.33. *Define the quadratic map $\psi\colon \mathbb{K}^2 \to \mathfrak{sl}_2(\mathbb{K})$ by $\psi(v) = v \cdot {}^t\!vS$, whose matrix expression is given for $v = {}^t(x\,y)$ by:*

$$\psi\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}\cdot\begin{pmatrix}y & -x\end{pmatrix} = \begin{pmatrix}xy & -x^2\\ y^2 & -yx\end{pmatrix} = \frac{(y^2-x^2)J - 2xyK + (x^2+y^2)S}{2}$$

It has image the set $\mathbb{X}_S = \{a \in \mathbb{X} \mid \langle a, S\rangle = x^2 + y^2\}$ of elements in the cone whose scalar product with S is a sum of squares, and is two-to-one outside the origin.

The map ψ sends the determinant square to the scalar product: for all $u, v \in \mathbb{K}^2$ we have $2\langle\psi(u), \psi(v)\rangle = \det(u, v)^2$.

Finally, the map ψ intertwines the tautological action of $\mathrm{SL}_2(\mathbb{K})$ on $\mathbb{K}^2$ to the restriction of its adjoint to $\mathbb{X}$.

Proof. If $A \in \mathrm{SL}_2(\mathbb{K})$ then ${}^tAS = SA^{-1}$, so $\psi(Av) = (Av)\cdot{}^t(Av)S = A\psi(v)A^{-1}$. $\square$

The quadratic map $\psi\colon \mathbb{K}^2 \to \mathbb{X}_S$ is in particular homogeneous and does not vanish outside the origin, so it has a well defined projectivization $\mathbb{P}(\psi)\colon \mathbb{P}(\mathbb{K}^2) \to \mathbb{P}(\mathbb{X}_S)$, which (from the polynomial nature of ψ) defines an algebraic morphism.

Notice that $\mathbb{P}(\mathbb{X}_S)$ equals $\mathbb{P}(\mathbb{X})$ because one can lift a representative of the later in $\mathbb{X}$ to obtain an element p whose entries satisfy $p_{11} = -p_{22}$ and $p_{11}^2 = -p_{12}p_{21}$ so multiplying p by p_{21} will sent it to $\mathbb{X}_S$.

Since $\mathbb{P}(\psi)$ is one-to-one, it defines an isomorphism of projective lines. Thus, cross-ratios of lines in $\mathbb{K}^2$ correspond to cross-ratios of points on the conic $\mathbb{P}(\mathbb{X})$.

Corollary 1.34. *The isomorphism of projective lines $\mathbb{P}(\psi)\colon \mathbb{P}(\mathbb{K}^2) \to \mathbb{P}(\mathbb{X})$ intertwines the tautological action of $\mathrm{PSL}_2(\mathbb{K})$ on $\mathbb{P}(\mathbb{K}^2)$ to its adjoint action on $\mathbb{P}(\mathbb{X})$.*

We now provide an intrinsic formulation for the map ψ.

Consider a non degenerate bilinear form ω on $\mathbb{V}$. Being non degenerate, it amounts to the isomorphism $\Omega\colon \mathbb{V} \to \mathbb{V}^*$ given by $\Omega(v)\colon u \mapsto \omega(u, v)$.

The diagonal map $\imath \otimes \Omega\colon \mathbb{V} \to \mathbb{V} \otimes \mathbb{V}^*$ given by $v \mapsto v \otimes \Omega(v)$ can be composed with the natural isomorphism $\mathfrak{gl}(\mathbb{V}) \mapsto \mathbb{V} \otimes \mathbb{V}^*$ to yield a map $\psi\colon \mathbb{V} \to \mathfrak{gl}(\mathbb{V})$. The map ψ is quadratic, in the sense that $\psi(\lambda v) = \lambda^2 \psi(v)$ for all $\lambda \in \mathbb{K}$.

By definition, for $v \in \mathbb{V}$ the element $\psi(v) \in \mathfrak{gl}(\mathbb{V})$ has image $\mathbb{K}v$ and kernel the ω-orthogonal $v^\perp$. In particular $\det \psi(v) = 0$ and $\psi(v) = 0 \iff v = 0$.

Moreover $\operatorname{Tr} \psi(v) = \omega(v, v)$ as one can see either from the intrinsic definitions of $\operatorname{Tr} \circ \psi\colon \mathbb{V} \to \mathbb{V} \otimes \mathbb{V}^* \to \mathbb{K}$ or by completing v in a basis of $\mathbb{V}$.

Remark 1.35. *Suppose ω is symmetric and anisotropic.*

Then $\det \psi(v) = 0$ and $\operatorname{Tr} \psi(v) = \omega(v, v) \neq 0$ so $\psi(v)$ has rank one and equals $\omega(v, v)$ times the ω-orthogonal projection on v (the unique idempotent element with image $\mathbb{K}v$ and kernel $v^\perp$). It belongs to the complement of the cone $\mathbb{X}$ in $\{\det = 0\}$.

From now on we suppose that ω is symplectic, it is unique to scaling by a factor. Its group of linear automorphisms $\operatorname{Sp}(\mathbb{V}, \omega)$ is naturally isomorphic to $\operatorname{SL}(\mathbb{V})$.

A *symplectic basis* of $(\mathbb{V}, \omega)$ is a pair $(u, v) \in \mathbb{V} \times \mathbb{V}$ such that $\omega(u, v) = 1$. Denote $S_{u,v} \in \operatorname{GL}(\mathbb{V})$ the unique element of order 4 sending u to v. Since $\det S_{u,v} = 1$ and $\operatorname{Tr} S_{u,v} = 0$ we have $S_{u,v} \in \operatorname{SL}(\mathbb{V}) \cap \mathfrak{sl}(\mathbb{V})$. Denote $\mathbb{H}_\omega$ the set of such elements $S_{u,v}$.

Lemma 1.36. *Let ω be a symplectic form on $\mathbb{V}$ and consider the quadratic map $\psi\colon \mathbb{V} \to \mathfrak{gl}(\mathbb{V})$ defined above. If we choose a symplectic basis (u, v) of $(\mathbb{V}, \omega)$ then we recover the map ψ in Lemma 1.33 with $S = S_{u,v}$.*

The map ψ has image the set $\mathbb{X}_\omega$ of elements $p \in \mathbb{X}$ such that for all $S_{u,v} \in \mathbb{H}_\omega$ the scalar product $\langle p, S_{u,v}\rangle = x^2 + y^2$ is a sum of squares of elements $x, y \in \mathbb{K}$. It is two-to-one outside the origin. For all $u, v \in \mathbb{V}$ we have $2\langle \psi(u), \psi(v)\rangle = \omega(u, v)^2$.

The map ψ intertwines the tautological action of $\operatorname{SL}(\mathbb{V})$ on $\mathbb{V}$ with the restriction of its adjoint action on $\mathbb{X}_\omega$.

Proof. The first statement follows from the Remark 1.6: a basis (u, v) of $\mathbb{V}$ yields canonical euclidean and symplectic forms, and the element $S_{u,v}$ is the polarisation of the later with respect to the former.

The rest follows from 1.33, but let us prove the second paragraph to add some geometric insight and make sure the signs are correct.

If $v \neq 0$ then $\psi(v)$ has image and kernel equal to $\mathbb{K}v$, therefore $\psi(v) \in \mathbb{X}$. Alternatively we already computed $\det \psi(v) = 0$ and $\operatorname{Tr} \psi(v) = \omega(v, v) = 0$, thus $\psi(v)$ is nilpotent, that is $\psi(v)^2 = 0$, and Proposition 1.11 says that $\psi(v) \in \mathbb{X}$.

Now consider $w \in \mathbb{V}$ and let us compute $\langle \psi(w), S_{u,v}\rangle = -\frac{1}{2} \operatorname{Tr}(\psi(w) \circ S_{u,v})$. Decomposing $w = xu + yv$ in the basis (u, v) we have $\omega(w, S_{u,v}u) = (xu + yv)x$

and $\omega(w, S_{u,v}v) = (xu + yv)y$. Thus, remembering that $\psi(w) = w.\omega(\cdot, w)$, we have $\mathrm{Tr}(-\psi(w) \circ S_{u,v}) = x^2 + y^2$. Hence $\mathrm{im}\,\psi \subset \mathbb{X}_\omega$.

Conversely, let $p \in \mathbb{X}_\omega$. There exists a pair $u, v \in \mathbb{V}$ defined over $\mathbb{K}$ such that $\omega(u, v) = 1$, so there exist $x, y \in \mathbb{K}$ such that $\langle p, S_{u,v}\rangle = x^2 + y^2$. Then $w = xu + yv$ has image $\psi(w) = p$ hence $\mathrm{im}(\psi) \supset \mathbb{X}_\omega$. We notice that if $\langle p, S_{u,v}\rangle$ is a sum of squares for one element $S_{u,v} \in \mathbb{H}_\omega$ then it is a sum of squares for all $S_{u',v'} \in \mathbb{H}_\omega$.

For distinct $u, v \in \mathbb{V}$ the composition $\psi(u) \circ \psi(v)$ is the projector on $\mathbb{K}u$ parallel to $\mathbb{K}v$ multiplied by $\mathrm{Tr}(\psi(v) \circ \psi(u)) = \omega(u, v)\omega(v, u) = -\omega(u, v)^2$. In particular $2\langle\psi(u), \psi(v)\rangle = \omega(u, v)^2$. □

Note, as in the previous paragraph, that $\mathbb{P}(\mathbb{X}_\omega) = \mathbb{P}(\mathbb{X})$. Recall that the algebraic correspondence $\Psi \colon \mathbb{P}(\mathbb{V} \times \mathbb{V}) \to \{\det = 0\}$ from Proposition 1.11 maps the diagonal . One may restrict Ψ to the diagonal $\mathbb{P}(\mathbb{V})$ and its image $\mathbb{P}(\mathbb{X})$, or to their complements. This expresses Ψ as the disjoint union of two maps $\Psi = \Psi_1 \sqcup \Psi_2$.

Corollary 1.37. *The projectivized map $\mathbb{P}(\psi) \colon \mathbb{P}(\mathbb{V}) \to \mathbb{P}(\mathbb{X})$ is the inverse of the isomorphism Ψ_1 of projective lines. In particular it conjugates the adjoint action of $\mathrm{PGL}(\mathbb{V})$ on $\mathbb{P}(\mathbb{X})$ to its tautological action on $\mathbb{P}(\mathbb{V})$ which are 3-transitive.*

The bi-quadratic map $\psi \circ \psi \colon v, w \mapsto \psi(v) \circ \psi(w)$ from $(\mathbb{V} \times \mathbb{V}) \setminus \{(v, v) \mid v \in \mathbb{V}\}$ to $\{\det = 0\} \setminus \mathbb{X}$ has a well defined projectivization: $\mathbb{P}(\psi \circ \psi)$ is the inverse of Ψ_2.

The disjoint union $\mathbb{P}(\psi) \sqcup \mathbb{P}(\psi \circ \psi)$ is inverse to $\Psi = \Psi_1 \sqcup \Psi_2$. In particular it defines an isomorphism of projective quadrics and conjugates the adjoint action of $\mathrm{PSL}(\mathbb{V})$ on $\mathbb{P}(\{\det = 0\})$ to its tautological diagonal action on $\mathbb{P}(\mathbb{V}) \times \mathbb{P}(\mathbb{V})$.

Remark 1.38. *For $u, v \in \mathbb{V}$ the element $\{\psi(u), \psi(v)\} \in \mathfrak{sl}(\mathbb{V})$ is $-\frac{1}{2}\omega(u, v)^2$ times the symmetry with respect to the line $\mathbb{K}u$ parallel to the line $\mathbb{K}v$. In formula:*

$$\psi(u)\psi(v) = -\omega(u, v)^2 \cdot \mathrm{Proj}_{u/v} \qquad \{\psi(u), \psi(v)\} = -\tfrac{1}{4}\omega(u, v)^2 \cdot \mathrm{Sym}_{u/v}$$

This follows from $\mathbb{1} + \mathrm{Sym}_{u/v} = 2\mathrm{Proj}_{u/v}$ and $\mathrm{Sym}_{v/u} = -\mathrm{Sym}_{u/v}$.

Maslov index of 3 lines and cross-ratio of 4 lines

We define the *Maslov index* $\epsilon \colon \mathbb{P}(\mathbb{V}) \times \mathbb{P}(\mathbb{V}) \times \mathbb{P}(\mathbb{V}) \to \{0\} \cup \mathbb{K}^\times/(\mathbb{K}^\times)^2$ as follows. It is alternate and non degenerate in the sense that $\epsilon(L_u, L_v, L_w) = 0$ if and only if at least two lines coincide. Let L_u, L_v, L_w be three distinct lines in $\mathbb{K}^2$ generated by u, v, w. We may suppose $u + v + w = 0$ by considering a linear dependance relation $xu + yv + zw = 0$ and replacing u, v, w by xu, yv, zw; such a triple is unique up to multiplication by a scalar. Then we have $\omega(u, v) = \omega(v, w) = \omega(w, u)$ and this yields a well defined quantity $\epsilon(L_u, L_v, L_w) \in \mathbb{K}^\times/(\mathbb{K}^\times)^2$.

We say that the triple of lines is *ordered* when $\epsilon(L_u, L_v, L_w) = 1$. Notice that there exists a triple of distinct lines such that $\epsilon(L_u, L_v, L_w) = \epsilon(L_w, L_v, L_u)$ if and only if $-1 \in (\mathbb{K}^\times)^2$.

Proposition 1.39. *The action of* $\mathrm{PGL}(\mathbb{V})$ *on* $\mathbb{P}(\mathbb{V})$ *is simply-transitive on triples of distinct lines. The action of* $\mathrm{PSL}(\mathbb{V})$ *on* $\mathbb{P}(\mathbb{V})$ *preserves the Maslov index* ϵ *and is simply-transitive on triples of distinct lines with a given Maslov index.*

Proof. Fix a symplectic basis of $\mathbb{V}$ in order to identify $\mathbb{P}(\mathbb{V})$ with $\mathbb{KP}^1$. Let L_u, L_v, L_w be three distinct lines in $\mathbb{K}^2$ and choose generators u, v, w such that $u+v+w=0$. The matrices of $\mathrm{GL}_2(\mathbb{K})$ sending the lines with inclinations $(0, 1, \infty) \in \mathbb{KP}^1$ to (L_u, L_v, L_w) are the scalar multiples of the matrix with columns (u, w). □

The *cross-ratio* of four points $\alpha', \beta', \alpha, \beta$ on a projective line (with respect to any affine chart yielding coordinates $\mathbb{KP}^1 = \mathbb{K} \cup \{\infty\}$) is given by:

$$\mathrm{bir}(\alpha', \alpha, \beta', \beta) = \frac{(\alpha - \alpha')(\beta - \beta')}{(\alpha - \beta')(\beta - \alpha')} \tag{bir}$$

which specializes to $\mathrm{bir}(z, 0, 1, \infty) = z$ and $\mathrm{bir}(\infty, w, 0, z) = z/w$.

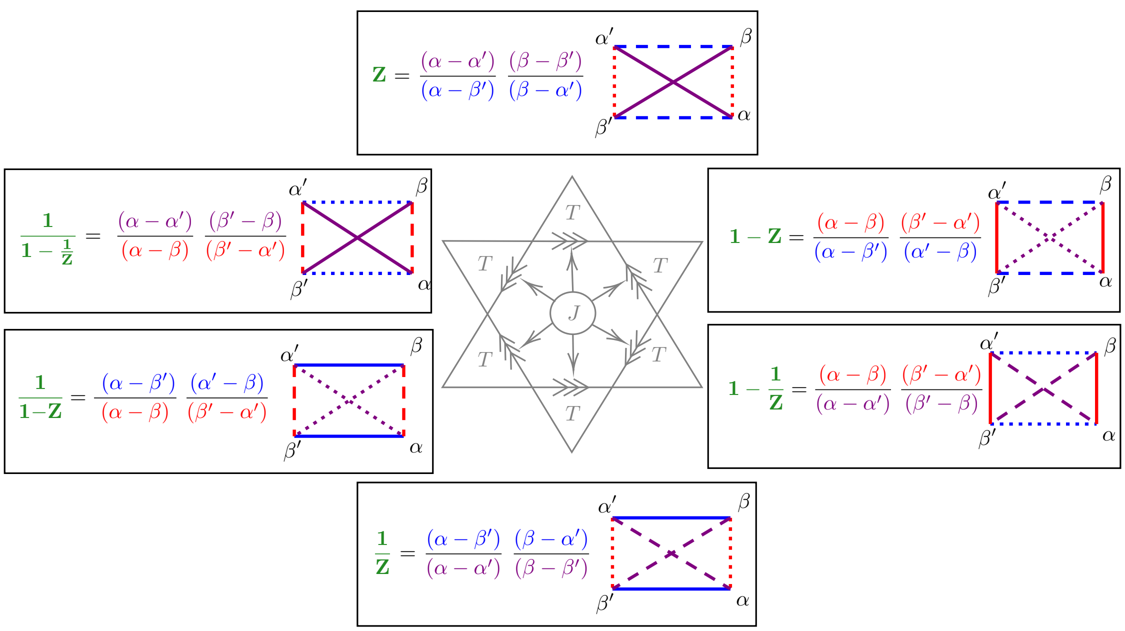


Figure 1.5: The cross-ratio and its tetrahedral group of transformations.

It satisfies the following symmetries, which we express as $\mathrm{L}\circ\sigma = \Sigma\circ\mathrm{L}$ where σ is a permutation of the four entries and $\Sigma \in \mathrm{PGL}_2(\mathbb{Z})$ is a linear fractional transformation belonging to the subgroup generated by $J\colon z \mapsto 1/z$ and $RK\colon z \mapsto 1 - z$.

$$\begin{aligned}
&\mathrm{bir}(\beta', \beta, \alpha', \alpha) = \mathrm{bir}(\alpha', \alpha, \beta', \beta) && (\alpha'\beta')(\alpha\beta) \mapsto \mathbb{1}\\
&\mathrm{bir}(\alpha, \alpha', \beta, \beta') = \mathrm{bir}(\alpha', \alpha, \beta', \beta) && (\alpha'\alpha)(\beta\beta') \mapsto \mathbb{1}\\
&\mathrm{bir}(\alpha', \beta, \beta', \alpha) = 1/\,\mathrm{bir}(\alpha', \alpha, \beta', \beta) && (\alpha\beta) \mapsto J\\
&\mathrm{bir}(\alpha', \beta', \alpha, \beta) = 1 - \mathrm{bir}(\alpha', \alpha, \beta', \beta) && (\alpha\beta') \mapsto RK
\end{aligned}$$

This map $\sigma \mapsto \Sigma$ described on the right yields the famous representation $\mathfrak{S}_4 \to \mathfrak{S}_3$. Note that the subgroup $\Sigma \subset \mathrm{PSL}_2(\mathbb{Z})$ generated by J and RK in $\mathrm{PSL}_2(\mathbb{Z})$ maps isomorphically onto $\mathrm{PSL}_2(\mathbb{Z}/2)$.

These symmetries imply that the cross-ratio also satisfies the addition rule:

$$\frac{1}{\mathrm{bir}(\alpha', \alpha, \beta', \beta)} + \frac{1}{\mathrm{bir}(\alpha', \alpha, \beta, \beta')} = 1. \tag{1.10}$$

The formula for the cross-ratio is designed to be equivariant under perspectives between projective lines in a projective space. In particular, it does not depend on the choice of affine coordinates, and is equivariant under the triply-transitive action of $\mathrm{PGL}_2(\mathbb{K})$ on $\mathbb{KP}^1$.

The division of two points in the affine line $\mathbb{K}$ satisfies a multiplicative cocycle identity on three points, namely for all $x, y, z \in \mathbb{K}$ we have $(z/x) = (z/y)(y/x)$. Using $\mathrm{bir}(\infty, x, 0, z) = z/x$ and the aforementioned triple transitivity, we find that the cross-ratio of four points in the projective line $\mathbb{KP}^1$ satisfies a multiplicative cocycle relation on five points, which is depicted in Figure 1.6:

$$\forall u, v, x, y, z \in \mathbb{KP}^1 \qquad \mathrm{bir}(u, x, v, y) \times \mathrm{bir}(u, y, v, z) = \mathrm{bir}(u, x, v, z) \tag{1.11}$$

Ptolemy's theorem for quadrilaterals inscribed in $\mathbb{P}(\mathbb{X})$

In this paragraph which is not necessary for the sequel, the tools developed to describe the quadratic geometry of $(\mathfrak{sl}(\mathbb{V}), \det)$ are applied to show the following analogue of Ptolemy's theorem for quadrilaterals inscribed in the projective conic $\mathbb{P}(\mathbb{X})$.

In fact, it is better formulated if we fix a symplectic form ω on $\mathbb{V}$ and consider vectors in the subset $\mathbb{X}_\omega$ of the isotropic cone $\mathbb{X}$. Since $\mathbb{P}(\mathbb{X}_\omega) = \mathbb{P}(\mathbb{X})$ we may always lift a quadrilateral to such a quadruple, and any lift will satisfy the identity.

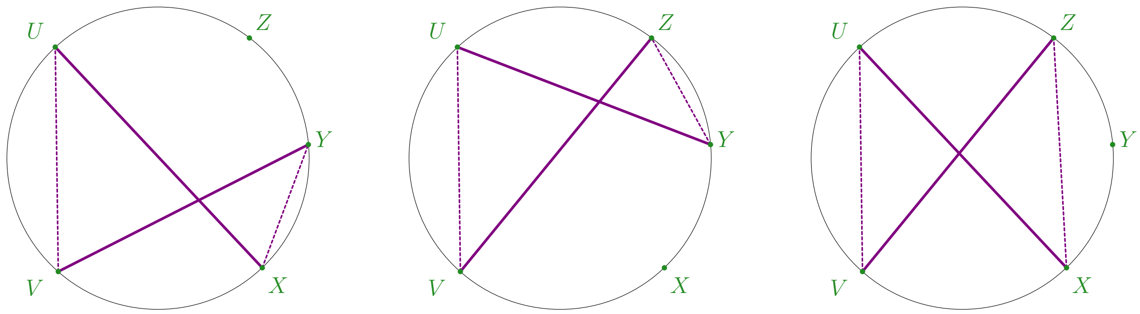


Figure 1.6: The cross-ratio satisfies a multiplicative cocyle property.

Proposition 1.40. *For distinct $u, v, x, y \in \mathbb{X}_\omega$, the following identity holds in $\sqrt{\mathbb{K}}$:*

$$\sqrt{\langle u, v\rangle \cdot \langle x, y\rangle} = \sqrt{\langle u, y\rangle \cdot \langle x, v\rangle} + \sqrt{\langle u, x\rangle \cdot \langle v, y\rangle} \tag{IPS}$$

Remark 1.41. *Formula IPS is invariant under the action of $(\mathbb{K}^\times)^2$ by individual dilatation of u, v, x, y, so we may suppose they lie on a conic $\{p \in \mathbb{X}_\omega \mid \langle S_{u,v}, p\rangle = 1\}$.*

Proof. The identity IPS is equivalent, after dividing by the left hand side, to:

$$\left(\frac{\langle u, v\rangle \cdot \langle x, y\rangle}{\langle u, y\rangle \cdot \langle x, v\rangle}\right)^{-\frac{1}{2}} + \left(\frac{\langle u, v\rangle \cdot \langle x, y\rangle}{\langle u, x\rangle \cdot \langle v, y\rangle}\right)^{-\frac{1}{2}} = 1.$$

But we know from the addition rule that $\operatorname{bir}(u, v, x, y)^{-1} + \operatorname{bir}(u, v, y, x)^{-1} = 1$ so the identity follows from the following lemma. □

Lemma 1.42. *For distinct $u, v, x, y \in \mathbb{X}_\omega$ we have in $\sqrt{\mathbb{K}}$:*

$$\operatorname{bir}(u, v, x, y) = \sqrt{\frac{\langle u, v\rangle \cdot \langle x, y\rangle}{\langle u, y\rangle \cdot \langle x, v\rangle}}. \tag{CRS}$$

Proof. We use the map $\psi\colon \mathbb{V} \to \mathbb{X}_\omega$ which preserves the cross-ratios of four lines and satisfies $2\langle \psi(\vec{u}) \mid \psi(\vec{v})\rangle = \omega(\vec{u}, \vec{v})^2$ for all $\vec{u}, \vec{v} \in \mathbb{V}$. Thus denoting $w = \psi(\vec{w})$:

$$\sqrt{\frac{\langle u, v\rangle \cdot \langle x, y\rangle}{\langle u, y\rangle \cdot \langle x, v\rangle}} = \frac{\omega(\vec{u}, \vec{v}) \cdot \omega(\vec{x}, \vec{y})}{\omega(\vec{u}, \vec{y}) \cdot \omega(\vec{x}, \vec{v})}$$

But the last equality equals the cross-ratio of the four lines generated by $\vec{u}, \vec{v}, \vec{x}, \vec{y}$ as one can see in the following Figure 1.7, which is equal to $\operatorname{bir}(u, v, x, y)$. □

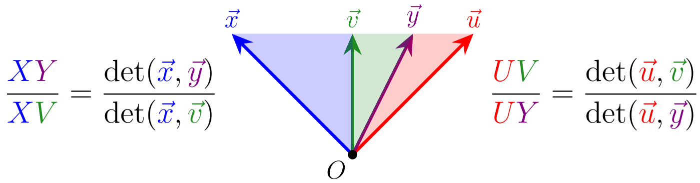


Figure 1.7: The cross-ratio of four lines in $(\mathbb{V}, \omega)$ in terms of the area form.

Ordering fixed points of $a, b \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ and cross-ratio $\mathrm{bir}(a, b)$

The object of this paragraph is to define the cross-ratio $\mathrm{bir}(a, b)$ of two elements $a, b \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ with the same determinant and find its expression in terms of $\langle a, b\rangle$.

Let $a \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ and fix the choice of a square root for its discriminant $\sqrt{\mathrm{disc}(a)}$. Since $a^2 = -\det(a)\mathbb{1} = \frac{1}{4}\mathrm{disc}(a)\mathbb{1}$, the eigen-values of a for its action on $\mathbb{V}$ are $\pm\frac{1}{2}\sqrt{\mathrm{disc}(a)}$ and we may denote $\alpha_\pm \in \mathbb{P}(\mathbb{V})$ the corresponding eigen-directions.

Now fix a symplectic form ω on $\mathbb{V}$. One may choose eigenvectors $v_\pm$ of a for the eigenvalues $\pm\frac{1}{2}\sqrt{\mathrm{disc}(a)}$ such that $\omega(v_-, v_+) = \sqrt{\mathrm{disc}(a)}$ and which are defined over $\mathbb{V} \otimes \mathbb{K}[\sqrt{\mathrm{disc}(a)}]$. Any other basis is obtained by the transformation $(\lambda^{-1}v_-, \lambda^{+1}v_+)$ for $\lambda \neq 0$. Besides, multiplying ω by $\mu \neq 0$ has the effect of multiplying ψ by μ and dividing $v_\pm$ by a same square root $\sqrt{\mu}$. Since ψ is quadratic, we find that $\psi(v_\pm) \in \mathbb{X}_\omega$ are invariant under multiplication of ω by any scalar μ. Moreover, the quantitity $\{\psi(v_-), \psi(v_+)\}$ does not depend on the choice of normalised eigen-basis since $\{\psi(\lambda^{-1}v_-), \psi(\lambda^{+1}v_+)\} = \{\psi(v_-), \psi(v_+)\}$, it only depends on the normalisation.

The intersection $a^\perp \cap \mathbb{X} \otimes \mathbb{K}[\sqrt{\mathrm{disc}(a)}]$ consists in two lines spanned by $\psi(v_\pm)$ hence the well defined quantity $\{\psi(v_-), \psi(v_+)\}$ belongs to the line $\mathbb{K}[\sqrt{\mathrm{disc}\, a}] \cdot a$. This scalar multiple is equal to that $\omega(v_-, v_+)$ appearing in the normalisation of the basis, that is the the square root $\sqrt{\mathrm{disc}(a)}$ chosen at the beginning:

$$\{\psi(v_-), \psi(v_+)\} = \omega(v_-, v_+) \cdot a = \sqrt{\mathrm{disc}(a)} \cdot a$$

Computation. We work on $\mathbb{K}^2$ with the standard symplectic form $u, v \mapsto \det(u, v)$.

Let $a \in \mathfrak{sl}_2(\mathbb{K})$ be given by the following expression, so that $\mathrm{disc}(a) = m^2 - 4lr$ and choose the following eigen-vectors $v_\pm$ normalised such that $\det(v_+, v_-) = \sqrt{\mathrm{disc}(a)}$.

$$a = \begin{pmatrix} -m/2 & -r \\ l & m/2 \end{pmatrix} \qquad \alpha_\pm = \frac{-m \pm \sqrt{\mathrm{disc}(a)}}{2l} \qquad v_\pm = \sqrt{l}\begin{pmatrix} \alpha_\pm \\ 1 \end{pmatrix}$$

Their images by ψ are given, according to Lemma 1.33, by:

$$\psi(v_\pm) = l\begin{pmatrix} \alpha_\pm & -\alpha_\pm^2 \\ 1 & -\alpha_\pm \end{pmatrix}$$

so using $l(\alpha_+ - \alpha_-) = \sqrt{\operatorname{disc}(a)}$ and $l(\alpha_+ + \alpha_-) = -m$ and $l\alpha_+\alpha_- = r$, we find:

$$\{\psi(v_-), \psi(v_+)\} = \tfrac{1}{2} l^2 (\alpha_+ - \alpha_-) \begin{pmatrix} (\alpha_+ + \alpha_-) & -2\alpha_+\alpha_- \\ 2 & -(\alpha_+ + \alpha_-) \end{pmatrix} = \sqrt{\operatorname{disc}(a)} \cdot a.$$

□

We recover the fact that given $a \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$, it is equivalent to choose a square root $\sqrt{\operatorname{disc}(a)}$, to order its fixed points $\alpha_-, \alpha_+ \in \mathbb{P}(\mathbb{V})$, or to order its fixed points $\mathbb{P}(\psi(v_-)), \mathbb{P}(\psi(v_+)) \in \mathbb{P}(\mathbb{X})$ which are well defined independently of ω.

Remark 1.43. *In some cases there is a preferred choice for the square root of $d \in \mathbb{K}$. This happens for instance when the field $\mathbb{K}[\sqrt{d}]$ is totally ordered.*

A theorem of Artin-Schreier says that a field admits a total order if and only if -1 is not a sum of squares (see [MH73, Chapter III, ğ2] for a proof).

Now given two elements $a, b \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$, we wish to order the sets of points $\mathbb{P}(a^\perp \cap \mathbb{X}) = \{\alpha', \alpha\}$ and $\mathbb{P}(b^\perp \cap \mathbb{X}) = \{\beta', \beta\}$ up to simultaneous inversion.

The square roots of two elements $d_a, d_b \in \mathbb{K}^\times$ generate isomorphic extensions when $d_a d_b \in (\mathbb{K}^\times)^2$. In that case the choice of a square root $\sqrt{d_a d_b} \in \mathbb{K}^\times$ determines $\sqrt{d_a}$ and $\sqrt{d_b}$ up to simultaneous change of sign, by imposing $\sqrt{d_a}\sqrt{d_a} = \sqrt{d_a d_b}$. This yields an identification between the extensions $\mathbb{K}[\sqrt{d_a}] = \mathbb{K}[\sqrt{d_b}]$. Note that if we are given the same element $d_a = d_b$ then we have a canonical choice for $\sqrt{d_a d_b}$. A choice for all such pairs d_a, d_b amounts to a determination of the square root $(\mathbb{K}^\times)^2 \to \mathbb{K}^\times$ and that is equivalent to a character of the group $\mathbb{K}^\times/(\mathbb{K}^\times)^2 \to \{\pm 1\}$.

Definition 1.44. *Consider $a, b \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ with a preferred choice of $\sqrt{\det(ab)}$. From the previous remarks we may order their fixed points (α', α) and (β', β) up to simultaneous inversion. Thus we may define their cross-ratio:*

$$\operatorname{bir}(a, b) \in \sqrt{\mathbb{K}}\mathbb{P}^1 \qquad \text{by} \qquad \operatorname{bir}(a, b) = \operatorname{bir}(\alpha', \alpha, \beta', \beta).$$

When $\det(a) = d = \det(b)$ we define this value as the preferred choice $d = \sqrt{\det(ab)}$.

We have $\operatorname{bir}(a, b) \in \mathbb{K}\mathbb{P}^1 \iff \det(ab) \in (\mathbb{K}^\times)^2$. The cross-ratio satisfies the symmetry relations $\operatorname{bir}(b, a) = \operatorname{bir}(a, b) = \operatorname{bir}(a^\#, b^\#)$ and the addition rule:

$$\frac{1}{\operatorname{bir}(a, b)} + \frac{1}{\operatorname{bir}(a, b^\#)} = 1.$$

which reflects the Galois symmetry acting on $\sqrt{\det(ab)}$.

Definition 1.45. *For $a, b \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ and a choice of $d = \sqrt{\det(ab)}$ we define:*

$$\cos(a,b) \in \sqrt{\mathbb{K}} \qquad \text{by} \qquad \cos(a,b) = \frac{\langle a,b\rangle}{\sqrt{\langle a,a\rangle\langle b,b\rangle}} = \frac{\langle a,b\rangle}{d}$$

When $\det(a) = \det(b)$ *we choose d equal to this common value.*

We have $\cos(a,b) \in \mathbb{K} \iff \det(ab) \in (\mathbb{K}^\times)^2$. Moreover $\cos(a,b)$ is invariant when multiplying a, b by a common factor, which may belong to any extension of $\mathbb{K}$.

Proposition 1.46. *Consider $a, b \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ with a choice for $d = \sqrt{\det(ab)}$. Then* $\operatorname{bir}(a,b)$ *and* $\cos(a,b)$ *are both defined and related by:*

$$\frac{1}{\operatorname{bir}(a,b)} = \frac{1+\cos(a,b)}{2} = \frac{d + \langle a,b\rangle}{2d}$$

We have $\operatorname{bir}(a,b) = \infty \iff \langle a,b\rangle = -d$ *and* $\operatorname{bir}(a,b) = 1 \iff \langle a,b\rangle = d$. *In particular* $\det\{a,b\} = d^2 - \langle a,b\rangle^2$ *is non zero if and only if* $\operatorname{bir}(a,b) \notin \{\infty, 1\}$.

Proof. The statement is invariant under multiplication of a, b by a common factor. To avoid ambiguity, we prove it in the case $\det(a) = d = \det(b)$ and choose a branch of $\sqrt{\cdot}$ over $\sqrt{\mathbb{K}}$ such that $\sqrt{d^2} = d$. We fix a symplectic form on $\mathbb{V}$ and a symplectic basis to work in $\mathfrak{sl}_2(\sqrt{\mathbb{K}})$.

The action of PGL_2 on $\mathbb{P}^1$ is triply transitive and preserves the cross-ratio so we may assume $(\alpha', \alpha, \beta', \beta) = (\infty, 1, 0, \beta)$. Our task is now to compute an corresponding pair (a,b), and express $\beta = \operatorname{bir}(a,b)$ in terms of $\cos(a,b)$.

We must set $a = \{\psi(u_-), \psi(u_+)\}$ and $b = \{\psi(v_-), \psi(v_+)\}$ for $u_\pm, v_\pm$ such that $\mathbb{P}(u_-) = \infty, \mathbb{P}(u_+) = 1$, $\mathbb{P}(v_-) = 0, \mathbb{P}(v_+) = \beta$ and $\det(u_-, u_+) = \det(v_-, v_+)$. The latter determinant equals $\sqrt{\operatorname{disc}(a)} = \sqrt{\operatorname{disc}(b)}$ and we normalise it to 1, choosing:

$$u_- = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad u_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad v_- = \begin{pmatrix} 0 \\ -1/\sqrt{\beta} \end{pmatrix} \quad v_+ = \begin{pmatrix} \sqrt{\beta} \\ 1/\sqrt{\beta} \end{pmatrix}$$

We compute $a = \frac{1}{2}(S - J - K)$ and $b = \frac{1}{2}((S+J)/\beta - K)$, whence $\langle a,b\rangle = \frac{1}{4}(1 - 2/\beta)$. Since $\det(a) = -1/4 = \det(b)$ we have $\cos(a,b) = -1 + 2/\beta$ as desired. □

Ordering fixed points of $A, B \in \mathrm{PGL}(\mathbb{V})$ and cross-ratio $\operatorname{bir}(A,B)$

Now let us extend the previous discussion concerning the ordering of fixed points and cross-ratios to elements in $\mathrm{PGL}(\mathbb{V})$.

An $M \in \mathbb{P}(\mathfrak{gl}(\mathbb{V}))$ has a well defined discriminant $\operatorname{disc}(M) \in \{0\} \cup \mathbb{K}^\times/(\mathbb{K}^\times)^2$. We call M semi-simple when $\operatorname{disc}(M) \neq 0$. Also $M \in \mathbb{P}(\mathfrak{gl}(\mathbb{V}))$ has a well defined determinant $\det(M) \in \{0\} \cup (\mathbb{K})^\times/(\mathbb{K}^\times)^2$. We have $M \in \mathrm{PGL}(\mathbb{V})$ when $\det(M) \neq 0$.

Definition 1.47. *For $A \in \mathrm{PGL}(\mathbb{V})$ we define $\delta(A) := \mathrm{disc}(A)/(\mathrm{Tr}\, A)^2 \in \mathbb{KP}^1$.*

We have $\delta(A) = \infty$ if and only if $A \in \mathbb{P}(\mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X})$, if and only if $A^2 \in \mathbb{K}^\times \mathbb{1}$, in other terms when $A \in \mathrm{PGL}(\mathbb{V})$ has order exactly two in the ring.

Let $A \in \mathrm{PGL}(\mathbb{V})$ with $\delta(A) \neq \infty$ and consider its action on $\mathbb{P}(\mathbb{V})$.

Its fixed points correspond to the eigen-directions in $\mathbb{V}$ of a lift in $\mathrm{GL}(\mathbb{V})$. If $\delta(A) = 0$ then A has only one fixed point. If $\delta(A) = 0$ then A has exactly two distinct fixed points, which can be defined over $\mathbb{K}[\sqrt{\delta(A)}]$.

We wish to distinguish, among its fixed points in $\mathbb{P}(\mathbb{V})$, one which is "repulsive" and one which is "attractive", in such a way that their roles are interchanged when A is inverted. As we shall see, this is equivalent to choosing a square root $\sqrt{\delta(A)}$. For this we shall reduce the problem to the previous discussion w had for $a \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$.

Let us assign to an element $A \in \mathrm{PGL}(\mathbb{V})$ such that $\mathrm{disc}(A) \neq 0$ and $\mathrm{tr}(A) \neq 0$ modulo $(\mathbb{K}^\times)^2$, a canonical element $a_0 \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ with the same fixed points in $\mathbb{P}(\mathbb{V})$.

Definition 1.48 (Normalised projection)**.** *For a semi-simple $A \in \mathrm{PGL}(\mathbb{V})$ which is not an involution, we define its normalised projection $a_0 \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ as follows. Choose a lift still denoted $A \in \mathrm{GL}(\mathbb{V})$, and set:*

$$a_0 = \frac{\mathrm{pr}(A)}{\mathrm{tr}(A)}$$

We have $A^\perp \cap \mathbb{X} = a_0^\perp \cap \mathbb{X}$ so A and a_0 act on $\mathbb{P}(\mathbb{V})$ with the same fixed points. Moreover $\mathrm{disc}(a_0) = \mathrm{disc}(A)/(\mathrm{tr}\, A)^2 = 4\delta(A)$ thus $\det(a_0) = -\delta(A)$.

The normalised projection commutes with the involution #, in other terms inverting A changes a_0 in its opposite.

Remark 1.49. *Note that a semi-simple non involutive $A \in \mathrm{PGL}(\mathbb{V})$ has always exactly two lifts in $\pm A \in \mathrm{SL}(\mathbb{V})$ defined over $\mathbb{K}[\sqrt{\det(A)}]$.*

Consider semi-simple non-involutive $A, B \in \mathrm{PGL}(\mathbb{V})$ and a choice of $\sqrt{\delta(A)\delta(B)}$. Denote $a_0, b_0 \in \mathfrak{sl}(\mathbb{V})$ their normalised projections. Since $\det(a_0) = -\delta(A)$ and $\det(b_0) = -\delta(B)$ we make opposite choices for $d = \sqrt{\det(ab)} = -\sqrt{\delta(A)\delta(B)} = -\delta$. We may thus define:

$$\mathrm{bir}(A, B) = \mathrm{bir}(a_0, b_0) \qquad \text{and} \qquad \cos(A, B) = \cos(a_0, b_0)$$

The cross-ratio satisfies the relations $\mathrm{bir}(A, B) = \mathrm{bir}(B, A) = \mathrm{bir}(A^{-1}, B^{-1})$ and the addition rule:

$$\frac{1}{\mathrm{bir}(A, B)} + \frac{1}{\mathrm{bir}(A, B^{-1})} = 1$$

Corollary 1.50. *Consider semi-simple non involutive* $A, B \in \mathrm{PGL}(\mathbb{V})$ *and a choice for* $\delta = \sqrt{\delta(A)\delta(B)}$. *Denoting* a_0 *and* b_0 *their normalised projections, we have:*

$$\frac{1}{\mathrm{bir}(A,B)} = \frac{1+\cos(A,B)}{2} = \frac{\delta - \langle a_0, b_0\rangle}{2\delta}$$

Remark 1.51. *We may use Corollary 1.15 to express the cosine after choosing lifts* $A, B \in \mathrm{GL}(\mathbb{V})$ *and the square root* $\sqrt{\mathrm{disc}(A)\,\mathrm{disc}(B)} = \mathrm{Tr}(A)\,\mathrm{Tr}(B)\sqrt{\delta(A)\delta(B)}$ *as:*

$$\cos(A,B) = \frac{\mathrm{Tr}(AB) - \mathrm{Tr}(AB^{\#})}{\sqrt{\mathrm{disc}(A)\,\mathrm{disc}(B)}} = \frac{\mathrm{disc}(AB) - \mathrm{disc}(AB^{\#})}{\mathrm{Tr}(A)\,\mathrm{Tr}(B)\sqrt{\mathrm{disc}(A)\,\mathrm{disc}(B)}}$$

Loosely speaking, the cosine $\cos(A,B)$ *measures the parallelism between the one-parameter subgroups of* $\mathrm{PGL}(\mathbb{V})$ *containing* A *and* B.

1.4 Adjoint action of $\mathrm{PSL}_2(\mathbb{K})$ on $\mathfrak{sl}_2(\mathbb{K})$

Groups of units and their conjugacy classes

The central quotient $\mathrm{GL}(\mathbb{V}) \to \mathrm{PGL}(\mathbb{V})$ and the determinant $\mathrm{GL}(\mathbb{V}) \to \mathbb{K}^\times$ fit into the following commutative diagram in which both columns and lines are short exact sequences (we omitted the trivial groups for presentation purposes).

$$\begin{array}{ccccc} \{\pm\mathbb{1}\} & \longrightarrow & \mathbb{K}^\times\mathbb{1} & \longrightarrow & (\mathbb{K}^\times)^2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{SL}(\mathbb{V}) & \longrightarrow & \mathrm{GL}(\mathbb{V}) & \longrightarrow & \mathbb{K}^\times \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{PSL}(\mathbb{V}) & \longrightarrow & \mathrm{PGL}(\mathbb{V}) & \longrightarrow & \mathbb{K}^\times/(\mathbb{K}^\times)^2 \end{array}$$

Let us recall the description of conjugacy classes in $\mathrm{GL}_2(\mathbb{K})$ and $\mathrm{PGL}_2(\mathbb{K})$, starting with the classical invariant theory of $\mathfrak{gl}_n(\mathbb{K})$ under the action of $\mathrm{GL}_n(\mathbb{K})$.

First recall that finite type modules over the principal ring $\mathbb{K}[X]$ are classified up to isomorphism by their unique invariant form: $\mathbb{K}[X]/(d_1) \oplus \cdots \oplus \mathbb{K}[X]/(d_l)$ where the so called invariant factors $d_k \in \mathbb{K}[X]$ are unitary, and satisfy $d_1 \mid \cdots \mid d_l$.

An endomorphism $M \in \mathfrak{gl}_n(\mathbb{K})$ endows the vector space $\mathbb{K}^n$ with the structure of a $\mathbb{K}[X]$-module. In this $\mathrm{GL}_n(\mathbb{K})$-equivariant correspondence, the characteristic polynomial of M is the order of the $\mathbb{K}[X]$-module, that is the product of all invariant factors, while its minimal polynomial is the exponent of the $\mathbb{K}[X]$-module, that is the last invariant factor.

In particular, conjugacy classes of semi-simple matrices in $\mathrm{GL}_2(\mathbb{K})$ correspond to equivalence classes of semi-simple torsion $\mathbb{K}[X]$-modules with length $l = 1$ (since the order, which is a degree two polynomial, must equal the exponent), so they are uniquely characterised by the determinant and trace.

Proposition 1.52. *The semi-simple conjugacy classes in* $\mathrm{PGL}(\mathbb{V})$ *are parametrized by the well defined quantity* $A \mapsto \mathrm{disc}(A)/\det(A) \in \mathbb{K}$.

We will recover this fact by studying the adjoint action of $\mathrm{PGL}(\mathbb{V})$ on $\mathfrak{sl}(\mathbb{V})$, without needing the classification of finite type modules over a principal ideal ring.

Moreover, we shall describe the conjugacy classes in $\mathrm{PSL}(\mathbb{V})$, while the classification of finite type modules over a principal ideal ring says nothing about those.

Adjoint action

We saw that the left adjoint linear action of $\mathrm{GL}(\mathbb{V})$ on $\mathfrak{gl}(\mathbb{V})$ preserves the involution, thus every structure which derives from it. It preserves in particular the restriction of the determinant form to the kernel $\mathfrak{sl}(\mathbb{V})$ of trace, hence its level sets like the *isotropic cone* $\mathbb{X}$ and the *unit hyperboloid* $\mathbb{H}$:

$$\mathbb{X} = \mathfrak{sl}(\mathbb{V}) \cap \{\det = 0\} \qquad \mathbb{H} = \mathfrak{sl}(\mathbb{V}) \cap \{\det = 1\}.$$

Remark 1.53. *The adjoint action* $\mathrm{GL}(\mathbb{V}) \to \mathrm{End}(\mathfrak{sl}(\mathbb{V}))$ *is the composition of left multiplication and right multiplication by the inverse, and these operations are conjugate (by the right adjoint action). In particular the adjoint action preserves the orientations of* $\mathfrak{sl}(\mathbb{V})$ *defined for a basis as the class of its determinant in* $\mathbb{K}^\times/(\mathbb{K}^\times)^2$.

Only the scalar matrices act trivially, and the maximal subspace on which the action is trivial equals $\mathbb{K}\mathbb{1}$. Therefore no information is lost after quotienting by these centers, and this yields a faithful representation $\mathrm{PGL}_2(\mathbb{K}) \to \mathrm{SO}(\mathfrak{sl}_2(\mathbb{K}), \det)$ into the group of orientation preserving isometries of $(\mathfrak{sl}_2(\mathbb{K}), \det)$.

Proposition 1.54. *For a plane* $\mathbb{V}$ *over a field* $\mathbb{K}$ *of characteristic different from* 2*, the adjoint action yields an isomorphism* $\mathrm{PGL}(\mathbb{V}) \to \mathrm{SO}(\mathfrak{sl}(\mathbb{V}), \det)$.

Proof. To prove surjectivity, we use a theorem of Cartan-Dieudonné [Die71]. It states that every isometry of a symmetric non-degenerate bilinear form over a d-dimensional $\mathbb{K}$-vector space is a product of at most d reflections. In particular, an element of $\mathrm{SO}(\mathfrak{sl}_2(\mathbb{K}), \det)$ is a product of at most 3 reflections, but since it has determinant 1 it is in fact a product of exactly two reflections. Thus we must express all products of two reflections as the conjugacy for some element.

If $q \in \mathfrak{gl}(\mathbb{V})$ is not isotropic, that is $\det(q) \neq 0$, then the orthogonal reflection $\sigma_q \in \mathrm{End}(\mathfrak{gl}(\mathbb{V}))$ of vector q across $q^\perp$ is given by:

$$\sigma_q(m) = m - 2\frac{\langle q, m\rangle}{\langle q, q\rangle} \cdot q$$

Notice that the orthogonal reflection of vector $\mathbb{1}$ across $\mathfrak{sl}(\mathbb{V})$ equals $\sigma_\mathbb{1} \colon m \mapsto -m^\#$. The endomorphism $\mu_q \in \mathrm{End}(\mathfrak{gl}(\mathbb{V}))$ corresponding to left multiplication by q, left conjugates $\sigma_\mathbb{1}$ to σ_q. In formulae, we have $\mu_q \colon m \mapsto qm$ and $\sigma_q = \mu_q \circ \sigma_\mathbb{1} \circ \mu_{q^{-1}}$. Thus

$$\sigma_q(m) = -q(qm)^\# = -\frac{qm^\# q}{\det(q)}$$

We now restrict our attention to $\mathrm{End}(\mathfrak{sl}(\mathbb{V}))$ and notice that for $q, m \in \mathfrak{sl}(\mathbb{V})$ this formula becomes $\sigma_q(m) = -qmq^{-1}$. Hence for $p, q \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ the reflection $\sigma_p \circ \sigma_q \in \mathrm{SO}(\mathfrak{sl}(\mathbb{V}), \det)$ coincides with the left adjoint action of $pq \in \mathrm{GL}(\mathbb{V})$. □

The adjoint action commutes with the projectivization map $\mathfrak{sl}(\mathbb{V}) \to \mathbb{P}(\mathfrak{sl}(\mathbb{V}))$. This realizes $\mathrm{PGL}(\mathbb{V})$ as a subgroup inside the automorphism group $\mathrm{PGL}(\mathfrak{sl}(\mathbb{V}))$ of the projective plane $\mathbb{P}(\mathfrak{sl}(\mathbb{V}))$, namely the stabiliser of the non-degenerate conic $\mathbb{P}(\mathbb{X})$.

The actions of $\mathrm{PGL}(\mathbb{V})$ and $\mathrm{PSL}(\mathbb{V})$ on $\mathbb{P}(\mathbb{X})$ follow from Corollary 1.37 & 1.39.

Action of A on $(\mathrm{pr}\, A)^\perp$

Let us describe the adjoint action of a non scalar element $A \in \mathrm{GL}(\mathbb{V})$ on $\mathfrak{sl}(\mathbb{V})$, which only depends on its class in $\mathrm{PGL}(\mathbb{V})$. It fixes the line through $\mathrm{pr}(A)$ on which it acts identically, and stabilizes the plane $\mathrm{pr}(A)^\perp$. Restricted to that orthogonal plane, it stabilizes the degenerate conic section $(\mathrm{pr}\, A)^\perp \cap \mathbb{X}$ as well as the quadric section $(\mathrm{pr}\, A)^\perp \cap \mathbb{H}$, the former being asymptotic to the latter.

If $\mathrm{disc}(A) = 0$, then $\mathrm{pr}\, A$ is an isotropic vector for the determinant and $\mathrm{pr}(A)^\perp$ is tangent to the cone $\mathbb{X}$ along the line $\mathbb{K} \cdot \mathrm{pr}(A)$, hence the conic section is a degenerate double line while the quadric section is empty. Otherwize $\mathrm{disc}(A) \neq 0$, and the shape of the conic and quadric sections depend on the class of $\mathrm{disc}(A)$ in $\mathbb{K}^\times/(\mathbb{K}^\times)^2$.

Corollary 1.55. *For $A \in \mathrm{SL}(\mathbb{V})$, the adjoint action of A on $\mathfrak{sl}(\mathbb{V})$ restricted to the plane $(\mathrm{pr}\, A)^\perp$ is equivalent over $\mathbb{K}$ to the tautological action of A^2 on $\mathbb{V}$.*

Remark 1.56. *The hypothesis that $A \in \mathrm{SL}(\mathbb{V})$ is important when considering the tautological action on $\mathbb{V}$.*

Proof 1. The line through $(\mathrm{pr}\, A)$ and its orthogonal plane $(\mathrm{pr}\, A)^\perp$ project to a point and a line in $\mathbb{P}(\mathfrak{sl}(\mathbb{V}))$ which are polars to one another with respect to the conic $\mathbb{P}(\mathbb{X})$ and are both fixed by the adjoint action of A.

Consider the map $\mathbb{P}(\mathbb{X}) \to \mathbb{P}((\mathrm{pr}\, A)^\perp)$ sending p to $p^\perp \cap (\mathrm{pr}\, A)^\perp$, as depicted in Figure 1.8. Being projective, it conjugates the adjoint actions of $\mathrm{PSL}(\mathbb{V})$ restricted to $\mathbb{P}(\mathbb{X})$ and restricted to $\mathbb{P}((\mathrm{pr}\, A)^\perp)$. Notice that it is two-to-one if $\mathrm{pr}\, A \notin \mathbb{X}$, but one-to-one if $\mathrm{pr}\, A \in \mathbb{X}$ in which case $\mathrm{disc}(A) = 0$ and A^2 is conjugate to A in $\mathrm{PGL}(\mathbb{V})$.

Precomposing with the quadratic map ψ, and the projectivization, we obtain a map $\mathbb{V} \to \mathbb{X}_\omega \to \mathbb{P}(\mathbb{X}) \to \mathbb{P}((\mathrm{pr}\, A)^\perp)$ which admits a unique lift $\mathbb{V} \to (\mathrm{pr}\, A)^\perp$ conjugating the tautological action of A^2 to the adjoint action of A. □

Proof 2. The Lemma 1.33 is valid over field of characteristic different from 2, so one may extend the scalars to any $\mathbb{K}$-algebra, including $\mathbb{K}$ and its quadratic closure $\sqrt{\mathbb{K}}$.

Over $\sqrt{\mathbb{K}}$, the fixed subspaces for the adjoint action of A on the plane $(\mathrm{pr}\, A)^\perp$ are precisely those obtained by intersecting it with $\mathbb{X}$. From Lemma 1.33, these eigenspaces pull back by $\psi \otimes \sqrt{\mathbb{K}}$ to those of A acting on the plane $\mathbb{V} \otimes \sqrt{\mathbb{K}}$, and

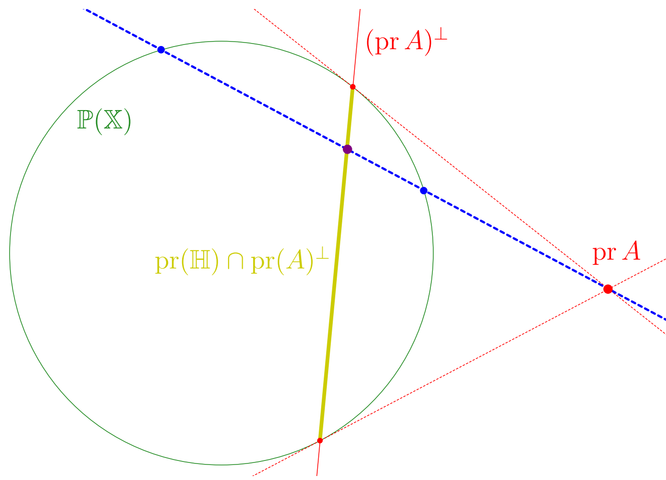


Figure 1.8: Projecting the conic $\mathbb{P}(\mathbb{X})$ onto the open segment $\mathrm{pr}(A)^{\perp} \cap \mathrm{pr}(\mathbb{H})$.

since ψ is quadratic the corresponding eigenvalues for the adjoint action are the squares of those for the tautological action.

Thus the adjoint action of A on $(\mathrm{pr}\, A)^{\perp}$ endows the later with the structure of a $\sqrt{\mathbb{K}}[A^2]$-module. Since A^2 is defined over $\mathbb{K}$, one may restrict the scalars to that field and obtain equivalent $\mathbb{K}[A^2]$-modules, and this proves the corollary. $\square$

Symmetries

We call *symmetry* of $\mathrm{PGL}(\mathbb{V})$ an element of order two (since it maps to an orthogonal symmetry in $\mathrm{SO}(\mathfrak{sl}(\mathbb{V}), \det)$ under the adjoint action).

A symmetry of $\mathrm{PGL}(\mathbb{V})$ is represented by a non-scalar $s \in \mathrm{GL}(\mathbb{V})$ such that s^2 belongs to the center, say $s^2 = \lambda \mathbf{1}$. This implies that $\det(s)^2 = \lambda^2$ so $\det(s) = \pm\lambda$, and dividing by s yields $s^{\#} = \pm s$, whence $s \in \mathfrak{sl}(\mathbb{V})$. Conversely, only those elements in $\mathrm{GL}(\mathbb{V}) \cap \mathfrak{sl}(\mathbb{V})$ represent symmetries of $\mathrm{PGL}(\mathbb{V})$.

Notice that $\mathrm{GL}(\mathbb{V}) \cap \mathfrak{sl}(\mathbb{V}) = \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$, so the symmetries of $\mathrm{PGL}(\mathbb{V})$ correspond by the projectivisation map to the complement $\mathbb{P}(\mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X})$ of the projective conic. This is an open projective variety whose irreducible components over $\mathbb{K}$ are indexed by the values of $\det\colon \mathrm{GL}(\mathbb{V}) \cap \mathfrak{sl}(\mathbb{V}) \to \mathbb{K}^{\times}/(\mathbb{K}^{\times})^2$.

We call this variety $\mathbb{P}(\mathfrak{sl}(\mathbb{X}) \setminus \mathbb{X})$ the *symmetric space* of $\mathrm{PGL}(\mathbb{V})$. Hence the group $\mathrm{PGL}(\mathbb{V})$ acts on its symmetric space $\mathbb{P}(\mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X})$ by the projectivised adjoint representation, and the elements of order two are the symmetries.

Since $s \in \mathrm{GL}(\mathbb{V}) \cap \mathfrak{sl}(\mathbb{V})$ maps to an element of order two in $\mathrm{SO}(\mathfrak{sl}(\mathbb{V}), \det)$ which fixes the line $\mathbb{K}s$, it acts like minus the identity on the orthogonal plane $s^{\perp}$, which may be called an orthogonal symmetry across the line $\mathbb{K}s$. In formula:

$$\forall x \in \mathfrak{sl}(\mathbb{V}) : \qquad sxs^{-1} + x = 2\frac{\langle s, x\rangle}{\langle s, s\rangle} \cdot s$$

and one may recognise from the proof of Proposition 1.54, the expression for the composition of reflections $\sigma_s \circ \sigma_{\mathbb{1}} \in \mathrm{SO}(\mathfrak{gl}(\mathbb{V}), \det)$ restricted to $\mathfrak{sl}(\mathbb{V})$.

The elements S, J, K of our favourite basis for $\mathfrak{sl}_2(\mathbb{K})$ have order two in $\mathrm{PGL}_2(\mathbb{K})$, so they are symmetries. Being orthogonal in $\mathfrak{sl}_2(\mathbb{K})$, each one conjugates the others in their opposites. In other terms the projective classes of S, J, K in $\mathbb{P}(\mathfrak{sl}_2(\mathbb{K}) \setminus \mathbb{X})$ are fixed under the adjoint actions of $S, J, K \in \mathrm{PGL}_2(\mathbb{K})$.

Acting on the symmetric space $\mathbb{P}(\mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X})$

The following proposition is the key to understanding the action of the group $\mathrm{PGL}(\mathbb{V})$ on its symmetric space $\mathbb{P}(\mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X})$ and will be used extensively in the following chapters, in particular to derive Propositions 1.82, 1.89 and 5.4.

Recall the Definition 1.44 for the cross-ratio $\mathrm{bir}(a, b)$ of $a, b \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ with the same determinant d and the Proposition 1.46 expressing it in terms of $\langle a, b\rangle/d$.

Proposition 1.57. *Consider distinct $a, b \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ with the same determinant $d \neq 0$ such that $d + \langle a, b\rangle \neq 0$ and let us work over the extension $\mathbb{K}' = \mathbb{K}[\sqrt{\mathrm{bir}(a, b)}]$.*

There exists a unique pair of opposite elements $\pm C$ with determinant 1 in the quadratic subalgebra $\mathbb{K}'[\{a, b\}]$ of $\mathfrak{gl}(\mathbb{V} \otimes \mathbb{K}')$ which conjugate a to b. They are:

$$C = \frac{1}{\sqrt{\mathrm{bir}(a, b)}} \cdot \mathbb{1} + \frac{\sqrt{\mathrm{bir}(a, b)}}{2} \cdot \frac{\{a, b\}}{d} = \frac{(d + \langle a, b\rangle)\mathbb{1} + \{a, b\}}{\sqrt{2d(d + \langle a, b\rangle)}} \tag{1.12}$$

There exists a unique element M in the quadratic subalgebra $\mathbb{K}[\{a, b\}]$ of $\mathfrak{gl}(\mathbb{V})$ with $\mathrm{tr}(M) = 1/\mathrm{bir}(a, b)$ which conjugates a to b, it is given by:

$$M = \frac{C}{\sqrt{\mathrm{bir}(a, b)}} = \frac{(d + \langle a, b\rangle)\mathbb{1} + \{a, b\}}{2d} \qquad \text{and} \quad \det(M) = \frac{1}{\mathrm{bir}(a, b)}.$$

Idea of the Proof. The computational proof translates the geometrical reasoning suggested (over the real field) in Figure 1.9, and relies on the following observations.

The plane through $a, b \in \mathfrak{sl}(\mathbb{V})$ is $\{a, b\}^{\perp}$ and it intersects the quadric $\{\det = d\}$ in a conic. (The projective line $\mathbb{P}(a, b) \subset \mathbb{P}(\mathfrak{sl}(\mathbb{V}))$ is polar to the point $\mathbb{P}(\{a, b\})$.)

For the adjoint action of $\mathrm{SL}(\mathbb{V})$ on $\mathfrak{sl}(\mathbb{V})$ the stabiliser of $\{a, b\}^{\perp}$ is the stabiliser of $\mathbb{K}\{a, b\}$. This contains $\mathbb{K}[\{a, b\}] \cap \mathrm{SL}(\mathbb{V})$ which forms a conic in $\mathrm{Span}(\mathbb{1}, \{a, b\})$, consisting of all matrices $C = t\mathbb{1} + u\{a, b\}$ such that $t^2 + u^2 \det(\{a, b\}) = 1$.

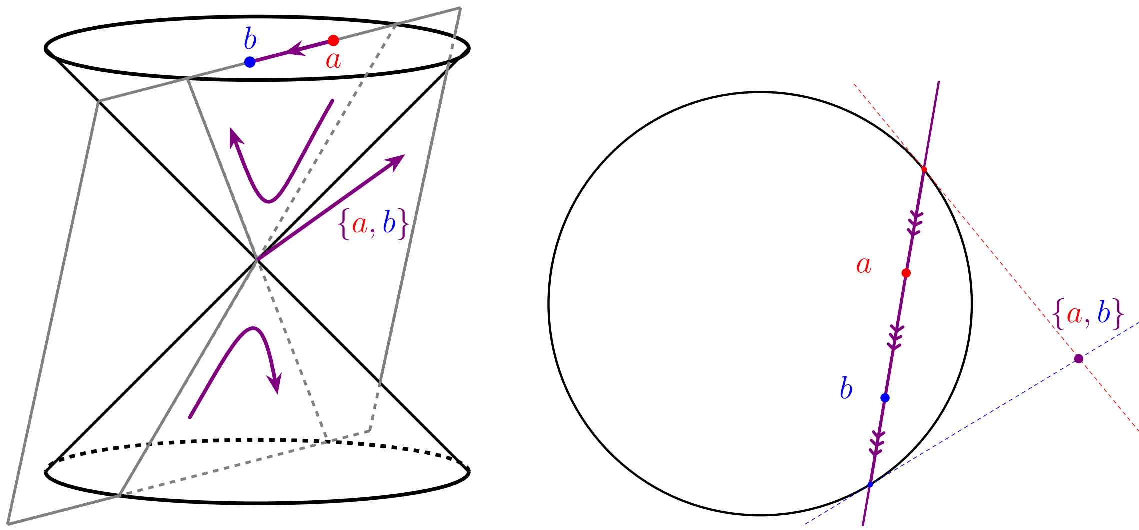


Figure 1.9: The one parameter group generated by $\{a, b\}$, which is contained in $\mathrm{Span}(\mathbf{1}, \{a, b\})$, acts by translation along the line (a, b).

If C maps a to b then $(\operatorname{tr} C)^2$ is determined by the distance between a and b, that is by $\langle a, b\rangle$. Hence, after a quadratic extension of the field, there is up to sign a unique element $C \in \mathrm{SL}(\mathbb{V})$ preserving the plane (a, b) and translating a to b. □

Proof. Suppose first that $d = 1$ so $a, b \in \mathbb{H}$. Let $x, y \in \mathbb{K}$ and $C = t + u\{a, b\}$. It has $\det(C) = t^2 + u^2 \det(\{a, b\})$. Since $ab \in \mathbb{H}$ we have $1 = \det(ab) = \langle a, b\rangle^2 + \det(\{a, b\})$ so $\det(\{a, b\}) = 1 - \langle a, b\rangle^2$. Consequently $\det(C) = t^2 + u^2(1 - \langle a, b\rangle^2)$ and we find:

$$C = t + u\{a, b\} \in \mathrm{SL}(\mathbb{V}) \iff t^2 + u^2(1 - \langle a, b\rangle^2) = 1$$

Now $Ca = bC \iff ta + u\{a, b\}a = tb + ub\{a, b\}$. But $\{a, b\}a = \frac{1}{2}(aba + b)$, and since $a \in \mathbb{H}$ acts like a symmetry across $\mathbb{K}a$, we have $aba = -aba^{-1} = b - 2\langle a, b\rangle a$ so $\{a, b\}a = b - \langle a, b\rangle a$. Similarly $b\{a, b\} = a - \langle a, b\rangle b$. Thus $Ca = bC$ is equivalent to $(a - b)(t - u(1 + \langle a, b\rangle)) = 0$ and since $a - b \neq 0$ we find

$$Ca = bC \iff t = u(1 + \langle a, b\rangle)$$

The conditions $C \in \mathrm{SL}(\mathbb{V})$ and $Ca = bC$ translate into a system of equations in (t, u) which has no solutions if $1 + \langle a, b\rangle = 0$ and otherwise it has a unique solution $\pm(t, u)$ up to simultaneous change of sign of the entries, given by:

$$t = \sqrt{\frac{1 + \langle a, b\rangle}{2}} = \frac{1}{\sqrt{\mathrm{bir}(a, b)}} \qquad u = \frac{1}{\sqrt{2 + 2\langle a, b\rangle}} = \frac{\sqrt{\mathrm{bir}(a, b)}}{2}$$

This proves the Lemma for $d = 1$.

Now suppose $a, b \in \mathfrak{sl}(\mathbb{V})$ have the same determinant $d \neq 0$. Divide them by $\sqrt{d}$, which may live in another quadratic extension of $\mathbb{K}$, to get $a', b' \in \mathbb{H}$ as before. Since $\langle a, b\rangle/d = \langle a', b'\rangle$ and $\{a, b\}/d = \{a', d'\}$ we have $\mathbb{K}'[\{a, b\}] = \mathbb{K}'[\{a, b\}]$.

Now for $C \in \mathbb{K}'[\{a, b\}]^\times$ an invertible element of this quadratic algebra, we have $Ca = bC \iff Ca' = b'C$ which completes the proof for all d. □

Remark 1.58. *Notice that, using the notations of Lemma 1.57, we have:*

$$dC^2 = \langle a, b\rangle + \{a, b\} = -ba \qquad \text{thus} \qquad C^2 = ba^{-1}.$$

Consequently $\pm C$ are the unique square roots of the product of symmetries ba^{-1} in the extended quadratic algebra $\mathbb{K}'[\{a, b\}]$.

Proof. Since $\det(ab) = \langle a, b\rangle^2 + \det\{a, b\}^2$, we have $\{a, b\}^2 = -\det\{a, b\} = \langle a, b\rangle^2 - d^2$ thus using the right hand side of identity 1.12 we develop:

$$C^2 = \frac{d + \langle a, b\rangle}{2d} + \frac{\{a, b\}}{d} + \frac{\langle a, b\rangle - d}{2d} = \frac{\langle a, b\rangle + \{a, b\}}{d}$$

Then use $ba = -\langle a, b\rangle - \{a, b\}$ and $a^{-1} = a^{\#}/d = -a/d$ to rewrite this $C^2 = ba^{-1}$. □

Scholium 1.59. *If we suppose that $\mathbb{K} = \mathbb{R}$ there are two cases of interest depending on the class of $d \in \mathbb{R}^*/(\mathbb{R}^*)^2$, which will be interpreted geometrically.*

If $d = 1$, we may write $\operatorname{tr}(C) = \cosh(\lambda)$ so that $\langle a, b\rangle = k = \cosh(2\lambda)$.

If $d = -1$, we may write $\operatorname{tr}(C) = \cos(\theta)$ so that $-\langle a, b\rangle = dk = \cos(2\theta)$.

Let us reap our first fruit from (the second part of) Lemma 1.57. This will enable us to describe the conjugacy classes in $\mathrm{PGL}(\mathbb{V})$.

Corollary 1.60. *Over a field $\mathbb{K}$ if characteristic different from 2, the group $\mathrm{PGL}_2(\mathbb{K})$ acts transitively on each irreducible component of $\mathbb{P}(\mathfrak{sl}_2(\mathbb{K}) \setminus \mathbb{X})$.*

Proof. Consider two elements of $\mathbb{P}(\mathfrak{sl}_2(\mathbb{K}) \setminus \mathbb{X})$ which lift to $a, b \in \mathfrak{sl}_2(\mathbb{K})$ with the same determinant $d \neq 0$. Multiplying b by -1 preserves its projective class and its determinant d, while changing $\langle a, b\rangle$ in its opposite so we may suppose $d + \langle a, b\rangle \neq 0$. Lemma 1.57 constructs $C \in \mathrm{PGL}_2(\mathbb{K})$ conjugating a to b. □

Corollary 1.61. *The semi-simple conjugacy classes in $\mathrm{PGL}_2(\mathbb{K})$ are classified by the value of $1/\delta(A) = (\operatorname{Tr} A)^2/\operatorname{disc}(A) \in \mathbb{K}$ which is 0 for the class of involutions.*

Now we focus on the action of $\mathrm{PSL}_2(\mathbb{K})$ on the components of $\mathbb{P}(\mathfrak{sl}_2(\mathbb{K}) \setminus \mathbb{X})$. There is a class of fields $\mathbb{K}$ over which we may immediately deduce the transitivity: those which are closed under square roots.

Corollary 1.62. *If* $\mathbb{K}^\times/(\mathbb{K}^\times)^2 = \{1\}$ *then* $\mathrm{PSL}_2(\mathbb{K})$ *acts transitively on* $\mathbb{P}(\mathfrak{sl}_2(\mathbb{K}) \setminus \mathbb{X})$.

The most general examples of such fields are obtained from an arbitrary given field with characteristic different from 2 by taking its universal quadratic closure: starting with $\mathbb{Q}$ we obtain the venerable field of numbers constructible by ruler and compas. Other examples include algebraically closed fields.

Remark 1.63. *We noticed in Remark 1.8 that* J *and* K *are conjugate by* $C \in \mathrm{GL}_2(\mathbb{K})$ *if and only if:*

$$C(x,y) = \begin{pmatrix} x & -x \\ y & y \end{pmatrix} \qquad \text{for} \quad x, y \in \mathbb{K} \quad \text{with} \quad \det(C) = 2xy \neq 0.$$

Applying Lemma 1.57 to J, K *of determinant* -1 *with* $\langle J, K\rangle = 0$ *and* $\{J, K\} = -S$ *yields* $C = \frac{1}{\sqrt{2}} C(1,1) \in \mathrm{SL}_2(\mathbb{K}[\sqrt{2}])$.

Yet, one may avoid the need of extending the field by choosing any $x, y \in \mathbb{K}$ *such that* $2xy = 1$*, for instance* $C(1/2, 1) \in \mathrm{SL}_2(\mathbb{K})$ *is valid over all* $\mathbb{K}$*. Notice that*

$$C(1/2, 1) = \tfrac{3}{4}(\mathbb{1} + S) + \tfrac{1}{4}(J + K)$$

has a balanced decomposition according to $\mathbb{K}[\{J, K\}] \oplus \{J, K\}^\perp$*, the summands being respectively the commutator and the anti-commutator of* $\{J, K\}$ *in* $\mathrm{GL}(\mathbb{V})$ *by 1.24.*

Consequently, Corollary 1.62 which only focuses on the transitivity of the action, is unsatisfying. The reason is that Lemma 1.57 only searches $C \in \mathbb{K}[\{a, b\}]$.

Theorem 1.64. *Consider* $a, b \in \mathbb{H}$ *such that* $\det\{a, b\} \neq 0$*, that is* $\mathrm{bir}(a, b) \notin \{0, \infty\}$*. The elements* $C \in \mathrm{SL}(\mathbb{V})$ *conjugating* a *to* b *correspond to the pairs* $(x, y) \in \mathbb{K} \times \mathbb{K}$ *such that* $\mathrm{bir}(a, b) = 4(x^2 + y^2)$ *by the formula:*

$$C(x, y) = x(\mathbb{1} - ba) + y(a + b) \tag{1.13}$$

In particular, a *and* b *are conjugate by an element of* $\mathrm{PSL}(\mathbb{V})$ *defined over* $\mathbb{K}$ *if and only if* $\mathrm{bir}(a, b)$ *is a sum of squares of two elements in* $\mathbb{K}$.

Proof. Since $\det\{a, b\} \neq 0$ the elements $\mathbb{1}, a, b, \{a, b\}$ form a basis of $\mathfrak{gl}(\mathbb{V})$. Notice moreover that the planes $\mathbb{K}[\{a, b\}]$ and $\mathrm{Span}(a, b)$ are orthogonal. Let $C \in \mathfrak{gl}(\mathbb{V})$ decomposed as $C = t\mathbb{1} + x\{a, b\} + ya + zb$ for $t, x, y, z \in \mathbb{K}$.

The condition $Ca = bC$ can be rewritten using $a^2 = -\mathbb{1} = b^2$ as well as $\{a,b\}a = b - \langle a,b\rangle a$ and $b\{a,b\} = a - \langle a,b\rangle b$. After grouping terms according to the basis $(\mathbb{1}, a, b, ab)$ we find:

$$Ca = bC \iff (t - x(1 + \langle a,b\rangle)) \cdot (a - b) + (z - y) \cdot (\mathbb{1} + ba) = 0$$

But $ba = -\langle a,b\rangle - \{a,b\} \in \mathbb{K}[\{a,b\}] \setminus \{0\}$ and $b - a \in \mathrm{Span}(a,b) \setminus \{0\}$ so by orthogonality of these planes we have $Ca = bC \iff t = x(1 + \langle a,b\rangle)$ & $y = z$.

Now for $x, y \in \mathbb{K}$ the determinant of $C = x(\mathbb{1} - ba) + y(a + b)$ can be computed using the orthogonality of $\mathrm{Span}(\mathbb{1}, ab)$ and $\mathrm{Span}(a,b)$ and the hypothesis $a, b \in \mathbb{H}$:

$$\det(C) = (x^2 + y^2) \cdot (2 + 2\langle a,b\rangle) = \frac{4(x^2 + y^2)}{\mathrm{bir}(a,b)}$$

so $C \in \mathrm{SL}(\mathbb{V}) \iff \mathrm{bir}(a,b) = (2x)^2 + (2y)^2$. □

Remark 1.65. *The expression obtained for $y = 0$ recovers an alternative expression of 1.12 which is valid for all $a, b \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ with the same determinant, namely:*

$$C = \tfrac{1}{2}\sqrt{\mathrm{bir}(a,b)}\left(\mathbb{1} + ba^{-1}\right).$$

Corollary 1.66. *Consider $a, b \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ defined over $\mathbb{K}$, of the same determinant d, such that $\det\{a,b\} \neq 0$. The elements $C \in \mathrm{SL}(\mathbb{V})$ defined over $\mathbb{K}$ which conjugate a to b correspond to the pairs $(x,y) \in \mathbb{K} \times \mathbb{K}$ such that:*

$$x^2 + dy^2 = \tfrac{1}{4}\,\mathrm{bir}(a,b) \qquad \text{by} \qquad C = x(\mathbb{1} + ba^{-1}) + y(a + b)$$

In particular, a and b are conjugate by an element of $\mathrm{PSL}(\mathbb{V})$ defined over $\mathbb{K}$ if and only if the Pell-Fermat equation $x^2 + dy^2 = \frac{1}{4}\,\mathrm{bir}(a,b)$ has a solution in $\mathbb{K}$.

Proof. The points $a' = a/\sqrt{d}$ and $b' = b/\sqrt{d}$ satisfy $\mathrm{bir}(a,b) = \mathrm{bir}(a',b')$ and the endomorphisms $C \in \mathrm{SL}(\mathbb{V})$ defined over $\mathbb{K}' = \mathbb{K}[\sqrt{d}]$ conjugating a to b are the same as those conjugating a' to b'.

By Theorem 1.64 these elements C correspond to the pairs $(x,y) \in \mathbb{K}' \times \mathbb{K}'$ such that $\mathrm{bir}(a,b) = 4(x'^2 + y'^2)$ by the formula $C = x'(\mathbb{1} + b'a'^{-1}) + y'(a' + b')$.

Setting $x = x'$ and $y = y'\frac{1}{\sqrt{d}}$ which satisfy $4(x^2 + dy^2) = \mathrm{bir}(a,b)$, we may rewrite:

$$C = x(\mathbb{1} + ba^{-1}) + y(a + b)$$

Now recall that a and b are defined over $\mathbb{K}$ and that $a + b \neq 0$ is orthogonal to $\mathbb{1} + ba^{-1} \neq 0$. Hence C is defined over $\mathbb{K}$ if and only if $x, y \in \mathbb{K}$. □

Remark 1.67. *If we are given $C \in \mathrm{SL}_2(\mathbb{K})$ conjugating a to b then the quickest way of computing its decomposition $C = x(\mathbb{1} + ba^{-1}) + y(a + b)$ is to notice that:*

$$x = \mathrm{Tr}(C)\tfrac{1}{4}\,\mathrm{bir}(a, b)$$

and then solve $x^2 + dy^2 = \frac{1}{4}\,\mathrm{bir}(a, b)$.

By definition, the Hilbert symbol $(\delta, \chi)_{\mathbb{K}}$ of $\delta, \chi \in \mathbb{K}^\times$ takes the value 1 or -1 according to whether the equation $x^2 = \delta y^2 + \chi z^2$ admits a solution in $\mathbb{KP}^2$ or not. Thus we have $(\delta, \chi)_{\mathbb{K}} = 1$ if and only if χ is the norm of an element in $\mathbb{K}(\sqrt{\delta})$.

Scholium 1.68 (Transitivity)**.** *Let $a, b, b \in \mathfrak{sl}_2(\mathbb{K}) \setminus \mathbb{X}$ have the same discriminant $\Delta \neq 0$ and be such that $\mathrm{bir}(a, b)$, $\mathrm{bir}(b, b)$ and $\mathrm{bir}(b, a)$ do not belong to $\{0, \infty\}$.*

Corollary 1.66 implies that if $\mathrm{bir}(a, b)$ and $\mathrm{bir}(b, b)$ are represented by the form $(2x)^2 - \Delta y^2$, then so is $\mathrm{bir}(a, b)$. This can be written using the Hilbert symbol as:

$$(\Delta, \mathrm{bir}(b, b))_{\mathbb{K}} = 1 \implies (\Delta, \mathrm{bir}(a, b))_{\mathbb{K}} = (\Delta, \mathrm{bir}(b, a))_{\mathbb{K}}$$

If we express the cross-ratio in terms of the fixed points using formula bir, we find that $\mathrm{bir}(a, b)\,\mathrm{bir}(b, b)\,\mathrm{bir}(a, b)$ equals $-\Delta$ times the norm of an element $\mathbb{K}(\sqrt{\Delta})$. This can be rewritten in terms of the Hilbert symbol as:

$$(\Delta, \mathrm{bir}(a, b)\,\mathrm{bir}(b, b)\,\mathrm{bir}(a, b))_{\mathbb{K}} = 1$$

We shall say more about the Hilbert symbol and this identity in the next section.

Remark 1.69. *Theorem 1.64 and its Corollary 1.66 provide a first step towards describing the orbits for the action on $\mathrm{PSL}(\mathbb{V})$ on the symmetric space $\mathbb{P}(\mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X})$ and the conjugacy classes in $\mathrm{PSL}(\mathbb{V})$.*

To complete such a description we must characterise the classes under the equivalence relation "$\mathrm{bir}(a, b)$ is a sum of squares", and more generally $(\Delta, \mathrm{bir}(a, b)) = 1$.

This depends on the arithmetic of $\mathbb{K}$ as we shall see for $\mathbb{Q}$ in the next section.

1.5 Application: binary quadratic forms & genera

In this section we reformulate the previous results in terms of binary quadratic forms, and apply them to study the group of genera.

Binary quadratic forms over $\mathbb{K}$ as elements of $\mathfrak{sl}_2(\mathbb{K})$

Let $Q\colon \mathbb{V} \to \mathbb{K}$ be a quadratic form. After choosing a basis of $\mathbb{V}$ this amounts to a homogeneous polynomial in two ordered variables with coefficients in $\mathbb{K}$.

One may polarise Q with respect to any non degenerate bilinear form on the plane $\mathbb{V}$, and one usually learns this for some euclidean scalar product, but we may also use a symplectic form: there exists a unique $\mathfrak{q} \in \mathfrak{sl}_2(\mathbb{K})$ such that $Q(v) = \omega(v, \mathfrak{q}v)$.

If we fix a basis $\mathbb{V} = \mathbb{K}^2$ and $\omega = \det$ we have the formula:

$$Q = lx^2 + mxy + ry^2 \in \mathcal{Q}(\mathbb{K}) \qquad \longleftrightarrow \qquad \mathfrak{q} = \frac{1}{2}\begin{pmatrix} -m & -2r \\ 2l & m \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{K})$$

This yields a one to one correspondence between $\mathcal{Q}(\mathbb{K})$ and $\mathfrak{sl}_2(\mathbb{K})$ through which the adjoint action of $\mathrm{PGL}_2(\mathbb{K})$ corresponds to the action by change of variables. It preserves the discriminant m^2-4lr and sends the Lie bracket $\{\mathfrak{q}_1, \mathfrak{q}_2\} = \frac{1}{2}(\mathfrak{q}_1\mathfrak{q}_2-\mathfrak{q}_2\mathfrak{q}_1)$ to the Poisson bracket of functions, under which quadratic forms are closed:

$$\{Q_1, Q_2\} = \tfrac{1}{4}\left((\partial_x Q_1)(\partial_y Q_2) - (\partial_x Q_2)(\partial_y Q_1)\right) = \{\mathfrak{q}_1, \mathfrak{q}_2\}.$$

Consequently, all the notions defined for single elements $\mathfrak{q} \in \mathfrak{sl}_2(\mathbb{K})$ or pairs of element $\mathfrak{a}, \mathfrak{b} \in \mathfrak{sl}_2(\mathbb{K})$ can be translated in terms of the corresponding binary quadratic forms $Q, Q_a, Q_b \in \mathcal{Q}(\mathbb{K})$.

Cross-ratio and cosine. After choosing a root $\sqrt{\mathrm{disc}(A)\,\mathrm{disc}(B)} = -4\sqrt{\det(\mathfrak{a}\mathfrak{b})}$, we may define the cosine:

$$\cos(Q_a, Q_b) = \frac{\mathrm{disc}(Q_a + Q_b) - (\mathrm{disc}(Q_a) + \mathrm{disc}(Q_b))}{2\sqrt{\mathrm{disc}(A)\,\mathrm{disc}(B)}} = \cos(\mathfrak{a}, \mathfrak{b})$$

and the cross-ratio of their roots, which are ordered up to simultaneous inversion:

$$\mathrm{bir}(Q_a, Q_b) = \mathrm{bir}(\alpha', \alpha, \beta', \beta) = \frac{(\alpha' - \alpha)(\beta' - \beta)}{(\alpha - \beta')(\beta - \alpha')} = \mathrm{bir}(\mathfrak{a}, \mathfrak{b})$$

For a common choice of root these are related by:

$$\frac{1}{\mathrm{bir}(Q_a, Q_b)} = \frac{1 + \cos(Q_a, Q_b)}{2}$$

In particular if Q_a, Q_b have the same discriminant Δ which is to be chosen as root of $\sqrt{\mathrm{disc}(Q_a)\,\mathrm{disc}(Q_b)}$, we have:

$$\frac{1}{\mathrm{bir}(Q_a, Q_b)} = \frac{\mathrm{disc}(Q_a + Q_b)}{4\Delta}$$

As for matrices, it is equivalent to say that Q_a and Q_b have non-degenerate cross-ratio $\mathrm{bir}(Q_a, Q_b) \notin \{0, \infty\}$ or non-degenerate Poisson bracket $\mathrm{disc}\{Q_a, Q_b\} \neq 0$.

The variables live in the cone. Conversely one may try to recover some notions defined for binary quadratic form in terms of the corresponding matrices. Let us explain the initial motivation leading to lemma 1.33, which was to recover the values that a form Q takes on $\mathbb{V}$ in terms of the geometry of $\mathfrak{q}$ with respect to $\mathbb{X}$.

Consider a binary quadratic form Q as above, and its corresponding $\mathfrak{q} \in \mathfrak{sl}_2(\mathbb{R})$ such that $Q(v) = \det(v, \mathfrak{q}v)$. Then Lemma 1.33 expresses $Q(v)$ only in terms of the geometry of $\mathfrak{sl}_2(\mathbb{R})$ as $Q(v) = \langle \mathfrak{q}, \psi(v) \rangle$.

Hence the elements of the cone $v \in \mathbb{X}$ play the role of the vector of variables (x, y) whereas the other elements $\mathfrak{q} \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ are the non degenerate binary quadratic forms. The value of $\mathfrak{q}$ at v is given by the scalar product, which may be interpreted in terms of the "distance" from $v \in \mathbb{X}$ to $\mathfrak{q}^\perp$, or the geometry of the pencil of degenerate conics generated by $\mathfrak{q}^\perp$ taken twice and $\mathbb{X}$.

Question 1.70. *One may hope to understand the geometry of the Gauss composition written as $Q_1(p_1) \times Q_2(p_2) = \langle \mathfrak{q}_1, p_1 \rangle \langle \mathfrak{q}_2, p_2 \rangle$ using an analog in Minkowski space of an appropriate "classical geometric theorem" on scalar products, to express this as $\langle \mathfrak{q}_3, p_3 \rangle$ for some $Q_3 = G(Q_1, Q_2)$ and a bilinear expression $p_3 = F_{Q_1,Q_2}(p_1, p_2)$.*

Question 1.71. *How to interpret the associative product of $\mathfrak{sl}_2(\mathbb{K})$ in terms of $\mathcal{Q}(\mathbb{K})$?*

$\mathrm{PSL}_2(\mathbb{K})$-equivalence of binary quadratic forms

Let $Q_a, Q_b \in \mathcal{Q}(\mathbb{K})$ and denote $\mathfrak{a}, \mathfrak{b} \in \mathfrak{sl}_2(\mathbb{K})$ the corresponding matrices. We call them $\mathbb{K}$-equivalent when there exists $C \in \mathrm{PSL}_2(\mathbb{K})$ such that $Q_a \circ C^{-1} = Q_b$. This amounts to saying that $C\mathfrak{a}C^{-1} = \mathfrak{b}$.

Remark 1.72. *If $\mathbb{K}$ is a subfield of $\mathbb{K}'$ then $\mathbb{K}$-equivalence implies $\mathbb{K}'$-equivalence. Over a field $\mathbb{K}$ such that $\mathbb{K}^\times/(\mathbb{K}^\times)^2 = 1$, two non-degenerate forms Q_a, Q_b are $\mathbb{K}$-equivalent if and only if they have the same discriminant.*

Corollary 1.73. *Consider $Q_a, Q_b \in \mathcal{Q}(\mathbb{K})$ with discriminant $4\delta \neq 0$ and cross-ratio $4\chi \notin \{0, \infty\}$. The $C \in \mathrm{PSL}_2(\mathbb{K})$ such that $Q_a = Q_b \circ C$ are parametrized by:*

$$(x, y) \in \mathbb{K} \times \mathbb{K} : \qquad x^2 - \delta y^2 = \chi \tag{Pell-Fermat}$$

according to the formula:

$$C(x, y) = x(\mathbb{1} + \mathfrak{b}\mathfrak{a}^{-1}) + y(\mathfrak{a} + \mathfrak{b})$$

where as usual:

$$\mathfrak{a} = \tfrac{1}{2}\begin{pmatrix} -m_a & -2r_a \\ 2l_a & m_a \end{pmatrix} \qquad \mathfrak{b} = \tfrac{1}{2}\begin{pmatrix} -m_b & -2r_b \\ 2l_b & m_b \end{pmatrix}.$$

Example 1.74. *The purely periodic irrational* $\alpha = 44 + \sqrt{2022} = \lfloor \overline{88, 1, 28, 1} \rfloor$ *is the first root of the primitive indefinite binary quadratic form* $Q_a = x^2 - 88xy - 86y^2$ *discriminant* $4 \times \delta$ *with* $\delta = 2022$*. It corresponds to the matrix* $\mathfrak{a} = \left(\begin{smallmatrix} 44 & 86 \\ 1 & -44 \end{smallmatrix}\right)$*.*

Performing a cyclic permutation of its period yields another purely periodic irrational $\beta = \lfloor \overline{28, 1, 88, 1} \rfloor$ *satisfying* $\beta = C_1 \cdot \alpha$ *for* $C_1 = R^{28}L =\in \mathrm{PSL}_2(\mathbb{Z})$*, which is the first root of* $Q_b = 3x^2 - 84xy - 86y^2$*. It corresponds to the matrix* $\mathfrak{b} = \left(\begin{smallmatrix} 42 & 86 \\ 3 & -44 \end{smallmatrix}\right)$*.*

Their cross-ratio is $4\chi = 2022/2021$*.*

Let us test the theorem in both directions.

In one direction, setting $x_1 = \mathrm{Tr}(C_1)\chi$ *and solving* $x_1^2 - \delta y_1^2 = \chi$ *in* y_1 *we recover:*

$$C_1 = 30 \times \tfrac{1}{4} \times \tfrac{2022}{2021} \times (\mathfrak{1} + \mathfrak{b}\mathfrak{a}^{-1}) + \tfrac{673}{2\times 2021} \times (\mathfrak{a} + \mathfrak{b}) = \left(\begin{smallmatrix} 29 & 28 \\ 1 & 1 \end{smallmatrix}\right).$$

Notice that (x_1, y_1) *become integral after multiplying by the denominator of* χ*.*

In the other direction, notice that $x^2 - 2022y^2$ *takes the values* $-2022, -2021, 4$ *on the pairs* $(0, 1), (1, 1), (2, 0)$*. We deduce a solution* $x_2^2 - 2022\, y_2^2 = \frac{-2022}{-2021\times 4}$ *by multiplying or dividing the corresponding numbers* $x + y\sqrt{2022}$*. This yields a matrix* $C_2 = \left(\begin{smallmatrix} 1 & 0 \\ 1/43 & 1 \end{smallmatrix}\right)$ *and we may indeed check that* $C_2\mathfrak{a} = \mathfrak{b}C_2$*, in other terms* $\beta = \frac{43\alpha}{\alpha+43}$*.*

Characterisation of $\mathbb{Q}$-equivalence

Now consider the set of $\mathrm{PSL}_2(\mathbb{Z})$-equivalence classes of primitive integral binary quadratic forms with non-square discriminant. Every field $\mathbb{K}$ yields a partition of this set into $\mathbb{K}$-classes, and we may observe how this partition varies with $\mathbb{K}$. This is true in particular for extensions of $\mathbb{Q}$. When $\mathbb{K} = \mathbb{R}$ we are considering binary quadratic forms with a same discriminant Δ, that is the class group $\mathrm{Cl}(\Delta)$.

Until the end of this paragraph we fix Δ a non-square discriminant, and let us investigate the partition of $\mathrm{Cl}(\Delta)$ given by $\mathbb{Q}$-equivalence. We abuse notations and identify classes in $\mathrm{Cl}(\Delta)$ with their representatives. Let Q_a, Q_b represent variable classes in $\mathrm{Cl}(\Delta)$ and Q_0 represent the principal class, that is the neutral element.

To characterise $\mathbb{Q}$-equivalence we look for obstructions to solving the Pell-Fermat equation locally at the various places of $\mathbb{Q}$, and if there are no obstructions then the local-to-global principle will ensure a solution exists over $\mathbb{Q}$. We describe an explicit method to compute some examples.

Method: local-to-global principle.

Denote $\mathcal{P} = \{-1, 2\} \cup \{3, 5, 7, \dots\}$ the set of rational primes, and $\mathbb{Q}_p$ the p-adic completion of $\mathbb{Q}$. The prime -1 refers (following Conway [CF97]) to the place at which the completion of $\mathbb{Q}$ is the Archimedean field $\mathbb{Q}_{-1} = \mathbb{R}$.

In what follows we rely on several properties of the Hilbert symbol, which we recall from [Ser70, Chapter III] when needed.

For $\delta, \chi \in \mathbb{Q}_p^\times$, the Hilbert symbol $(\delta, \chi)_p$ equals 1 or -1 according to whether the homogenised Pell-Fermat equation $x^2 - \delta y^2 = \chi z^2$ admits a solution in $\mathbb{Q}_p\mathbb{P}^2$ or not. Thus $(\delta, \chi)_p = 1$ if and only if χ is the norm of an element in $\mathbb{Q}_p(\sqrt{\delta})$.

Let us define the set of prime obstructions to solving the Pell-Fermat equation $(2x)^2 - \Delta y^2 = \operatorname{bir}(Q_a, Q_b)$ by $\mathcal{P}(Q_a, Q_b) = \{p \in \mathcal{P} \mid (\Delta, \operatorname{bir}(Q_a, Q_b))_p = -1\}$.

Theorem 1.75. *$Q_a, Q_b \in \mathrm{Cl}(\Delta)$ are $\mathbb{Q}$-equivalent if and only if $\mathcal{P}(Q_a, Q_b) = \emptyset$.*

Proof. Apply the Hasse-Minkowski theorem [Ser70, Chapter IV, Theorème 8] to the ternary quadratic form $(2x)^2 - \Delta y^2 - \operatorname{bir}(Q_a, Q_b)z^2$: it represents 0 over $\mathbb{Q}\mathbb{P}^2$ if and only if it represents 0 over $\mathbb{Q}_p\mathbb{P}^2$ for all $p \in \mathcal{P}$. □

The following Lemma enable us to turn the previous proposition into an explicit method for computing $\mathbb{Q}$-classes.

Lemma 1.76. *If $p \in \mathcal{P} \setminus \{2\}$ divides δ and χ to even powers, then $(\delta, \chi)_p = 0$.*

In other terms $\mathcal{P}(Q_a, Q_b) \setminus \{2\}$ is contained in the set of primes appearing with odd valuations in the factorisation of Δ or $\operatorname{bir}(Q_a, Q_b)$. In particular it is finite.

Proof. A pedestrian method is to reduce the equation $\bmod p$, argue that there exists a solution by a counting procedure, and lift it to $\mathbb{Q}_p$ using Hensel's lemma.

Alternatively, one may use the explicit formulae [Ser70, Theorem III.1] of the Hilbert symbol at p in terms of the Legendre symbols of $\delta, \chi \in \mathbb{Q}_p$ at -1 and p. □

Remark 1.77. *The set $\mathcal{P}(Q_a, Q_b) \setminus \{2\}$ determines $\mathcal{P}(Q_a, Q_b)$.*

Proof. Hilbert proved a global relation among the local symbols: $\prod_{p\in\mathcal{P}} (\delta, \chi)_p = 1$, which is a reformulation of the quadratic reciprocity law. Hence if we know the symbols at all primes except one of them, then we know the last one. □

Our final Proposition implies that Q_a and Q_b are $\mathbb{Q}$-equivalent if and only if $\mathcal{P}(Q_0, Q_a) = \mathcal{P}(Q_0, Q_b)$. This simplifies the determination of all sets $\mathcal{P}(Q_a, Q_b)$ to those involving Q_0.

Proposition 1.78. *For $Q_a, Q_b, Q_c \in \mathrm{Cl}(\Delta)$ the set $\mathcal{P}(Q_a, Q_b)$ is equal to the symmetric difference of $\mathcal{P}(Q_c, Q_a)$ and $\mathcal{P}(Q_b, Q_c)$.*

Proof. The Hilbert symbol of $\mathbb{Q}_p$ defines, by [Ser70, Theorem III.2], a non-degenerate symmetric bilinear form on the $\mathbb{F}_2$-vector space $(\mathbb{Q}_p^\times)/(\mathbb{Q}_p^\times)^2$. The Lemma can thus be reformulated as $(\Delta, \chi_{a,b,c})_p = 1$ where $\chi_{a,b,c} = \operatorname{bir}(Q_c, Q_a)\operatorname{bir}(Q_a, Q_b)\operatorname{bir}(Q_b, Q_c)$.

We must therefore compare $\chi_{a,b,c} \in \mathbb{Q}^\times$ with the subgroup generated by the norms of elements in $\mathbb{Q}(\sqrt{\Delta})^\times$. Using the explicit formula for the cross-ratio we find that:

$$\mathrm{bir}(Q_a, Q_b) = \frac{-\Delta/(l_a l_b)}{\mathrm{Norm}_\Delta(\alpha' - \beta)} \quad \text{hence} \quad \chi_{a,b,c} = \frac{-\Delta^3/(l_a l_b l_c)^2}{\mathrm{Norm}_\Delta((\gamma' - \alpha)(\alpha' - \beta)(\beta' - \gamma))}$$

Consequently $(\Delta, \chi_{a,b,c}) = (\Delta, -\Delta)_p = 1$ as desired. □

Examples for fundamental $\Delta > 0$.

In the following examples, we fix a positive non-square discriminant Δ and describe the partition of $\mathrm{Cl}(\Delta)$ into $\mathbb{Q}$-classes.

For this we choose a set of reduced representatives $Q_j = l_j x^2 + m_j xy + r_j y^2$ so that the roots $\alpha_j = (-m_j + \sqrt{\Delta})/(2r_j)$ have purely periodic continued fraction expansions and for each j we compute the set $\mathcal{P}'(Q_j, Q_0)$. The indices j are meant to reflect the structure of the class group, in particular 0 refers to the neutral element.

$\mathrm{Cl}(\Delta) = \mathbb{Z}/4$ for $\Delta = 4 \times 2022$. Since $\delta = 2022 = 2 \times 3 \times 337$ is square-free and $\equiv 2 \bmod 4$ the ring of integers of the field $\mathbb{Q}(\sqrt{2022})$ has discriminant $\Delta = 4 \times \delta$. The fundamental solution to the Pell-Fermat equation $t^2 - \delta u^2 = 1$ is $(t, u) = (1349, 30)$.

The ideal class group $\mathrm{Cl}(\Delta)$ is isomorphic to $\mathbb{Z}/4$. Its partition into genera is $\{\alpha_0, \alpha_2\}, \{\alpha_1, \alpha_3\}$ and this coincides with its partition into $\mathbb{Q}$-classes as shown by the following table.

$Q_j = (l_j, m_j, r_j)$	Period of α_j	$\mathcal{P}(Q_0, Q_j)$
$(1, -88, -86)$	$[88, 1, 28, 1]$	$\emptyset$
$(66, -72, -11)$	$[1, 4, 2, 2, 3, 1, 2, 7]$	$\{2, 337\}$
$(43, -84, -6)$	$[2, 44, 2, 14]$	$\emptyset$
$(34, -60, -33)$	$[2, 4, 1, 7, 2, 1, 3, 2]$	$\{2, 337\}$

Partition of $\mathrm{Cl}(8088) = \mathbb{Z}/4$ into $\mathbb{Q}$-classes: $\{0, 2\}, \{1, 3\}$.

$\mathrm{Cl}(\Delta) = \mathbb{Z}/5$ for $\Delta = 4 \times 439$. Since $\delta = 439$ is square-free and $\equiv 3 \bmod 4$ the ring of integers of the field $\mathbb{Q}(\sqrt{439})$ has discriminant $\Delta = 4 \times \delta$. The fundamental solution to the Pell-Fermat equation $t^2 - \delta u^2 = 1$ is $(t, u) = (440, 21)$.

The ideal class group $\mathrm{Cl}(\Delta)$ is isomorphic to $\mathbb{Z}/5$. Its partition into genera is trivial: there is only one genus since all elements of $\mathbb{Z}/5$ are squares. The partition into $\mathbb{Q}$-classes is $\{\alpha_0, \alpha_2, \alpha_4\}, \{\alpha_1, \alpha_3\}$ as shown by the following table.

$Q_j = (l_j, m_j, r_j)$	Period of α_j	$\mathcal{P}(Q_0, Q_j)$
$(2, -38, -39)$	$[19, 1, 40, 1]$	$\emptyset$
$(15, -14, -26)$	$[1, 1, 6, 3, 13, 1]$	$\{2, 439\}$
$(18, -10, -23)$	$[1, 2, 3, 1, 3, 1, 7, 1]$	$\emptyset$
$(30, -34, -5)$	$[1, 3, 1, 3, 2, 1, 1, 7]$	$\{2, 439\}$
$(13, -40, -3)$	$[3, 6, 1, 1, 1, 13]$	$\emptyset$

Partition of $\mathrm{Cl}(1756) = \mathbb{Z}/5$ into $\mathbb{Q}$-classes: $\{0, 2\}, \{1, 3, 5\}$.

$\mathrm{Cl}(\Delta) = \mathbb{Z}/2 \times \mathbb{Z}/3$ for $\Delta = 4 \times 427$. Since $\delta = 7 \times 61$ is square-free and $\equiv 3 \bmod 4$ the ring of integers of the field $\mathbb{Q}(\sqrt{427})$ has discriminant $\Delta = 4 \times \delta$. The fundamental solution to the Pell-Fermat equation $t^2 - \delta u^2 = 1$ is $(t, u) = (62, 3)$.

The ideal class group $\mathrm{Cl}(\Delta)$ is isomorphic to $\mathbb{Z}/4$. Its partition into genera is $\{\alpha_0, \alpha_2\}, \{\alpha_1, \alpha_3\}$ and this coincides with its partition into $\mathbb{Q}$-classes as shown by the following table.

$Q_j = (l_j, m_j, r_j)$	Period of α_j	$\mathcal{P}(Q_0, Q_j)$
$(14, -14, -27)$	$[1, 1, 40, 1]$	$\emptyset$
$(23, -34, -6)$	$[1, 1, 1, 1, 3, 6]$	$\{2, 61\}$
$(9, -32, -19)$	$[4, 13, 1, 1]$	$\emptyset$
$(7, -28, -33)$	$[4, 1, 19, 1]$	$\{2, 61\}$
$(22, -6, -19)$	$[1, 13, 4, 1]$	$\emptyset$
$(11, -38, -6)$	$[3, 1, 1, 1, 1, 6]$	$\{2, 61\}$

Partition of $\mathrm{Cl}(1708) = \mathbb{Z}/6$ into $\mathbb{Q}$-classes: $\{0, 2, 4\}, \{1, 3, 5\}$.

$\mathrm{Cl}(\Delta) = \mathbb{Z}/7$ for $\Delta = 4 \times 1087$. Since $\delta = 1087$ is square-free and $\equiv 1 \bmod 4$ the ring of integers of the field $\mathbb{Q}(\sqrt{1087})$ has discriminant $\Delta = 4 \times \delta$. The fundamental solution to the Pell-Fermat equation $t^2 - \delta u^2 = 1$ is $(t, u) = (1088, 33)$.

The ideal class group $\mathrm{Cl}(\Delta)$ is isomorphic to $\mathbb{Z}/7$. Its partition into genera is trivial: there is only one genus since all elements in $\mathbb{Z}/7$ are squares. Its partition into $\mathbb{Q}$-classes is $\{\alpha_0, \alpha_2, \alpha_4, \alpha_6\}, \{\alpha_1, \alpha_3, \alpha_5\}$ as shown by the following table.

$\mathrm{Cl}(\Delta) = \mathbb{Z}/2 \times \mathbb{Z}/2$ for $\Delta = 4 \times 195$. Since $\delta = 195 = 3 \times 5 \times 13$ is square-free and $\equiv 3 \bmod 4$ the ring of integers of the field $\mathbb{Q}(\sqrt{195})$ has discriminant $\Delta = 4 \times \delta$. The fundamental solution to the Pell-Fermat equation $t^2 - \delta u^2 = 1$ is $(t, u) = (14, 1)$.

$Q_j = (l_j, m_j, r_j)$	Period of α_j	$\mathcal{P}(Q_0, Q_j)$
$(2, -62, -63)$	$[31, 1, 64, 1]$	$\emptyset$
$(23, -22, -42)$	$[1, 1, 10, 3, 21, 1]$	$\{2, 1087\}$
$(18, -46, -31)$	$[3, 9, 7, 4, 1, 1]$	$\emptyset$
$(11, -50, -42)$	$[5, 3, 1, 2, 2, 1, 1, 1, 2, 1]$	$\{2, 1087\}$
$(34, -42, -19)$	$[1, 1, 1, 2, 2, 1, 3, 5, 2, 1]$	$\emptyset$
$(7, -64, -9)$	$[9, 3, 1, 1, 4, 7]$	$\{2, 1087\}$
$(41, -58, -6)$	$[1, 1, 1, 21, 1, 10]$	$\emptyset$

Partition of $\mathrm{Cl}(4348) = \mathbb{Z}/7$ into $\mathbb{Q}$-classes: $\{0, 2, 4, 6\}, \{1, 3, 5\}$.

The ideal class group $\mathrm{Cl}(\Delta)$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. Its partition into genera corresponds to the cosets for the subgroup of squares which is trivial: each genus is represented by one class. This coincides with the partition into $\mathbb{Q}$-classes as shown by the following table.

$(1, -26, -26)$ $[26, 1]$ $\emptyset$	$(19, -18, -6)$ $[1, 4, 1, 3]$ $\{2, 13\}$
$(17, -10, -10)$ $[1, 8, 1, 1]$ $\{3, 5\}$	$(2, -26, -13)$ $[13, 2]$ $\{2, 3, 5, 13\}$

Partition of $\mathrm{Cl}(780) = \mathbb{Z}/2 \times \mathbb{Z}/2$ into $\mathbb{Q}$-classes: $\{00\}, \{01\}, \{10\}, \{11\}$.

$\mathrm{Cl}(\Delta) = \mathbb{Z}/2 \times \mathbb{Z}/4$ **for** $\Delta = 4 \times 399$**.** Since $\delta = 399 = 3 \times 7 \times 19$ is square-free and $\equiv 3 \bmod 4$ the ring of integers of the field $\mathbb{Q}(\sqrt{399})$ has discriminant $\Delta = 4 \times \delta$. The fundamental solution to the Pell-Fermat equation $t^2 - \delta u^2 = 1$ is $(t, u) = (20, 1)$.

The ideal class group $\mathrm{Cl}(\Delta)$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/4$. Its partition into genera corresponds to the cosets for the subgroup of squares $\{\alpha_{00}, \alpha_{02}\}$. This coincides with the partition into $\mathbb{Q}$-classes as shown by the following table.

Remark 1.79 (Observations)**.** *Genus equivalence does not imply $\mathbb{Q}$-equivalence: there exist forms of the same genus which are not $\mathbb{Q}$-equivalent.*

The $\mathbb{Q}$-equivalence does not control the period lengths: there exist $\mathbb{Q}$-equivalent forms whose roots have euclidean periods of different length.

Inverse elements in the class group can remain in different $\mathbb{Q}$-classes.

$(1,-38,-38)$ $[38,1]$ $\emptyset$	$(10,-34,-11)$ $[3,1,2,3]$ $\{7,19\}$	$(25,-14,-14)$ $[1,12,1,1]$ $\emptyset$	$(10,-26,-23)$ $[3,3,2,1]$ $\{7,19\}$
$(2,-38,-19)$ $[19,2]$ $\{3,19\}$	$(5,-34,-22)$ $[7,2,2,1]$ $\{3,7\}$	$(29,-30,-6)$ $[1,4,1,5]$ $\{3,19\}$	$(5,-36,-15)$ $[7,1,1,2]$ $\{3,7\}$

Partition of $\mathrm{Cl}(1596) = \mathbb{Z}/2 \times \mathbb{Z}/4$ in $\mathbb{Q}$-classes: $\{00, 02\}, \{01, 03\}, \{10, 12\}, \{11, 13\}$.

Question 1.80 (Conjecture)**.** *The $\mathbb{Q}$-equivalence implies genus-equivalence.*

More precisely, the $\mathbb{Q}$-equivalence classes seem to be described as follows. Decompose the class group into a product of primary cyclic groups:

$$\mathrm{Cl}(\Delta) = \prod_{p \in \mathcal{P}} \prod_{j \in \mathbb{N}} (\mathbb{Z}/p^e)^{n_{p,e}}$$

and denote $Q_{p,e,k} \in \mathbb{Z}/p^e$ where $1 \leq k \leq n_{p,e}$ the coordinates of Q. Then the $Q_{p,e,k} \bmod 2$ provide a complete set of invariants for the $\mathbb{Q}$-classes.

1.6 Hyperbolic geometry of $\mathrm{PSL}_2(\mathbb{R})$

In this section, we apply the previous results to the the field $\mathbb{R}$ of real numbers, to expand them in this geometrical setting. We provide a few alternative proofs by recasting them in the language of hyperbolic geometry.

Almost everything can be adapted to real closed fields (ordered fields in which all positive elements are squares, like the set of real numbers constructible by ruler and compass) and even to formally real fields (fields which admit an order, which is equivalent to saying that -1 is not a sum of squares).

Consider a two-dimensional real vector space endowed with a basis denoted $\mathbb{R}^2$ and write $\mathfrak{gl}_2(\mathbb{R})$ its algebra of endomorphisms.

Topology of the quadratic space

Quadratic form. On $\mathfrak{gl}_2(\mathbb{R})$ the determinant form det has signature $(2, 2)$, in particular the isotropic vectors form a cone over a torus, and the set of units $\mathrm{SL}_2(\mathbb{R})$ is an affine quadric which is homeomorphic to a solid torus.

The restriction of the determinant to $\mathfrak{sl}_2(\mathbb{R})$ has signature $(1, 2)$, the isotropic vectors form a cone $\mathbb{X}$ over the circle, and the set of units, that is the intersection $\mathbb{H} = \{\det = 1\} \cap \mathfrak{sl}_2(\mathbb{R})$, consists in a double-sheeted hyperboloid.

The *upper* and *lower* connected components of $\mathbb{H}$ refer to the subsets of elements whose scalar product with S are respectively positive and negative. We denote $\mathbb{H}' = \{\det = -1\} \cap \mathfrak{sl}_2(\mathbb{R})$, it is a single-sheeted hyperboloid.

Adjoint action. By Proposition 1.54, the adjoint action provides an isomorphism

$$\mathrm{PGL}_2(\mathbb{R}) \to \mathrm{SO}(\mathfrak{sl}_2(\mathbb{R}), \det)$$

to the group of orientation preserving isometries for the determinant form over $\mathfrak{sl}_2(\mathbb{R})$, which is isomorphic to $\mathrm{SO}(1, 2)$. As we shall see, the subgroup $\mathrm{PSL}_2(\mathbb{R})$ maps to the connected component $\mathrm{SO}^+(1, 2)$ of the identity which preserves the connected components of $\mathbb{X} \setminus \{0\}$ or of the double-sheeted hyperboloid $\mathbb{H}$.

The adjoint actions commute with the projectivization map $\mathfrak{sl}_2(\mathbb{R}) \to \mathbb{P}(\mathfrak{sl}_2(\mathbb{R}))$. This represents $\mathrm{PGL}_2(\mathbb{R}) \subset \mathrm{PGL}_3(\mathbb{R})$ as the stabiliser of the non-degenerate conic $\mathbb{P}(\mathbb{X})$ which is homeomorphic to a circle, the subgroup $\mathrm{PSL}_2(\mathbb{R})$ corresponds to the elements preserving each orientation of the circle. The interior of the conic is $\mathbb{P}(\mathbb{H})$, homeomorphic to a disc, and inherits the orientation from $\mathfrak{sl}_2(\mathbb{R})$ restricted to the upper component of the double-sheeted hyperboloid.

Group of units. Denote $\mathrm{SL}_2^{\pm}(\mathbb{R})$ the preimage of $\{\pm 1\}$ under the morphism $\det\colon \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^*$. This gives the short exact sequence $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2^{\pm}(\mathbb{R}) \to \{\pm 1\}$, which is split: the element -1 in the cokernel is represented by any matrix of determinant -1, like J. The central extension $\mathbb{R}^*\mathbb{1} \to \mathrm{GL}_2(\mathbb{R}) \to \mathrm{PGL}_2(\mathbb{R})$ restricts to give a double cover $\{\pm\mathbb{1}\} \to \mathrm{SL}_2^{\pm}(\mathbb{R}) \to \mathrm{PGL}_2(\mathbb{R})$.

Symmetries. The elements of $\mathrm{PGL}_2(\mathbb{R})$ acting as symmetries on $\mathfrak{sl}_2(\mathbb{R})$ correspond to those in $\mathrm{SL}_2^{\pm}(\mathbb{R}) \cap \mathfrak{sl}_2(\mathbb{R}) = \mathbb{H} \cup \mathbb{H}'$. Hence there are two types of symmetries depending on the sign of their determinant, or on the relative position of their axis with respect to $\mathbb{X}$. Accordingly, they preserve or exchange the components of $\mathbb{X} \setminus \{0\}$. For instance the element $S \in \mathbb{H}$ preserves them, whereas $J \in \mathbb{H}'$ exchanges them.

The action on $\mathbb{P}(\mathbb{H})$ is obtained by composing the action on $\mathbb{H}$ with $\mathbb{H} \to \mathbb{P}(\mathbb{H})$. If $\det(a) = 1$, then $a \in \mathbb{H}$ projects to a point in $\mathbb{P}(\mathbb{H})$ through which it acts by symmetry, preserving the orientation. If $\det(a) = -1$, then $\mathbb{P}(a^{\perp})$ intersects $\mathbb{P}(\mathbb{H})$ along a line, across which a acts by reflection, reversing the orientation.

Exponential mapping. Let us describe the adjoint actions of the one parameter subgroups of $\mathrm{SL}_2(\mathbb{R})$ generated by $S, K, S + J \in \mathfrak{sl}_2(\mathbb{R})$.

As $S^2 = -\mathbb{1}$, we have $A := \exp(\theta S) = x + yS$ with $x = \cos(\theta)$ and $y = \sin(\theta)$. We have $ASA^{-1} = S$ and $AJA^{-1} = (x^2 - y^2)J + 2xyK = \cos(2\theta)J + \sin(2\theta)K$, $AKA^{-1} = (x^2 - y^2)K - 2xyJ = \cos(2\theta)K - \sin(2\theta)J$, so A acts by an elliptic rotation of angle 2θ on the plane $S^{\perp}$ oriented by (J, K).

As $K^2 = \mathbb{1}$ we have $A := \exp(\lambda K) = x + yK$ with $x = \cosh(\lambda)$ and $y = \sinh(\lambda)$. We have $AKA^{-1} = K$ and $ASA^{-1} = (x^2 + y^2)S + 2xyJ = \cosh(2\lambda)S + \sinh(2\lambda)J$, $AJA^{-1} = (x^2 + y^2)J + 2xyS = \cosh(2\lambda)J + \sinh(2\lambda)S$, so A acts like a hyperbolic rotation of angle 2λ on $K^{\perp}$ with stable and unstable eigenvectors $S + J$ and $S - J$.

Let $L = \mathbb{1} + \frac{1}{2}(S + J)$. As $(L - \mathbb{1})^2 = 0$ we have $\exp(t(L - \mathbb{1})) = \mathbb{1} + t(L - \mathbb{1}) = L^t$. We leave it as an exercise to contemplate the adjoint action of L^t on $\frac{1}{2}(S \pm K)$.

Remark 1.81. *For $a \in \mathfrak{sl}_2(\mathbb{R})$ and $t \in \mathbb{R}$, the adjoint action of the exponential* $\exp(ta)$ *on* $\mathfrak{sl}_2(\mathbb{R})$ *expands like:*

$$\exp(ta) \cdot \exp(-ta) = \mathbb{1} + t\{a, \cdot\} + o(t) \quad \text{thus} \quad \{a, b\} = \frac{d}{dt}\left[\exp(ta) b \exp(-ta)\right]_{t=0}$$

as expected from the classical relationship between a Lie group and its Lie algebra.

This enables one to guess the orientation of the rotation under the adjoint action of $\exp(\theta S)$ *from the relations* $\{S, J\} = K$ *and* $\{S, K\} = -J$.

Similarly, the expanding an contracting directions under the adjoint action of $\exp(\lambda K)$ *follow from the relations* $\{K, S\} = J$ *and* $\{K, J\} = S$.

Action on the symmetric space

Proposition 1.82. *The adjoint action of* $\mathrm{PGL}_2(\mathbb{R})$ *is free and transitive on the unit tangent bundle of* $\mathbb{H}$*. The isomorphism* $\mathrm{PGL}_2(\mathbb{R}) \to \mathrm{SO}(1,2)$ *sends the subgroup* $\mathrm{PSL}_2(\mathbb{R})$ *to the connected component of the identity* $\mathrm{SO}^+(1,2)$ *preserving each component of* $\mathbb{H}$*.*

The restriction of det *to the tangent planes of* $\mathbb{H}$ *defines a metric with constant curvature* -1 *for which* $\mathrm{PGL}_2(\mathbb{R})$ *is the group of orientation preserving isometries. This is the linear or hyperboloid model for (a double copy of) the hyperbolic plane.*

The disc $\mathbb{P}(\mathbb{H})$ *inherits the metric pushforewarded by the projection map: this is the projective model for the hyperbolic plane.*

Proof. We first prove that $\mathrm{PGL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$. As $JSJ^{-1} = -S$, the action of J exchanges both components of $\mathbb{H}$, so we may restrict to the action of $\mathrm{PSL}_2(\mathbb{R})$ on a single component. Since two element $a, b \in \mathbb{H}$ belonging to a same component have $\langle a, b\rangle \geq 1$, the Lemma 1.57 provides C conjugating a to b.

The stabiliser of S is its centraliser, that is the subgroup $\mathrm{PSO}(2) \subset \mathrm{PSL}_2(\mathbb{R})$, and it acts freely transitively on the unit vectors of $T_S\mathbb{H} = \mathrm{Span}(J, K)$. This shows that the action is free and transitive on the unit tangent bundle. Moreover $\mathrm{PGL}_2(\mathbb{R})$ preserves the determinant, hence the metric induced on $\mathbb{H}$, of which it is thus the group of all orientation preserving isometries.

We are left to compute the curvature of $\mathbb{H}$ at S. The tangent space $T_S\mathbb{H}$ is spanned by J, K on which the first fundamental form has determinant -1. If we parametrize the neighbourhood of S by $x = (1 + y^2 + z^2)^{1/2}$ we find that the second fundamental form has determinant 1. The curvature is by definition the ratio of these determinants which is therefore -1. □

Proposition 1.83. *The group* $\mathrm{PSL}_2(\mathbb{R})$ *acts transitively on the single-sheeted hyperboloid* $\mathbb{H}'$*. The stabiliser of* K *is the subgroup of diagonal matrices.*

The restriction of det *to the tangent planes of* $\mathbb{H}$ *defines a metric with constant curvature* $+1$ *for which* $\mathrm{PGL}_2(\mathbb{R})$ *is the group of orientation preserving isometries.*

Proof. It is clear from the description of the one parameter subgroups generated by S and K that every element in $\mathbb{H}'$ is the image of J by an element of the form $\exp(\theta S)\exp(\lambda K)$. Note that J acts on $\mathbb{H}'$ by orientation preserving isometries.

The computation of the curvature is similar. The tangent space $T_J\mathbb{H}'$ is spanned by S, K on which the first fundamental form has determinant $+1$. If we parametrize the neighbourhood of S by $y = (x^2 - z^2 - 1)^{1/2}$ we find that the second fundamental form has determinant $+1$. The ratio yields the curvature. □

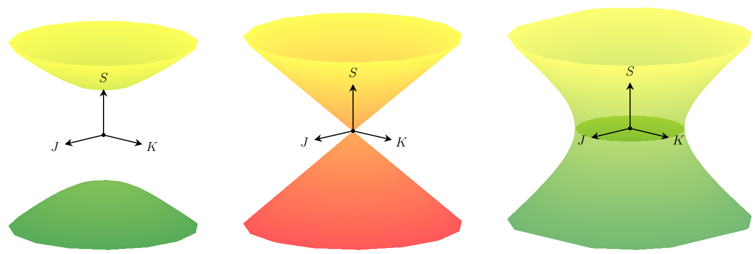


Figure 1.10: Inside $\mathfrak{sl}_2(\mathbb{R})$: isotropic cone, unit quadrics, and preferred basis.

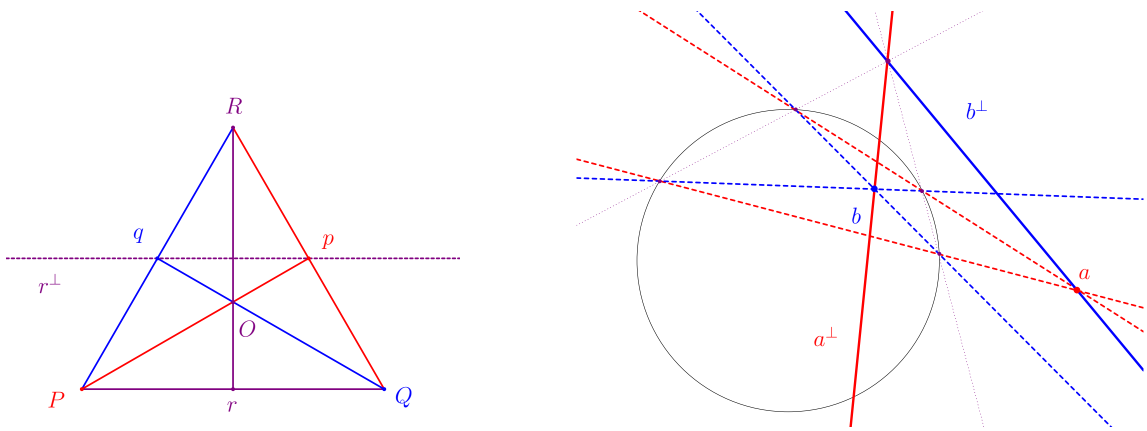


Figure 1.11: Constructing polarities with respect to the conic. See Figure 1.3.

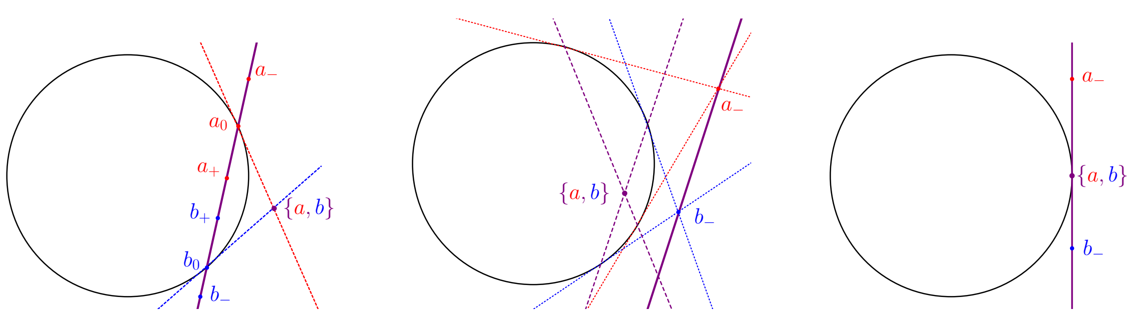


Figure 1.12: Configurations of $(\mathbb{P}a, \mathbb{P}b, \mathbb{P}\{a,b\})$ with respect to $\mathbb{P}\mathbb{X}$.

Discriminant and projection

The matrix $A \in \mathrm{SL}_2(\mathbb{R})$ is called *elliptic*, *parabolic* or *hyperbolic* according to the sign of its discriminant. Recall that $\operatorname{disc}(\operatorname{pr} A) = -4\det(\operatorname{pr} A)$ whose sign can be read off the relative position of $\operatorname{pr} A$ with respect to the isotropic cone $\mathbb{X} = \mathfrak{sl}_2(\mathbb{R}) \cap \{\det = 0\}$, or the intersection of $(\operatorname{pr} A)^\perp$ with $\mathbb{X}$, as in Figure 1.12. We say that $A \in \mathrm{SL}_2(\mathbb{R})$ is *semi-simple* when $\operatorname{disc}(A) \neq 0$, thus when A is not parabolic.

Remark 1.84. *The projection* $\operatorname{pr}\colon \mathrm{SL}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{R})$ *is not surjective. Its image is the region* $\{\det \leq 1\}$ *outside the hyperboloid* $\mathbb{H}$*. It is a regular degree two covering above* $\{\det < 1\}$*, and presents fold singularities above* $\mathbb{H}$*. The parabolic elements map to the cone, the hyperbolic elements cover the outside of the cone, and the elliptic elements map to the region between the cone and the hyperboloid.*

Notice that for $A \in \mathrm{PSL}_2(\mathbb{R})$, its discriminant makes sense independently of its lift in $\mathrm{SL}_2(\mathbb{R})$, and so does the point $\operatorname{sign}(\operatorname{tr} A)(\operatorname{pr} A) \in \mathfrak{sl}_2(\mathbb{R})$ which changes into its opposite under inversion of A. For the special case where A is involutive, we have two possible lifts with $\operatorname{tr} A = 0$ which coincide with their projections $\pm A = \pm \operatorname{pr} A \in \mathbb{H}$.

The discriminant defines an algebraic stratification of the space $\mathfrak{sl}_2(\mathbb{R})$ consisting in its regular levels, along with $\mathbb{X} \setminus \{0\}$ and $\{0\}$. We may refine this into a semi-algebraic stratification using the sign of the scalar product with S, which distinguishes the connected components of a double-sheeted hyperboloid and of $\mathbb{X} \setminus \{0\}$.

Corollary 1.85 (Conjugacy classes in $\mathrm{PSL}_2(\mathbb{R})$)**.** *Let* $A \in \mathrm{PSL}_2(\mathbb{R})$*. If* A *is an involution it is conjugate to* $\pm S$*. Otherwize its conjugacy class corresponds to the semi-algebraic stratum containing* $\operatorname{sign}(\operatorname{tr} A)\operatorname{pr}(A)$*.*

Proof. Suppose A is non trivial (for which the statement is obvious), and not an involution (for which the statement is contained in Proposition 1.82).

Recall from Proposition 1.52 that the conjugacy class of $A \in \mathrm{PSL}_2(\mathbb{R}) \setminus \{\mathbb{1}\}$ inside $\mathrm{PGL}_2(\mathbb{R})$ is characterised by its discriminant. The map $A \mapsto \operatorname{sign}(\operatorname{tr} A)\operatorname{pr} A$ preserves the discriminant and is equivariant under conjugacy. Consequently, the statement is a corollary of the descriptions of the orbits under the adjoint action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathfrak{sl}_2(\mathbb{R})$, given in 1.82 for $\operatorname{disc}(A) < 0$, and in 1.83 for $\operatorname{disc}(A) > 0$.

For $\operatorname{disc}(A) = 0$, the action of $\mathrm{PSL}_2(\mathbb{R})$ preserves the components of $\mathbb{X} \setminus \{0\}$, and the one parameter groups generated by S, K show that it is transitive on each. $\square$

Explicitly, if $A \in \mathrm{PSL}_2(\mathbb{R})$ is neither trivial nor involutive, then one may lift it to $\mathrm{SL}_2(\mathbb{R})$ and decompose it as $x + ya$ with $x, y > 0$ and $a \in \mathfrak{sl}_2(\mathbb{R})$ such that $\det(a) = -1$ or $\det(a) = 1$. Thus $\operatorname{sign}(\operatorname{tr} A)(\operatorname{pr} A) = \sqrt{|\det(\operatorname{pr} A)|}a$.

Then there exists $P \in \mathrm{PSL}_2(\mathbb{R})$ whose adjoint action sends a to K or $\pm S$, and thus conjugates A to $\exp(\lambda K)$ for $\lambda \in \mathbb{R}$ or to $\exp(\pm\theta S)$ for $\theta \in]0, \pi[$.

Orientations

We orient $\mathfrak{sl}_2(\mathbb{R})$ according to the sign of the volume form defined in Corollary 1.20. Thus $[K, S, J] = \langle\{K, J\}, S\rangle = \det(S) = 1$ implies that (S, J, K) is a positive basis.

Consider a semi-simple $A \in \mathrm{PSL}_2(\mathbb{R})$. We wish to explicit the orientations involved in the action of A on $\mathfrak{sl}_2(\mathbb{R})$ in terms of the decomposition $\mathbb{R}\cdot(\mathrm{pr}\, A)\oplus(\mathrm{pr}\, A)^\perp$.

We know that it restricts to the identity on $\mathbb{R}\cdot(\mathrm{pr}\, A)$ and its action on $(\mathrm{pr}\, A)^\perp$ is equivalent to the tautological action of A^2 on $\mathbb{R}^2$.

The following proposition says that for $\mathrm{disc}(A) > 0$, the orientation of the rotation is positive on $\mathbb{H}\cap(\mathrm{pr}\, A)^\perp$ when the plane is seen from the point $\mathrm{sign}(\mathrm{tr}\, A)(\mathrm{pr}\, A)$.

Proposition 1.86. *Let* $A \in \mathrm{PSL}_2(\mathbb{R})$ *have* $\mathrm{disc}\, A > 0$*. Then for all* $v \in (\mathrm{pr}\, A)^\perp \cap \mathbb{H}$ *the triple* $(v, AvA^{-1}, \mathrm{sign}(\mathrm{tr}\, A)\,\mathrm{pr}\, A)$ *forms a positive basis of* $\mathfrak{sl}_2(\mathbb{R})$*.*

Proof. Lift $A \in \mathrm{SL}_2(\mathbb{R})$ and decompose it as $A = x + ya$ with $x, y \in \mathbb{R}$ and $a \in \mathfrak{sl}_2(\mathbb{R})$ such that $|\det(a)| = 1$ and $\mathrm{sign}(x) = \mathrm{sign}(y)$. Thus $\mathrm{sign}(\mathrm{tr}\, A)(\mathrm{pr}\, A) = \sqrt{|\det(\mathrm{pr}\, A)|}a$.

The plane $a^\perp$ intersects the cone in two lines $L^\pm$ which are the eigenspaces for the action of A, denote L^+ and L^- the expanding and contracting directions respectively. If $v_\pm \in L^\pm$ are eigenvectors in the same half-cone, the proposition amounts to showing that (v_-, v_+, a) is a positive basis for $\mathfrak{sl}_2(\mathbb{R})$, that is $[v_+, a, v_-] = \langle\{v_+, v_-\}, a\rangle > 0$.

From Corollary 1.85, we have some $P \in \mathrm{PSL}_2(\mathbb{R})$ diagonalising A to $B = x + yK$. Since the adjoint action of $\mathrm{PSL}_2(\mathbb{R})$ preserves both the orientation of $\mathfrak{sl}_2(\mathbb{R})$ and the components of $\mathbb{H}$, it conjugates the action of A on $a^\perp$ to that of B on $b^\perp$, preserving the orientations on $a^\perp$ and $b^\perp$. Hence the proposition is equivariant under conjugacy by $P \in \mathrm{PSL}_2(\mathbb{R})$ and we are reduced to showing it for B.

For $b = K$, the point $S \in b^\perp \cap \mathbb{H}$ is conjugated by $B = x + yK$ to the element $(x^2 + y^2)S + (2xy)J$, which is between S and J since $xy > 0$. More precisely, $B = x + yK$ has eigenvectors $v_\pm = S \pm J$ for the eigenvalues $x \pm y > 0$ and the basis (v_-, v_+, K) is positive indeed. □

Proposition 1.87. *One may similarly prove that an elliptic element* A *which is not an involution acts like a rotation on* $(\mathrm{pr}\, A)^\perp$ *turning in the trigonometric direction:* $(v, AvA^{-1}, \mathrm{sign}(\mathrm{tr}\, A)\,\mathrm{pr}\, A)$ *is a positive basis for all non zero* $v \in (\mathrm{pr}\, A)^\perp$*.*

Finally if $A = \mathrm{pr}\, A$ *is an involution, then the infinitesimal action* $v \mapsto \{A, v\}$ *preserves* $A^\perp$ *and acts in like a rotation of order four in the trigonometric direction: for all non zero* $v \in A^\perp$*, the triple* $(v, \{A, v\}, A)$ *is a positive basis.*

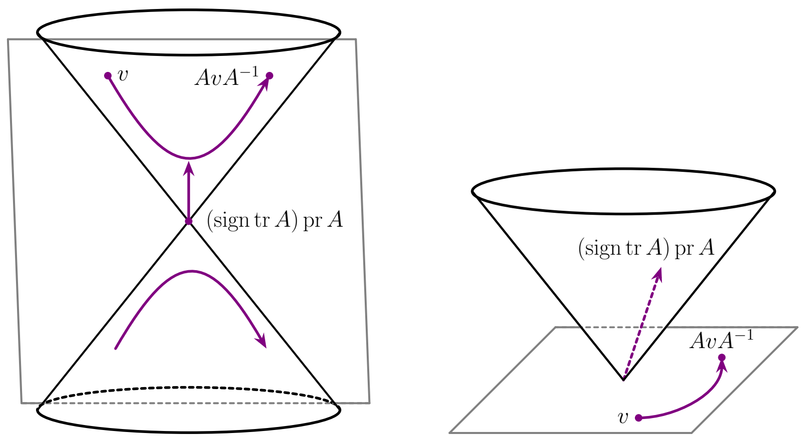


Figure 1.13: Orientations of $(v, AvA^{-1}, \operatorname{sign}(\operatorname{tr} A) \operatorname{pr} A)$ for semi-simple A.

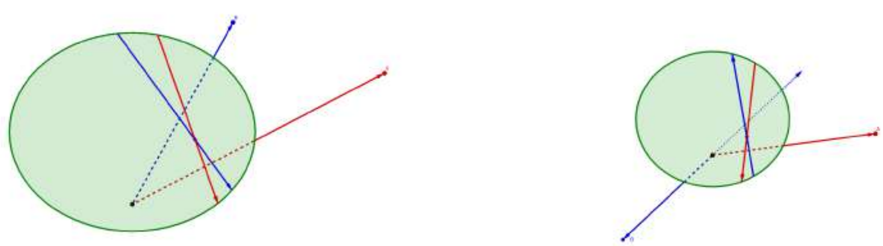


Figure 1.14: Sign of the cosine between almost parallel hyperbolic geodesics (orthogonal planes to vectors, almost equal or opposite, two extremal cases)

The sign of the cosine

A hyperbolic $A \in \mathrm{PSL}_2(\mathbb{R})$ acts by translation on the hyperbolic plane $\mathbb{P}(\mathbb{H})$ along an axis which we orient from the repulsive to the attractive fixed point of A on the boundary $\mathbb{P}(\mathbb{X})$. We may define the *normalised projection* $a \in \mathbb{H}'$ of A as:

$$a = \operatorname{sign}(\operatorname{tr}(A)) \frac{2\operatorname{pr}(A)}{\sqrt{|\operatorname{disc}(A)|}}$$

Scholium 1.88. *Note that this normalised projection is a positive multiple of that in definition 1.48. This one is only defined for hyperbolic elements in* $\mathrm{PSL}_2(\mathbb{R})$ *whereas the other was defined for all non involutive semi-simple elements in* $\mathrm{PGL}_2(\mathbb{K})$*. Here we have no qualms about taking square roots in order to obtain an element in* $\mathbb{H}'$*.*

Proposition 1.89. *For hyperbolic* $A, B \in \mathrm{PSL}_2(\mathbb{R})$ *whose axes intersect in* $\mathbb{P}(\mathbb{H})$*, denote* $a, b \in \mathfrak{sl}_2(\mathbb{R})$ *their normalised projections, and call* $\theta \in]-\pi, \pi[$ *the angle between their oriented axes, going from* $a^\perp$ *to* $b^\perp$*.*

Then θ *is determined by* $\operatorname{sign}(\theta)$ *given by the orientation of the basis* $(a, b, \{a, b\})$*, and* $\cos(\theta)$ *which we denote* $\cos(A, B)$*. Those are given by:*

$$\cos(A,B) = -\langle a, b\rangle \qquad \text{and} \qquad \operatorname{sign}\cos(A,B) = \operatorname{sign}\left(\operatorname{disc}(AB) - \operatorname{disc}\left(AB^{-1}\right)\right).$$

Proof. Lemma 1.57 provides a square root of ba^{-1} in $\mathrm{SL}_2(\mathbb{R}) \cap \operatorname{Span}(\mathbf{1}, \{a, b\})$ which conjugates a to b while fixing $\{a, b\}$. The restriction of this adjoint action to $\{a, b\}^\perp$ must be a rotation of angle θ, but by Corollary 1.55 it is conjugate to the action of its square ba on $\mathbb{R}^2$: hence the half-trace $\cos\theta$ of that rotation must equal $\operatorname{tr} ba = -\langle a, b\rangle$.

The first part of Proposition 1.87 applied to the square root of ba provided in Lemma 1.57 says that $(a, b, \{a, b\})$ forms a positive basis if and only if $0 < \theta < \pi$.

Corollary 1.15 implies that $\operatorname{sign}(-\langle a, b\rangle) = \operatorname{sign}(\operatorname{disc} AB - \operatorname{disc} AB^{-1})$. □

Intuitive proofs. Let us provide some other geometric intuitions supporting the proof.

Conjugation by ba is the composition of two symmetries across the axes $a^\perp$ and $b^\perp$ which meet in a point $a^\perp \cap b^\perp$, yielding a rotation of angle 2θ around that point. This conjugation is quadratic in ba so one must have $\operatorname{tr} ba = \cos\theta$.

Here is another geometric explanation for why $\operatorname{sign}(\cos\theta) = \operatorname{sign}(-\langle a, b\rangle)$, relying on Figure 1.14. By Proposition 1.86 the elements a, b are close in $\mathfrak{sl}_2(\mathbb{R})$ if and only if their oriented axes $a^\perp$ and $b^\perp$ are close. If they are close enough then θ is small enough to have a positive cosine while $\langle a, b\rangle$ is close enough for $\det(a) \approx \det(b)$ to be negative. On the contrary, if a and $-b$ come close then θ approaches π, so its cosine is negative, whereas $\langle a, b\rangle$ approaches $-\det(a) \approx -\det(b)$ which is positive.

The last two paragraphs lead to another slick geometric proof that $\operatorname{sign}(\cos\theta)$ equals $\operatorname{sign}(\operatorname{disc}(AB) - \operatorname{disc}(AB^{-1}))$ by composing symmetries. □

Remark 1.90. *The Proposition 1.89 recovers the fact that the angle between the oriented axes $a^\perp$ and $b^\perp$ is equal to the angle $\cos(a,b)$ defined in 1.45.*

Proposition 1.91. *Consider hyperbolic elements $A, B \in \mathrm{PSL}_2(\mathbb{R})$ with normalised projections $a, b \in \mathfrak{sl}_2(\mathbb{R})$.*

Then according to whether $a^\perp \cap b^\perp$ lies inside or outside the conic and denoting θ or λ the angle or distance between their oriented axes respectively, we have:

$$\frac{1}{\mathrm{bir}(A,B)} = \cos(\theta/2)^2 = \frac{1+\cos(\theta)}{2} \qquad \frac{1}{\mathrm{bir}(A,B)} = \cosh(\lambda/2)^2 = \frac{1+\cosh(\lambda)}{2}$$

Proof. Since $\mathrm{PGL}_2(\mathbb{R})$ acts on $\mathbb{RP}^1$ triple-transitively, and preserving the cross-ratio of quadruples of points, we are reduced to expressing the angle θ or distance d between the geodesics $(0z)$ and $(\infty 1)$ in terms of $\mathrm{bir}(z,0,1,\infty) = z$.

Working in the half plane model for hyperbolic geometry, we may compute $\cos(\theta)$ from the conformal property and trigonometric geometry, and $\cosh(d)$ from a trigonometric integral involving the explicit metric. We find $-1 + 2/z$ in both cases. □

Scholium 1.92. *We provided several proofs for Proposition 1.89 because it will play a key role in what follows, especially to relate the hyperbolic geometry and combinatorics of translation axes for pairs of matrices $A, B \in \mathrm{PSL}_2(\mathbb{Z})$. In particular, it must be compared with Proposition 2.44.*

We could have derived it independently from Propositions 1.91 and 1.46, implying that for hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{R})$ with normalised projections $a, b \in \mathbb{H}'$ we have:

$$\frac{1}{\mathrm{bir}(A,B)} = \frac{1-\langle a,b\rangle}{2}.$$

The crossing function

Definition 1.93. *Let $\mathrm{cord}\colon \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \to \{-1,0,1\}$ be the cyclic order function of three points in an oriented circle $\mathbb{S}^1$. Define $\mathrm{cross}\colon (\mathbb{S}^1 \times \mathbb{S}^1 \setminus \Delta)^2 \to \pm\{0, 1/2, 1\}$ by*

$$\mathrm{cross}(u,v,x,y) = \tfrac{1}{2}\left(\mathrm{cord}(u,x,v) - \mathrm{cord}(u,y,v)\right)$$

which takes the values ± 1 when the chords uv and xy are linked on the boundary, $\pm 1/2$ if they share one extremity, and 0 otherwize.

If the circle is realised as the boundary of the hyperbolic plane $\mathbb{S}^1 = \partial\mathbb{H}$, then $\mathrm{cross}(u,v,x,y)$ is the oriented intersection number of the geodesics uv and xy, which is ± 1 when they intersect inside $\mathbb{H}$, $\pm 1/2$ when they meet on the boundary $\partial\mathbb{H}$ at one point, and 0 otherwize.

We denote $|\mathrm{cross}|(u,v,x,y) \in \{0, 1/2, 1\}$ the absolute value of $\mathrm{cross}(u,v,x,y)$. When $A, B \in \mathrm{PSL}_2(\mathbb{R})$ are hyperbolic with fixed points (α', α) and (β', β), we also write $\mathrm{cross}(A,B) = \mathrm{cross}(\alpha', \alpha, \beta', \beta)$ and $|\mathrm{cross}|(A,B) = |\mathrm{cross}|(\alpha', \alpha, \beta', \beta)$.

Following Iverson [Knu92], denote $[\![P]\!] \in \{0,1\}$ the truth value of a property P, it satisfies the usual rules of boolean algebra.

Proposition 1.94. *Consider hyperbolic elements $A, B \in \mathrm{PSL}_2(\mathbb{R})$ whose fixed points $\alpha', \alpha, \beta', \beta$ are all distinct. If a, b denote their normalised projections in $\mathfrak{sl}_2(\mathbb{R})$, and $t = \mathrm{tr}[A,B]$ the half trace of their well defined commutator $[A,B] \in \mathrm{SL}_2(\mathbb{R})$, then:*

$$|\mathrm{cross}|(A,B) = [\![\mathrm{bir}(\alpha', \alpha, \beta', \beta) > 1]\!] = \frac{1 + \mathrm{sign}\det\{a,b\}}{2} = \frac{1 - \mathrm{sign}(t - t^{-1})}{2}$$

Proof. The first three quantities equal 0 or 1 according to whether (α', α) and (β', β) cross inside or outside the conic. The last equality follows from Corollary 1.15. □

Scholium 1.95. *For hyperbolic elements $A, B \in \mathrm{PSL}_2(\mathbb{R})$ with normalised projections $a, b \in \mathfrak{sl}_2(\mathbb{R})$, we shall make use of the function:*

$$\frac{[\![\mathrm{bir}(A,B) > 1]\!]}{\mathrm{bir}(A,B)} = \frac{1 + \mathrm{sign}\det\{a,b\}}{2} \times \frac{1 - \langle a, b\rangle}{2}$$

which equals 0 unless the oriented axes of A and B intersect in the hyperbolic plane.

If the oriented axes $a^\perp$ and $b^\perp$ intersect at an angle θ approaching to $0 \bmod \pi$, meaning that $a \pm b \to 0$, then this quantity approaches:

$$|\mathrm{cross}|(A,B) \times \mathrm{sign}\cos(A,B) = \frac{1 + \mathrm{sign}\det\{a,b\}}{2} \times \frac{1 - \mathrm{sign}\langle a, b\rangle}{2}$$

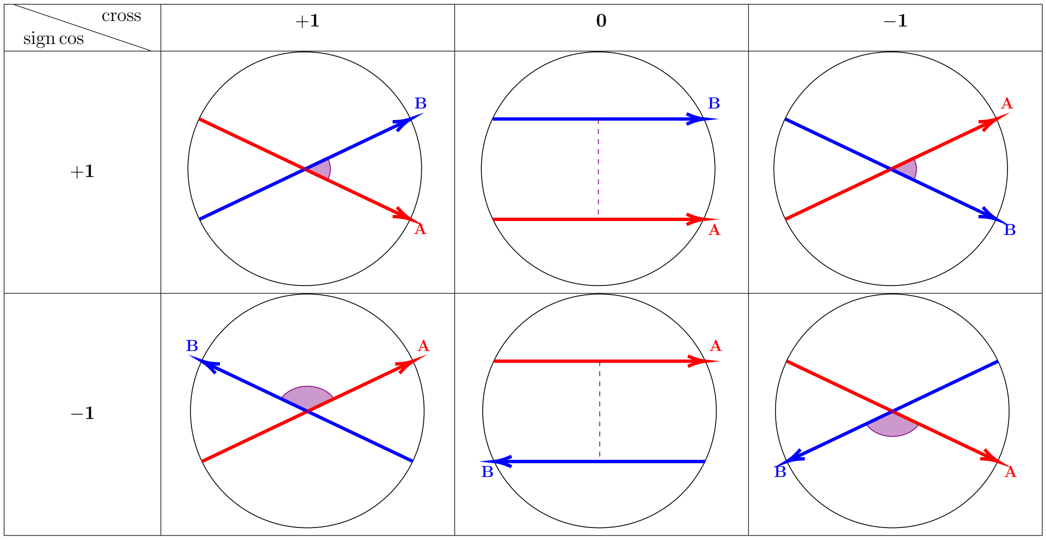


Figure 1.15: Drawing configurations: cross and sign cos.

Composing symmetries

Let us compose the symmetries of $\mathrm{PGL}_2(\mathbb{R})$ acting on $\mathbb{P}(\mathbb{H})$ to express distances and angles between fixed points and lines of isometries in terms of scalar products. Before describing the geometry, we recall the algebra. We saw that symmetries lift to (pairs of opposite) elements in $\mathbb{H} \cup \mathbb{H}' = \mathrm{SL}_2^{\pm}(\mathbb{R}) \cap \mathfrak{sl}_2(\mathbb{R})$. In particular $a, b \in \mathbb{H} \cup \mathbb{H}'$ have product $ab = -\langle a, b\rangle + \{a, b\}$, which belongs to $\mathrm{SL}_2^{\pm}(\mathbb{R})$. We shall assume $\det(a) = \det(b)$ and $a \neq \pm b$, so that $ab \in \mathrm{SL}_2(\mathbb{R})$ fixes only $\mathbb{P}(\{a, b\}) = a^{\perp} \cap b^{\perp}$ and is either elliptic, parabolic or hyperbolic depending on $\operatorname{sign}\operatorname{disc}\{a, b\}$.

Composing symmetries in $\mathbb{H}$. The adjoint action of $a \in \mathbb{H}$ is by hyperbolic reflection through a, also an order two rotation around a. Figure 1.17 represents such an action on the conic $\mathbb{P}(\mathbb{X})$. The elements $a, b \in \mathbb{H}$ compose to give a hyperbolic element, acting by translation along the axis passing through them, that is the line $\{a, b\}^{\perp}$. Figure 1.17 represents such a composition acting on the conic $\mathbb{P}(\mathbb{X})$. The displacement length of ab is twice the distance $d(a, b)$ separating them:

$$\langle a, b\rangle = -\operatorname{tr}(ab) = \pm \cosh d(a, b)$$

the last sign being $+1$ if a and b belong to the same connected component of $\mathbb{H}$ and -1 otherwize. Note in passing that Lemma 1.57 provides an element sending a to b, which can be characterised as the unique square root of ba in $\mathrm{PSL}_2(\mathbb{R})$.

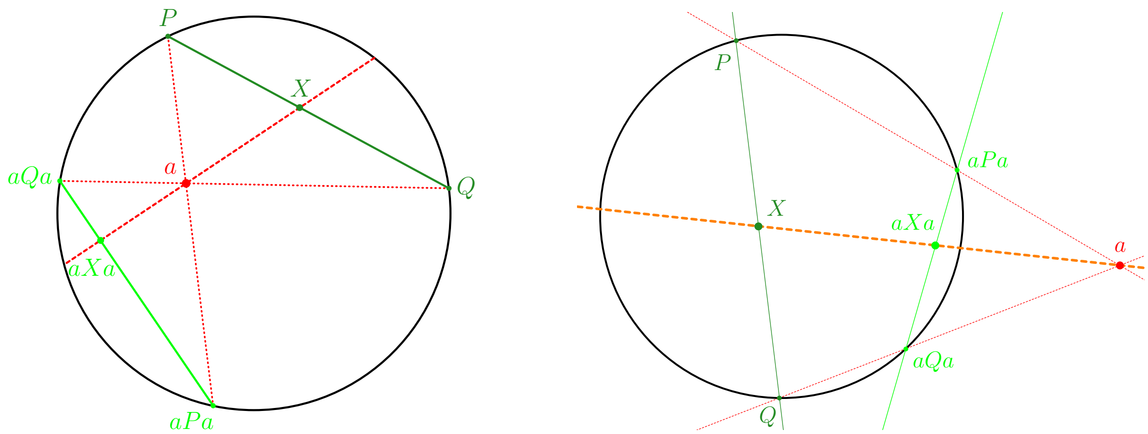


Figure 1.16: Constructing symmetries trough points inside and outside the conic.

Composing symmetries in $\mathbb{H}'$. The adjoint action of $a \in \mathbb{H}'$ is by hyperbolic symmetry across $a^\perp$, and restricts to the conic $\mathbb{P}(\mathbb{X})$ as represented in Figure 1.18. Now consider $a, b \in \mathbb{H}'$ acting as symmetries accros $a^\perp$ and $b^\perp$. The composition ab can be elliptic, parabolic or hyperbolic depending on the position of $\mathbb{P}(\{a, b\}) = a^\perp \cap b^\perp$ with respect to the conic $\mathbb{P}(\mathbb{X})$.

If $\operatorname{disc}\{a, b\} > 0$, then ab acts by translation along the axis $\{a, b\}^\perp$ which is (the unique common) perpendicular to $a^\perp$ and $b^\perp$. The displacement length is twice the distance $d(a^\perp, b^\perp)$, and given by

$$\langle a, b\rangle = -\operatorname{tr}(ab) = \pm \cosh d(a^\perp, b^\perp)$$

where (as before) the last sign equals $+1$ if the line (a, b) intersects twice the same component of $\mathbb{H}$ and -1 if it intersects both components.

If $\operatorname{disc}\{a, b\} < 0$, then ab acts by rotation around their intersection $a^\perp \cap b^\perp$. The angle of rotation is twice the angle θ at which $a^\perp$ and $b^\perp$ meet, and given by:

$$\langle a, b\rangle = -\operatorname{tr}(ab) = \pm \cos(\theta)$$

where again the sign is -1 if (a, b) intersects both components of $\mathbb{H}$ and $+1$ otherwize.

We leave the case $\operatorname{disc}\{a, b\} = 0$ as an exercise (see Figure 1.18).

Decomposing products of translations

Let us now decompose the product of two hyperbolic translations $A, B \in \mathrm{PSL}_2(\mathbb{R})$ in products of symmetries. Denote c the intersection of their axes $(\operatorname{pr} A)^\perp \cap (\operatorname{pr} B)^\perp$ in $\mathbb{P}(\mathfrak{sl}_2(\mathbb{R}))$, and suppose it does not lie on the conic $\mathbb{P}(\mathbb{X})$.

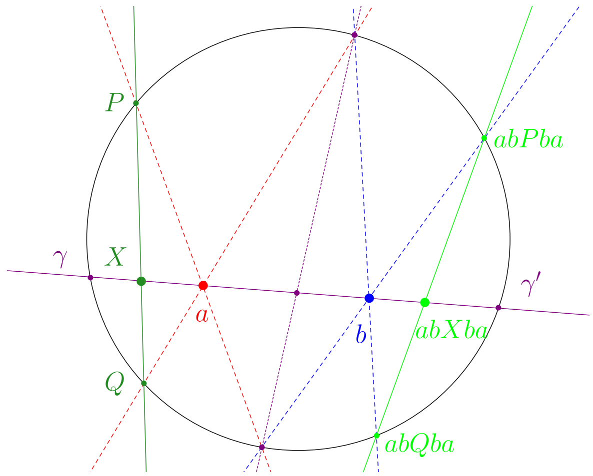


Figure 1.17: Composing symmetries trough points inside the conic.

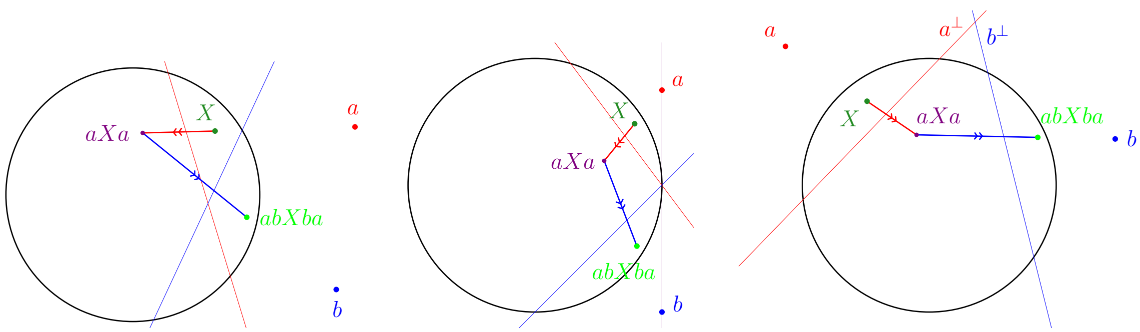


Figure 1.18: Composing symmetries trough points outside the conic.

Axes intersect inside $\mathbb{P}(\mathbb{X})$. If c lies inside the conic, we may lift $c \in \mathbb{H}$ and then decompose $A = ac$ and $B = cb'$ for some unique $a, b' \in \mathbb{H}$ in the same component as c. Thus $AB = -ab'$ is a hyperbolic translation along (a, b') of length $2d(a, b')$. Notice that $A^{-1} = ca'$ and $B^{-1} = bc$ where $a' = cac^{-1}$ and $b = cb'c^{-1}$ are the symmetric points of a and b' through c. The parallelogram with vertices a, b, a', b' and center c may have three possible shapes: long, square or short; according to the value of the $\mathrm{sign}(\cos\theta) \in \{+1, 0, -1\}$ where $\theta \in]-\pi, \pi[$ is the angle between the oriented diagonals $a'a$ and $b'b$. This $\mathrm{sign}(\cos\theta)$ also equals:

$$\mathrm{sign}(d(a,b) - d(a,b')) = \mathrm{sign}(\mathrm{tr}(AB)^2 - \mathrm{tr}(AB^{-1})^2) = \mathrm{sign}(\mathrm{disc}(AB) - \mathrm{disc}(AB^{-1}))$$

Remark 1.96. *The aforementioned "parallelogram" refers to the notion which makes sense in a uniquely geodesic metric space: a quadrilateral whose diagonals intersect in* *their midpoint. In the hyperbolic metric space* $\mathbb{P}(\mathbb{H})$*, the sides of such parallelograms may well intersect inside the conic* $\mathbb{P}(\mathbb{X})$*, once extended in both directions.*

Axes intersect outside $\mathbb{P}(\mathbb{X})$**.** If c lies outside the conic, we may lift $c \in \mathbb{H}'$, and decompose $A = ac$ and $B = cb'$ for some $a, b' \in \mathbb{H}'$. Then $AB = ab'$ is a hyperbolic translation with axis (a, b') and displacement length $2d(a^\perp, b'^\perp)$. Notice that $A^{-1} = ca'$ and $B^{-1} = bc$ where $a' = cac^{-1}$ and $b = cb'c^{-1}$ are the symmetric points of a and b' through c.

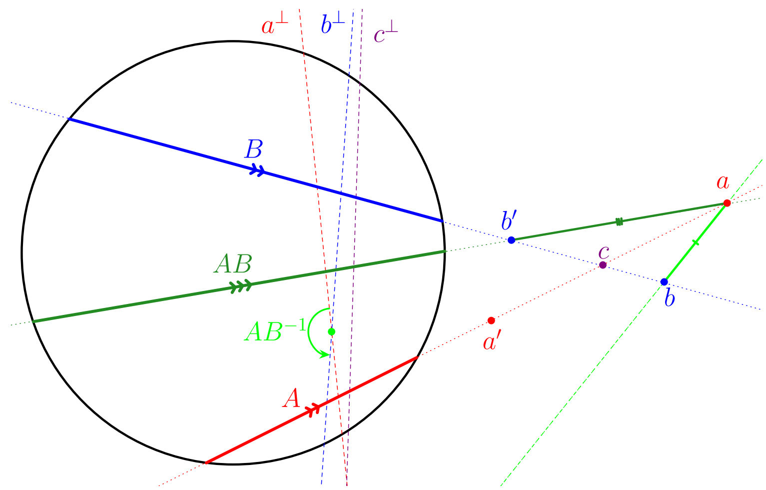


Figure 1.19: Decomposing products of translations whose axes intersect inside $\mathbb{P}(\mathbb{X})$.

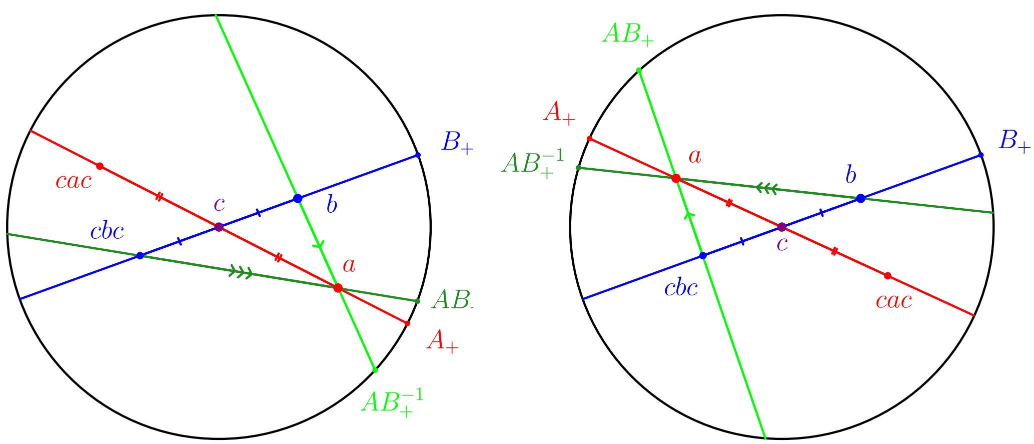


Figure 1.20: Decomposing products of translations whose axes intersect outside $\mathbb{P}(\mathbb{X})$.

Geometric interpretations of the main theorem

Consider primitive integral binary quadratic forms Q_a, Q_b with non-square discriminant Δ.

If $\Delta > 0$ then they correspond to pairs of complex conjugate points in $\mathbb{C}$ or equivalently (by ordering the roots up to simultaneous inversion) to points α, β in the upper half plane $\mathbb{HP}$. The $\mathrm{PSL}_2(\mathbb{Z})$-classes correspond to points $[\alpha], [\beta]$ modular orbifold $\mathbb{M} = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{HP}$ often called singular moduli in the study of elliptic curves.

The geodesic arc from α to β in $\mathbb{HP}$ has length λ given in terms of the cross-ratio $\mathrm{bir}(\alpha', \alpha, \beta', \beta)$ by the formula:

$$\left(\cosh \tfrac{\lambda}{2}\right)^2 = \frac{1+\cosh(\lambda)}{2} = \frac{1}{\mathrm{bir}(Q_a, Q_b)}$$

The Corollary 1.73 implies the following.

Corollary 1.97. *Two singular moduli $[\alpha], [\beta] \in \mathbb{Q}(\sqrt{\Delta})$ are $\mathbb{Q}$-equivalent if and only if there exists a hyperbolic geodesic arc in $\mathbb{M}$ from $[\alpha]$ to $[\beta]$ whose length λ is of the form:*

$$\cosh\left(\tfrac{\lambda}{2}\right) = \frac{1}{\sqrt{(2x)^2 - \Delta y^2}} \qquad \text{for} \quad x, y \in \mathbb{Q}$$

in which case all geodesic arcs from $[\alpha]$ to $[\beta]$ have this property.

If $\Delta < 0$ then Q_a and Q_b correspond to oriented geodesic axes $(\alpha', \alpha), (\beta', \beta)$ in the upper half-plane model $\mathbb{HP}$ of the hyperbolic plane. Their $\mathrm{PSL}_2(\mathbb{Z})$-classes correspond to the primitive closed oriented geodesics γ_a, γ_b in the modular orbifold $\mathbb{M} = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{HP}$.

Consider the oriented hyperbolic axes (α', α) and (β', β) in $\mathbb{HP}$. If they intersect, then their angle θ is given in terms of the cross-ratio $\mathrm{bir}(\alpha', \alpha, \beta', \beta)$ by the formula:

$$\left(\cos \tfrac{\theta}{2}\right)^2 = \frac{1+\cos(\theta)}{2} = \frac{1}{\mathrm{bir}(Q_a, Q_b)}$$

If they do not intersect, then they have a unique common perpendicular geodesic arc, which may receive compatible co-orientations from each axis or not. When it is the case, its length λ is given in terms of the cross-ratio $\mathrm{bir}(\alpha', \alpha, \beta', \beta)$ by the formula:

$$\left(\cosh \tfrac{\lambda}{2}\right)^2 = \frac{1+\cosh(\lambda)}{2} = \frac{1}{\mathrm{bir}(Q_a, Q_b)}$$

The Corollary 1.73 implies the following.

Corollary 1.98. *Two modular geodesics are $\mathbb{Q}$-equivalent if and only if we have one of the following equivalent conditions:*

- θ *There exists one intersection point with angle $\theta \in]0, \pi[$ such that:*

$$\left(\cos \tfrac{\theta}{2}\right)^2 = \tfrac{1}{(2x)^2 - \Delta y^2} \qquad \text{for} \quad x, y \in \mathbb{Q}$$

in which case all intersection points have this property.

- λ *There exists one co-oriented ortho-geodesic of length λ such that:*

$$\left(\cosh \tfrac{\lambda}{2}\right)^2 = \tfrac{1}{(2x)^2 - \Delta y^2} \qquad \text{for} \quad x, y \in \mathbb{Q}$$

in which case all co-oriented ortho-geodesics have this property.

In other terms, the geometric quantities on the left hand sides belong to the group of norms of the quadratic extension $\mathbb{Q}(\sqrt{\Delta})/\mathbb{Q}$ (which is stable by inversion).

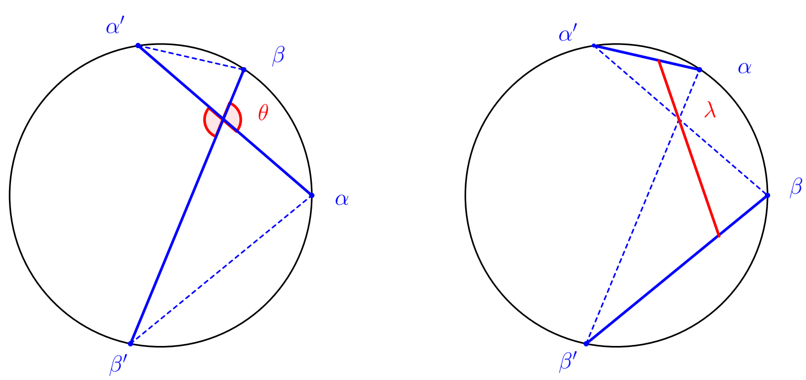
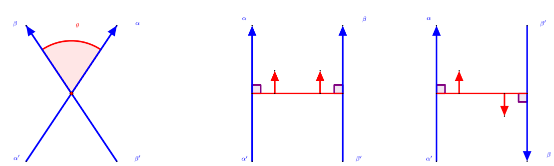


Figure 1.21: Cross-ratios and cosines in the real case.



Angle well defined in $]0, \pi[$. Ortho-geodesics well and badly co-orientated.

Chapter 2

The modular group $\mathrm{PSL}_2(\mathbb{Z})$

Outline of the chapter

This chapter concerns the combinatorics of the modular group $\mathrm{PSL}_2(\mathbb{Z})$, represented as the automorphism group of a planar trivalent tree. The aim is to describe its conjugacy classes and define functions of pairs of conjugacy classes. It is divided in three sections, all of which are highly connected to the rest of the thesis.

The first one constructs the action of $\mathrm{PSL}_2(\mathbb{Z})$ on the infinite planar trivalent tree $\mathcal{T}$ from an intrinsic viewpoint, in the spirit of the Bruhat-Tits theory of buildings.

The second one concerns hyperbolic matrices in $\mathrm{SL}_2(\mathbb{Z})$ and takes advantage of their relation with real quadratic irrationals to compare their action on the euclidean plane $\mathbb{R}^2$ and the hyperbolic plane $\mathbb{HP}$. In particular we determine the intersection patterns of their eigen-directions and their translation axes with certain triangulations of those planes. This will serve in Chapter 3 to describe the geometry of modular geodesics.

The third one exploits the combinatorial action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{T}$ to extract conjugacy-invariants for pairs of hyperbolic matrices. It ends with a general method for constructing functions of pairs of conjugacy classes by averaging such invariants. We shall specify those functions in later chapters to recover intersection numbers of modular geodesics and linking numbers of modular knots.

Action of $\mathrm{PSL}_2(\mathbb{Z})$ on the space of lines $\mathbb{P}(\mathbb{Z}^2)$

The first section provides an intrinsic construction of the trivalent tree $\mathcal{T}$. For this we consider the following simplicial complex $\triangle_2$ of dimension 2. Its vertex set is the space of lines $\mathbb{P}(\mathbb{Z}^2)$ of the plane $\mathbb{Z}^2$. Two vertices are connected by an edge when the corresponding lines generate $\mathbb{Z}^2$. Then we fill in each triangle by a face. The dual graph to the 1-skeleton of $\triangle_2$ is the first barycentric subdivision $\mathcal{T}'$ of the trivalent tree $\mathcal{T}$. The group $\mathrm{PSL}_2(\mathbb{Q})$ acts on $\mathbb{QP}^1$ and the stabiliser of the incidence relation defining $\triangle_2$ is $\mathrm{PSL}_2(\mathbb{Z})$. In fact, $\mathrm{PSL}_2(\mathbb{Z})$ acts freely transitively on the edges of $\mathcal{T}'$, from which we deduce that it is isomorphic to the free amalgam $\mathbb{Z}/2 * \mathbb{Z}/3$.

All this is well known. One may consult [CF97, Hat22] for similar descriptions of the trivalent tree (and much more concerning its relation to binary quadratic forms). The intrinsic construction of the complex $\triangle_2$ is inspired by the book [Ser77] which christened the Bass-Serre theory of groups acting on trees.

We also propose a geometric realisation of $\triangle_2$ as an ideal triangulation of the hyperbolic plane $\mathbb{HP}$. For this we construct another triangle complex $\Delta_4 \subset \mathbb{R}^2$ whose vertices are the primitive vectors of the lattice $\mathbb{Z}^2$, the edges connect pairs of primitive vectors forming a basis of $\mathbb{Z}^2$, and three vertices form a triangle when the corresponding vectors have an alternating sum equal to zero. Then we use the quadratic map $\psi\colon \mathbb{R}^2 \to \mathbb{X}$ from 1.33 to send $\triangle_4$ into the isotropic cone of $(\mathfrak{sl}_2(\mathbb{R}), \det)$. After "rectifying the edges" of $\psi(\triangle_4)$ we obtain a polytope $\bar{\psi}(\triangle_4)$ inscribed in the double sheeted hyperboloid $\mathbb{H}$, which is isomorphic to the simplicial complex $\triangle_2 \sqcup \triangle_2$. Finally we projectify to obtain an ideal triangulation of the hyperbolic plane $\mathbb{HP}$ by $\triangle_2$. The first barycentric subdivision $\triangle_4'$ maps to a tesselation of $\mathbb{HP}$ under $\mathrm{PSL}_2(\mathbb{Z})$, which is thus a geometric realisation of the first barycentric subdivision $\triangle_2'$.

This construction connects the geometric study of $\mathfrak{sl}_2(\mathbb{R})$ and its symmetric space $\mathbb{HP}$ of Chapter 1 with the combinatorics of $\mathcal{T}'$ and the arithmetics of $\triangle_4$. If the details are original, the ideas are largely superseded by the Bruhat-Tits theory of buildings, to which [Ser77] can again serve as a reference for the case of SL_2.

Conjugacy classes of matrices in $\mathrm{PSL}_2(\mathbb{Z})$

The second section opens with the dictionary between hyperbolic matrices in $\mathrm{PSL}_2(\mathbb{Z})$ and real quadratic irrationals, as briefly summarised in the introductory Section 0.2. We emphasize the importance of the elements A in the euclidean monoid $\mathrm{PSL}_2(\mathbb{N})$ which is freely generated by the parabolic matrices $L\&R$. They correspond to the real numbers α whose continued fraction expansions are periodic from the first or second entry onward (the numbers $-1/\alpha'$ also have the same form, with mirror image periods). The article [Lac88] surveys this arithmetic dictionary in detail.

A hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathbb{HP}$ by translation along its geometric axis γ_A and on $\mathcal{T}$ by translation along its combinatorial axis g_A. Both axes join the boundary points $\alpha', \alpha \in \partial\mathcal{T} = \partial\mathbb{HP} = \mathbb{RP}^1$. Besides, the lifts of a hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ with stable eigen-direction V_α. We recall in Proposition 2.28

why the intersections of V_α with $\triangle_4 \subset \mathbb{R}^2$ and of γ_A with $\triangle_2 \subset \mathbb{HP}$ follow the same patterns, given by the continued fraction expansion of α. Finally, we describe the intersection patterns of V_α with $\triangle_4'$ in Proposition 2.33 and of γ_A with $\triangle_2 \subset \mathbb{HP}$ in Proposition 2.40. These patterns are different, and both will serve in Chapter 3 to describe the geometry of modular geodesics.

One may consult [Smi77, Hum16] and [Ser85b, KU06] for related discussions concerning various continued fraction expansions of real numbers, and the intersection patterns of hyperbolic geodesics with the modular tessellations of $\mathbb{HP}$ under (certain subgroups of) the modular group.

Conjugacy invariants of matrices in $\mathrm{PSL}_2(\mathbb{Z})$

In the third section, we construct conjugacy invariants for pairs of hyperbolic matrices $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ by studying the relative positions of their combinatorial axes g_A, g_B in the trivalent tree $\mathcal{T}$. One invariant is given by the order of their endpoints on the boundary. Another is given by the length of their intersection, or of the combinatorial geodesic which connects them. We combine these two invariants to form the relevant quantities $\mathrm{cross}(g_A, g_B)$ and $\mathrm{cosign}(g_A, g_B)$ for computing intersection numbers and linking numbers later on. The important facts are Lemma 2.43 saying that A, B can be simultaneously conjugated in $\mathrm{PSL}_2(\mathbb{N})$ if and only if $\mathrm{cosign}(A, B) = 1$, and Proposition 2.44 which computes $\mathrm{cosign}(A, B) = \mathrm{len}(AB) - \mathrm{len}(AB^{-1})$. The latter proposition refers to [Pau89, CP20], which one may consult for similar discussions of groups acting on trees, and much more can be found in [GdlH90, Pau97].

Then we recall in Proposition 2.48 that hyperbolic matrices commute if and only if their non-oriented combinatorial axes coincide, if and only if they are integral powers of a same element in $\mathrm{PSL}_2(\mathbb{Z})$. This enables us to describe the stabiliser of hyperbolic matrices under conjugacy. Finally, we provide a general method for constructing functions $F([A], [B])$ of pairs of conjugacy classes in $\mathrm{PSL}_2(\mathbb{Z})$ by summing conjugacy invariants $f(A, B)$ of pairs in $\mathrm{PSL}_2(\mathbb{Z})$. The sum is performed over all representatives A, B modulo the diagonal action of $\mathrm{PSL}_2(\mathbb{Z})$ by conjugacy, and modulo the stabilisers of A and B. This construction will serve in Chapters 3 and 4 where we shall specify the function f as linear combinations of cross & cosign to recover functions F computing the intersection numbers of modular geodesics and linking numbers of modular knots. It will serve again in Chapter 5 where f will be given by $\mathrm{bir}(A, B)$.

The functions F can be considered as Poincaré series in two variables, especially when f is a function of $\mathrm{Tr}(A), \mathrm{Tr}(B), \mathrm{Tr}(AB)$, such as $\mathrm{bir}(A, B)$. Special cases of such bivariate Poincaré series have appeared in [For23, Chapter V] and [Pau13].

2.1 Action of $\mathrm{PSL}_2(\mathbb{Z})$ on the space of lines $\mathbb{P}(\mathbb{Z}^2)$

Let us recall here the following elements of $\mathrm{GL}_2(\mathbb{Z})$, all in $\mathrm{SL}_2(\mathbb{Z})$ except J. They satisfy the relations $S^2 = T^3 = -\mathbb{1}$ and $J^2 = \mathbb{1}$ as well as $L = T^{-1}S$ and $R = TS^{-1}$.

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

To prevent confusion, notice that although L and R are not conjugate in $\mathrm{PSL}_2(\mathbb{Z})$, the identities $LJ = JR$, and $LS = SR^{-1}$ show that L is conjugate to R in $\mathrm{PGL}_2(\mathbb{Z})$ and to R^{-1} in $\mathrm{PSL}_2(\mathbb{Z})$.

The Lagrangian complex $\triangle_2$

Consider an integral plane with a fixed basis (v_∞, v_0), that is a free $\mathbb{Z}$-module of rank 2 decomposed as $\mathbb{Z}v_\infty \oplus \mathbb{Z}v_0$, which we can write as $\mathbb{Z}^2$. In particular the basis induces an orientation and a symplectic form $(u, v) \mapsto \det(u, v)$.

A *line* $V \subset \mathbb{Z}^2$ is a maximal submodule of rank 1, that is a lagrangian subspace. The set of lines $\mathbb{P}(\mathbb{Z}^2)$ forms a rational projective line, which is cyclically ordered by the symplectic form on $\mathbb{Z}^2$ so that if three lines $U, V, W \in \mathbb{QP}^1$ are generated by vectors $u, v, w \in \mathbb{Q}^2$ with sum $u + v + w = 0$, the cyclic order $\mathrm{cord}(U, V, W)$ is given by the common value $\operatorname{sign}\det(u, v) = \operatorname{sign}\det(v, w) = \operatorname{sign}\det(w, u)$.

Remark 2.1. *As a matter of convention, we represent oriented planes in such a way that the positive direction induced on their unit circle is the trigonometric one, but we represent the associated projective lines with the induced cyclic order so that the positive direction is the clockwise one.*

This is because we parametrize the set of lines according to their inclination using the affine chart $x \in \mathbb{Q} \mapsto [x : 1] \in \mathbb{QP}^1$ instead of their slope which is given by the affine chart $y \in \mathbb{Q} \mapsto [1 : y] \in \mathbb{QP}^1$. We will often denote V_x the line of inclination $[x : 1]$ and v_x a primitive vector on this line.

The *Lagrangian complex* $\triangle_2$ is the simplicial complex of dimension two whose vertices are the lines in $\mathbb{Z}^2$, whose edges are the pairs of lines whose sum is $\mathbb{Z}^2$, and such that every triangle is filled by a face. Its orientation is given by the cyclic order on $\mathbb{P}(\mathbb{Z}^2)$. The *base edge* of $\triangle_2$ connects the lines generated by the basis vectors.

A simplex is *labelled* when we fix a linear order on its vertices. An oriented triangle has three positive labellings and three negative labellings.

Proposition 2.2. *The group* $\mathrm{PSL}_2(\mathbb{Z})$ *acts on the Lagrangian complex* $\triangle_2$*, freely transitively on the labelled edges, as well as on the positively labelled triangles. The stabilisers of an edge or triangle permute their labels cyclically and are respectively conjugate to the subgroups* $\mathbb{Z}/2$ *or* $\mathbb{Z}/3$ *generated by* S *or* T*.*

Proof. The group $\mathrm{PSL}_2(\mathbb{Z})$ acts on the space of lines $\mathbb{P}(\mathbb{Z}^2)$, hence on its cartesian products $\mathbb{P}(\mathbb{Z}^2)^k$ modulo the alternate groups $\mathfrak{A}_k$. We must check that the simplices of Δ_2 are preserved under these actions.

A line is generated by a unique vector up to change of sign (this vector is primitive, meaning not a non trivial multiple of another one, or visible from the origin). The group $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the set of primitive vectors, so $\mathrm{PSL}_2(\mathbb{Z})$ acts transitively on the space of lines.

The lines U_1, U_2 generate a submodule $U_1 + U_2$ equal to $\mathbb{Z}^2$ if and only if they admit generators u_j forming a positive basis, in which case the tuple (u_1, u_2) is unique up to change of sign. The group $\mathrm{SL}_2(\mathbb{Z})$ acts freely transitively on the set of oriented bases, so $\mathrm{PSL}_2(\mathbb{Z})$ acts freely transitively on the set of ordered pairs of lines.

In particular $\mathrm{PSL}_2(\mathbb{Z})$ preserves the set of vertices and edges of $\triangle_2$ so it acts by simplicial automorphisms.

Recall the basis (v_∞, v_0) and let $v_{\pm 1} = v_\infty \pm v_0$. Denote V_j the line $\mathbb{Z}v_j$. The triples (V_0, V_1, V_∞) and (V_0, V_{-1}, V_∞) form positive and negative triangles in $\triangle_2$. Up to relabelling, these are the only triangles containing the base edge (because the only solutions to the system $\det(v_\infty, v)^2 = 1 = \det(v, v_0)^2$ are $v_{\pm 1}$).

We deduce from the transitive action on labelled edges that each one is the first of a unique positively labelled triangle, thus $\mathrm{PSL}_2(\mathbb{Z})$ acts transitively on positively labelled triangles. The assertion about stabilisers follows from the fact that the order two matrix S acts by switching the labels of the edge (V_∞, V_0), whereas the order three matrix T acts by cyclically permuting the labels of the triangle (V_0, V_1, V_∞). □

Remark 2.3. *The action of* $\mathrm{PGL}_2(\mathbb{Z})$ *on* $\triangle_2$ *is freely transitive on labelled triangles, and its elements preserve or reverse the orientation according to the sign of their determinant. Indeed, the matrix* J *generates the kernel of* $\det\colon \mathrm{PGL}_2(\mathbb{Z}) \to \{\pm 1\}$ *and acts like a symmetry of* $\mathbb{P}(\mathbb{Z}^2)$ *exchanging* v_∞ *and* v_0 *while fixing* $v_{\pm 1}$*.*

The lotus and the spiderweb

The *universal lotus* $\triangle_1$ is an oriented simplicial complex with a base edge (v_∞, v_0), which can be embedded in $\mathbb{R}^2$ as represented in figure 2.1, and constructed as follows. The vertices are all primitive vectors in $\mathbb{N}^2$, every pair of vertices forming a basis of $\mathbb{Z}^2$ yields an edge, and every edge (u, v) yields a positively oriented triangle $(u, u+v, v)$. The boundary of $\triangle_1$ consists in the base edge (v_∞, v_0) and the set of all vertices.

Scholium 2.4. *P. Popescu-Pampu baptised the* universal Lotus *in [PP11] while investigating resolutions of plane curve singularities, and used it to relate several of their combinatorial invariants (see [GBGPPP20] for a recent account).*

We define the oriented simplicial complex $\triangle_4$ from the embedding $\triangle_1 \subset \mathbb{R}^2$ by taking symmetries in the axes. The quotient by the antipodal map of $\mathbb{R}^2$ sends it to an oriented CW-complex which is not simplicial, but after identifying the base edges one recovers an oriented simplicial complex obtained by attaching two copies of the lotus along their base edges, which is isomorphic to the Lagrangian complex $\triangle_2$.

The map $\mathbb{P}(\psi)\colon \mathbb{P}(\mathbb{R}^2) \to \mathbb{P}(\mathbb{X})$ in Corollary 1.34 which intertwines the tautological and adjoint actions of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{P}(\mathbb{R}^2)$ and $\mathbb{P}(\mathbb{X})$, quotients $\triangle_4$ to $\triangle_2$. As before, this quotient is in the category of cell complexes: let us rectify $\psi(\triangle_4)$ to obtain a geometric realisation of $\triangle_4$ in $\mathfrak{sl}_2(\mathbb{R})$, which projectivizes to a geometric realisation of $\triangle_2$ in $\mathfrak{sl}_2(\mathbb{R})$. The piecewize linear map $\bar{\psi}\colon \triangle_4 \to \mathfrak{sl}_2(\mathbb{R})$ is defined like ψ on the vertices of $\triangle_4$ and extends linearly on each triangle.

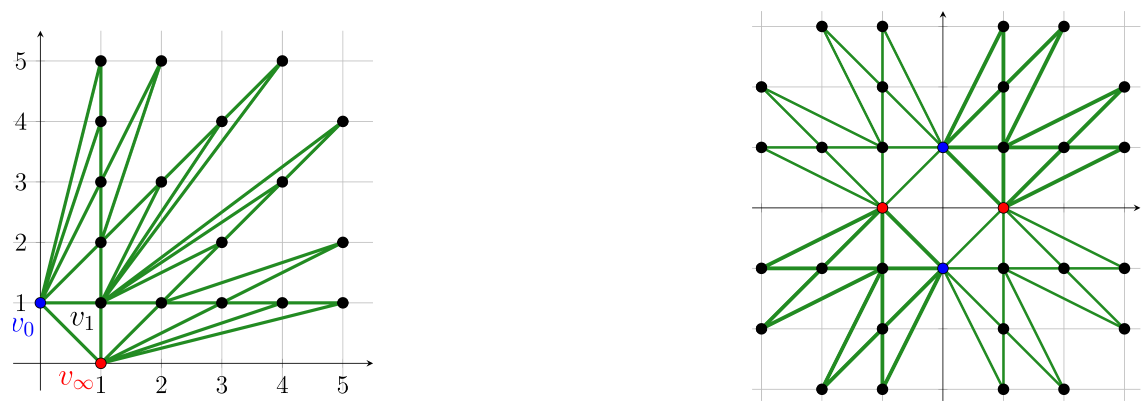


Figure 2.1: The lotus $\triangle_1$ and its quadruple $\triangle_4$.

The *spiderweb* is the simplicial complex embedded in $\mathfrak{sl}_2(\mathbb{R})$ as represented in figure 2.2, and constructed as follows. Vertices are all primitive vectors of $\mathfrak{sl}_2(\mathbb{Z})$ belonging to the isotropic cone $\mathbb{X}$, edges are all segments connecting pairs of vertices p, q such that $\langle p, q\rangle = 1$, and every triangle is filled by a face.

Proposition 2.5. *The spiderweb is a geometric realisation of $\triangle_2 \sqcup \triangle_2$, whose upper half-component is $\bar{\psi}(\triangle_4)$, obtained from $\psi(\triangle_4)$ by rectifying its edges.*

The map $\mathfrak{sl}_2(\mathbb{Z}) \to \mathbb{P}(\mathfrak{sl}_2(\mathbb{Z}))$ sends the spiderweb to a geometric realisation of $\triangle_2$ yielding an ideal triangulation of the projective model for the hyperbolic plane $\mathbb{HP}$.

We deduce an adjoint action of $\mathrm{PGL}_2(\mathbb{Z})$ on the geometric realisation of $\triangle_2$. It is equivalent to its tautological action on the Lagrangian complex $\triangle_2$.

Proof. Recall the quadratic map $\psi\colon \mathbb{R}^2 \to \mathbb{X}$ from Lemma 1.33 whose image is the upper half cone, and which sends $\det(u,v)^2$ to $\langle \psi(u), \psi(v)\rangle$.

The image of $\triangle_4$ by ψ defines a simplicial complex in the upper half cone: the vertices are the primitive vectors of $\mathfrak{sl}_2(\mathbb{Z})$ lying in the upper half cone, and the edges connect the pairs of vertices p, q such that $\langle p, q\rangle = 1$. If the arcs of ellipses forming its edges are rectified to segments with the same endpoints, we obtain a new simplicial complex which is the one defining the upper half component of the spiderweb.

Notice that the image by ψ of the square formed by the base edges of $\triangle_4$ consists in a circular bigon: after rectification it gets folded on the segment between its vertices $\psi(v_\infty) = \psi(1,0) = -\operatorname{pr}(R)$ and $\psi(v_0) = \psi(0,1) = \operatorname{pr}(L)$. Hence the upper half component of the spiderweb, which we know equal to the rectification of $\psi(\triangle_4)$, is the quotient of $\psi(\triangle_4)$ by the identification of the two base edges, that is $\triangle_2$.

The adjoint action of $\mathrm{PGL}_2(\mathbb{Z})$ on $\mathfrak{sl}_2(\mathbb{Q})$ preserves the lattice $\mathfrak{sl}_2(\mathbb{Z})$ and its subset of primitive vectors, as well as the isotropic cone $\mathbb{X}$ and the scalar product. Consequently, $\mathrm{PGL}_2(\mathbb{Z})$ acts on the spiderweb and its projectivization, that is the geometric realisation of $\triangle_2$ in $\mathbb{P}(\mathfrak{sl}_2(\mathbb{R}))$.

It follows from Corollary 1.34 to Lemma 1.33 that this adjoint action is equivalent to the tautological action of $\mathrm{PGL}_2(\mathbb{Z})$ on the Lagrangian complex. $\square$

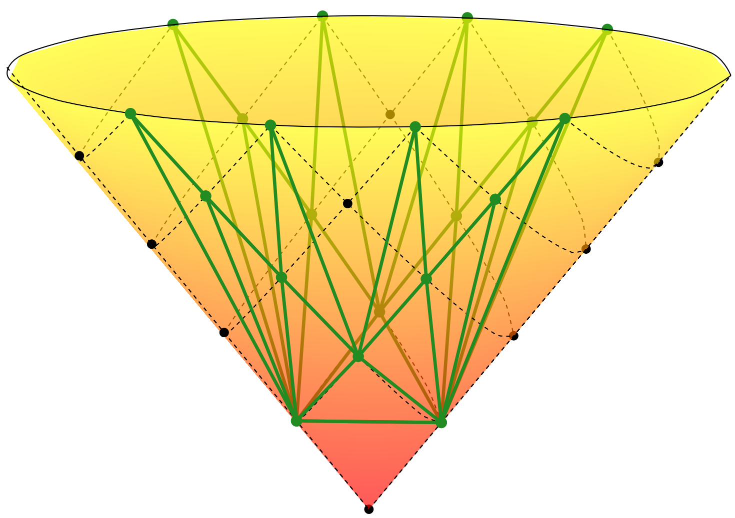


Figure 2.2: The spiderweb $\bar{\psi}(\triangle_1)$ inscribed in the half-cone $\psi(\mathbb{R}^2)$.

Remark 2.6. *Our conventions concerning the orientations of* $\mathbb{P}(\mathbb{R}^2)$ *and* $\mathbb{P}(\mathbb{H} \cup \mathbb{X})$ *imply that* $\mathbb{P}(\bar{\psi})\colon \triangle_4 \subset \mathbb{R}^2 \to \triangle_2 \subset \mathbb{P}(\mathbb{H} \cup \mathbb{X})$ *is orientation reversing.*

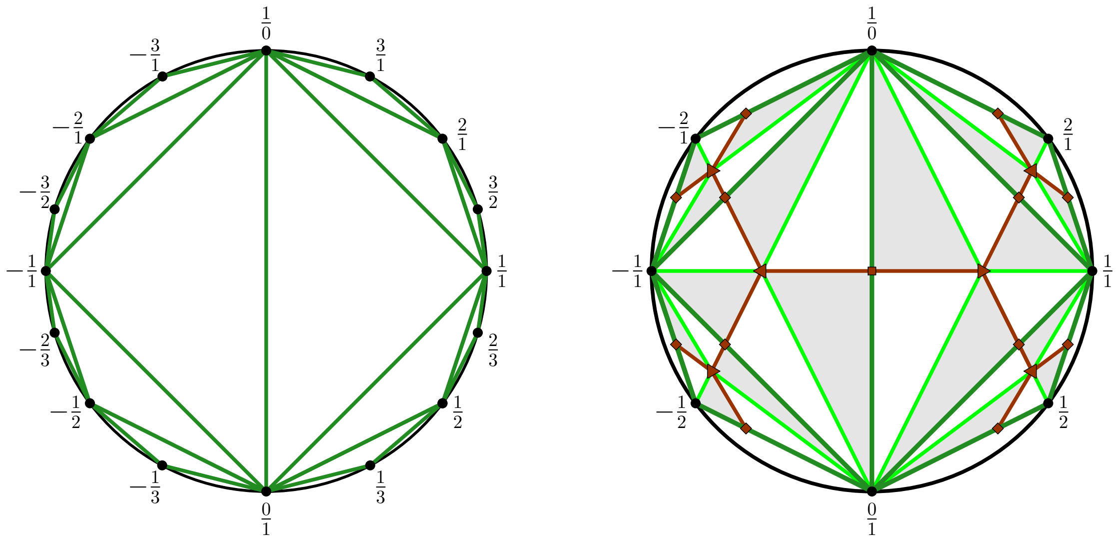


Figure 2.3: The lagrangian complex $\triangle_2$ and its first barycentric subdivision $\triangle_2'$.

The trivalent tree $\mathcal{T}$

Let us construct the first barycentric subdivision $\triangle_2'$ of the Lagrangian complex, represented in figure 2.3. The aim is to derive a corollary to Proposition 2.2, which will be our starting point to classify conjugacy classes in $\mathrm{PSL}_2(\mathbb{Z})$.

The k-simplices of $\triangle_2'$ correspond to the $(k+1)$-flags of simplices in $\triangle_2$, and their incidence relations are given by the inclusion of flags. In particular there is one vertex per simplex of $\triangle_2$ called its barycenter. Therefore $\triangle_2'$ has three kinds of vertices, three kinds of edges and one kind of face. Its vertices correspond to barycenters of vertices, edges or faces; its edges connect the barycenters of vertices and edges, of edges and faces, or of vertices and faces; its faces are triangles with one vertex and one edge of each kind.

Remark 2.7. *To draw the first barycentric subdivision $\triangle_2'$, consider each edge of $\triangle_2$ in turn and join the two vertices forming its adjacent triangles. In this way, every triangle of $\triangle_2$ gets cut by three medians to the mid-point of the edge, and concurrent at the barycenter of the triangle. For instance the edge $(\infty, 0)$ is incident to the triangles $(0, 1, \infty)$ and $(0, -1, \infty)$ so one must join the vertices -1 and 1.*

One may also construct $\triangle_2'$ using the geometric realisation of $\triangle_2$ in $\mathbb{HP}$. The hyperbolic ideal triangle $(0, 1, \infty)$ has a unique inscribed circle tangent at each edge: it is centered at the fixed point of T and tangent to $(\infty, 0)$ at the fixed points of S.

The edges of $\triangle_2'$ joining the barycenters of edges and faces of $\triangle_2$ form a bipartite tree $\mathcal{T}'$. This tree $\mathcal{T}'$ is the first barycentric subdivision of a trivalent tree $\mathcal{T}$, which is by construction the dual graph to the 1-skeletton of $\triangle_2$. Those trees inherit a cyclic order from their planar embeddings, meaning that around each vertex, the set of its incident edges has a cyclic order. We endow them with simplicial metrics for which all edges are isometric to a segment of the same length.

Corollary 2.8. *The action of* $\mathrm{PSL}_2(\mathbb{Z})$ *on* $\mathcal{T}'$ *is freely transitive on its set of edges. The stabilisers of vertices with degree* 2 *and* 3 *are cyclic groups of order* 2 *and* 3*, conjugate to those generated by* S *and* T *respectively. Thus* $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$.

Remark 2.9. *We may define the sign of an order* 3 *element* $A \in \mathrm{PSL}_2(\mathbb{Z})$ *according to the cyclic order of* (x, Ax, A^2x) *for any* $x \in \mathbb{QP}^1$*, or equivalently for any edge* x *incident to the vertex of* $\mathcal{T}$ *stabilized by* A*. For instance* T *is a positive rotation whereas* $-T^2 = T^{-1}$ *is a negative rotation.*

The sign of an order 3 *rotation is invariant by conjugation, and the cyclic groups appearing in the presentation of* $\mathrm{PSL}_2(\mathbb{Z})$ *are cyclically ordered.*

In $\mathcal{T}'$ we define an *edge-path* to be a sequence of edges, indexed by an interval of $\mathbb{Z}$ whose length may be finite or infinite, such that any two successive elements share an extremity. When such adjacent edges are distinct, they share exactly one extremity (at which point one may describe the edge-path as being locally non-constant). An edge-path yields a sequence of S and $T^{\pm 1}$ indicating how to turn the edges around their bivalent and trivalent vertices to get from one to the next.

An edge-path of $\mathcal{T}'$ is called *locally geodesic* when the corresponding sequence of S and $T^{\pm 1}$ alternates between both letters (so we are never turning around vertices). This amounts to saying that any three consecutive edges in the sequence are such that the first and the last share no common extremities (neither equal nor adjacent). Since $\mathcal{T}'$ is a tree, all edges in a local geodesic are distinct.

Any two edges e_0, e_l are connected by a unique edge-path $(e_0, \dots, e_l)$ that is locally geodesic: it is the shortest of them all so we may call it a global geodesic, denote it (e_0, e_l). The distance $d(e_0, e_l) = l$ is the length of the corresponding $S\&T^{\pm 1}$-sequence. There is a unique shortest path connecting an edge to (an edge in) a geodesic, or to (an edge around) a given vertex. There is a unique semi-infinite geodesic leading from an edge to a boundary point and any two boundary points are connected by a unique bi-infinite geodesic.

A *horocycle* of $\mathcal{T}'$ is a bi-infinite geodesic which bounds a connected component of $\triangle \setminus \mathcal{T}$. These connected components are indexed by the vertices of $\triangle$, and the horocycle corresponding to $p/q \in \mathbb{QP}^1 = \partial\triangle$ converges either way to that point. A geodesic is a horocycle when it turns the same way at every trivalent vertex, that is when the elements $T^{\pm 1}$ always appear with the same power in the corresponding sequence of $S\&T^{\pm 1}$.

Remark 2.10. *One may also define paths of oriented edges in $\mathcal{T}$, and consider the corresponding sequences of L, R and S. A locally geodesic oriented edge-path of $\mathcal{T}$ can only contain S at its extremities.*

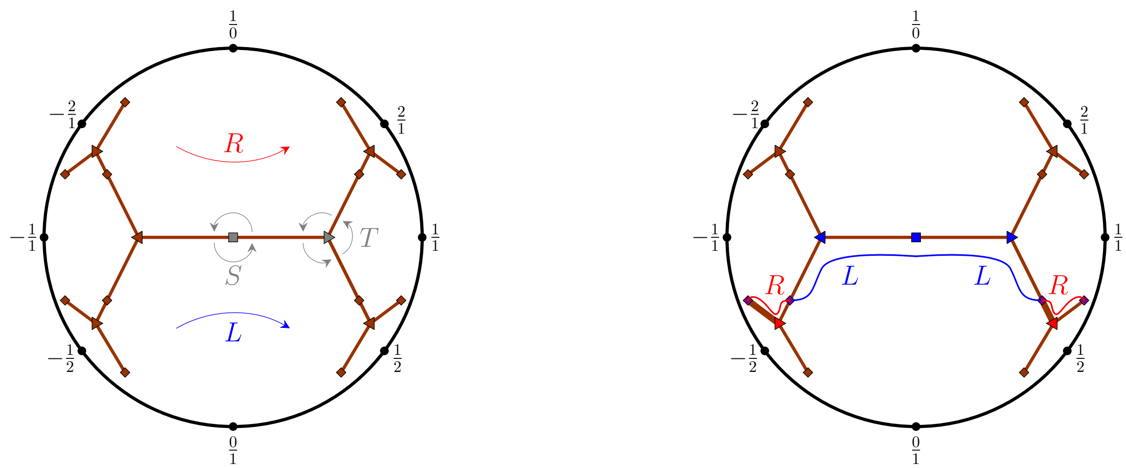


Figure 2.4: The action of S, T, L, R and LRL on the dual tree $\mathcal{T}$ of $\triangle_2$.

Conjugacy classes in the modular group $\mathrm{PSL}_2(\mathbb{Z})$

We now show the structure theorems for the amalgam $\mathbb{Z}/2 * \mathbb{Z}/3$, providing a reduced factorisation of elements and normal forms for their conjugacy classes.

Theorem 2.11. *Every element in $\mathrm{PSL}_2(\mathbb{Z})$ has a unique factorisation as a product of the form $\prod_k T^{\sigma_k} S$, where $\sigma_k \in \{\pm 1\}$ except the first and last which may also be 0.*

An element of finite order is conjugate to a power of S or T, whereas an element of infinite order is conjugate to a non-empty product of $L = T^{-1}S$ and $R = TS^{-1}$ which is unique up to cyclic permutation.

Remark 2.12. *Recall the partition of non trivial matrices $A \in \mathrm{PSL}_2(\mathbb{Z})$ into three types: elliptic, parabolic, hyperbolic. These have been defined by the sign of the discriminant of their characteristic polynomial which equals $\operatorname{disc}(A) = \operatorname{Tr}(A)^2 - 4$.*

The non-trivial elements of finite order are the elliptic ones whereas those of infinite order partition into parabolic and hyperbolic.

A trace computation shows that parabolic elements are conjugate to positive powers of L or R whereas hyperbolic elements are conjugate to products of L and R in which each letter appears at least once.

Remark 2.13 (Unicity of elliptic generators)**.** *The structure theorem implies that $\{S, T\}$ is the unique pair, up to independent inversions and simultaneous conjugation, of torsion elements which generate the modular group.*

Scholium 2.14. *In fact we shall later see that if a pair of elliptic elements in $\mathrm{PSL}_2(\mathbb{R})$ generates a subgroup conjugate to $\mathrm{PSL}_2(\mathbb{Z})$, then up to independent inversions and simultaneous conjugation, it is $\{S, T\}$.*

Proof of Theorem 2.11. The action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{T}'$ is freely transitive on its edges, so every A is identified with $e_A = A \cdot e_\mathbb{1}$ where $e_\mathbb{1}$ denotes the base edge. The geodesic $(e_\mathbb{1}, e_A)$ yields the unique reduced sequence of $S\&T^{\pm 1}$ whose product equals A.

Now conjugate this reduced $S\&T^{\pm 1}$ factorisation by the longest suffix whose inverse appears as a prefix (they cannot overlap nor meet half way because the word is reduced). If we obtain S or $T^{\pm 1}$ then we are done, otherwize we find a new sequence whose first and last letters are not mutually inverse.

Notice that if this sequence starts with $T^{\pm 1}$ and ends with S then the geodesic in $\mathcal{T}'$ can be read as a geodesic in $\mathcal{T}$ yielding a sequence of $L\&R$. Similarly if it starts with an S and ends with a $T^{\pm 1}$ then one may read it in $\mathcal{T}$ and find a sequence of L^{-1} and R^{-1}, but this is conjugate by S to the transpose of its inverse which is a sequence in L and R. If it starts and ends with the same letter $T^{\pm 1}$ then one may conjugate by that element $T^{\pm 1}$ to obtain one of the above. □

Proposition 2.15. *Consider the action of a non trivial $A \in \mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{T}'$.*

If A has finite order then it fixes a unique vertex in $\mathcal{T}'$ whose valence 2 or 3 equals the order of the element.

If A has infinite order then it stabilises a unique bi-infinite geodesic of $\mathcal{T}'$ on which it acts by translation.

Proof. For $A \in \mathrm{PSL}_2(\mathbb{Z})$, let l_A be the minimum distance $d(e, A \cdot e)$ over all edges e. We have $l_A = 0$ if and only if $A = \mathbb{1}$.

If $l_A > 0$ then $l_A = 1$ if and only if A fixes exactly one vertex. The geodesic leading from the base edge to (an appropriate edge belong to the link of) that vertex yields a matrix which conjugates A to S or $T^{\pm 1}$.

If $l_A > 1$ then the edges e such that $d(e, A \cdot e) = l_A$ form a bi-infinite geodesic, which can be obtained as the limit when n goes to infinity of the intersection $(e_{A^{-n}}, e_{A^n}) \cap (e_{A^{-n-1}}, e_{A^{n+1}})$. Indeed this defines a stable geodesic g_A under the action of A along which it acts by translation, and every edge e of $\mathcal{T}'$ is sent to an edge $A \cdot e$ at distance $d(e, A \cdot e) = l_A + 2 \cdot d(e, g_A)$. A geodesic from the base edge to (an appropriate edge in) the geodesic g_A yields a matrix which conjugates A to a matrix whose axis passes through the base edge, and which factorises as a product of $L = T^{-1}S$ and $R = TS^{-1}$.

Note that one must be a little careful in adjusting the edge-paths leading from the base edge to the link of a vertex or to a geodesic, in order to yield the matrix conjugating A to S, $T^{\pm 1}$ or a product of $L\&R$. □

Gromov boundary of the tree $\mathcal{T}$

The technical details in this subsection will not be needed in the sequel so we refer to [GdlH90] for background about Gromov hyperbolic spaces and their boundaries.

A graph becomes a metric space by declaring that all its edges are isometric to a given segment of $\mathbb{R}$. In this way $\mathcal{T}$ and $\mathcal{T}'$ become real trees, the latter being identified with the former after subdividing each edge in two edges of equal length.

The automorphism group of a simplicial graph equals the isometry group of the corresponding metric graph, so the isomorphism group of the cyclically ordered trivalent real tree $\mathcal{T}$ is $\mathrm{PSL}_2(\mathbb{Z})$.

A real tree is a 0-hyperbolic space, meaning that for any triangle every edge is contained in the union of the other two. One may define its Gromov boundary: its elements are the classes of half-infinite geodesics modulo the equivalence relation of admitting equal tails; and to describe its topology it is convenient to take advantage of the identification with its space of ends.

The space of ends of a topological space is the direct limit with respect to inclusion of the connected components in the complement of compact sets, endowed with the direct limit topology. The ends of $\mathcal{T}$ form a totally disconnected compact space $\partial\mathcal{T}$.

Denote $\mathcal{G}$ the product $\partial\mathcal{T} \times \partial\mathcal{T}$ minus the diagonal, which corresponds to the set of bi-infinite oriented geodesics in $\mathcal{T}$. Notice that the set of primitive infinite order elements of $\mathrm{PSL}_2(\mathbb{Z})$ injects in $\mathcal{G}$ by the map $A \mapsto g_A$. Those are precisely the bi-infinite periodic geodesics.

The isometric action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{T}$ extends to a continuous action on $\partial\mathcal{T}$. A non trivial element A has 0 fixed points if it has finite order, and 2 fixed points if it has infinite order given by the endpoints of its translation axis.

If A is parabolic then its axis is a horocycle, which as a subset of $\triangle_2'$ converges both ways to the same vertex of $\triangle_2$. If A is hyperbolic its axis has two distinct endpoints in the completion of $\triangle_2$ by $\partial\mathcal{T}$, which do not belong to $\triangle_2$.

2.2 Conjugacy classes of matrices in $\mathrm{PSL}_2(\mathbb{Z})$

Continued fractions, Euclidean monoid, Binary tree

Every positive real number x admits a *Euclidean continued fraction expansion*:

$$\lfloor n_0, n_1, n_2, \dots \rfloor = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \dots}} \quad \text{with} \quad n_j \in \mathbb{N} \quad \text{and} \quad \forall j > 0,\ n_j > 0.$$

Such an expansion is infinite if and only if x is irrational, in which case it is unique. A rational x has two expansions $x = \lfloor n_0, \dots, n_k \rfloor$: one for which $n_k = 1$, the other for which $n_k > 1$, and exactly one of these has even *length* $k+1$.

To represent negative real numbers we apply the involution $x \mapsto S(x) = -1/x$. Thus every number $x \in \mathbb{RP}^1$ admits exactly one representation $x = \lfloor n_0, \dots \rfloor$ or $x = -1/\lfloor n_0, \dots \rfloor$, except for the rationals which have two, including $0 = \lfloor\ \rfloor = \lfloor 0 \rfloor$ and $\infty = -1/\lfloor\ \rfloor = -1/\lfloor 0 \rfloor$. This corresponds to the partition $\mathbb{RP}^1 = [0, \infty[\sqcup S \cdot [0, \infty[$.

Remark 2.16. *If we wished to work only with $x > 1$ so that all $n_j \in \mathbb{N}^*$, then we must use both involutions $x \mapsto -x$ and $x \mapsto 1/x$ to obtain the four intervals of $\mathbb{RP}^1$. This corresponds to the partition $\mathbb{RP}^1 = \mathbf{1}\cdot]1, \infty] \sqcup S\cdot]1, \infty] \sqcup J\cdot]1, \infty] \sqcup K\cdot]1, \infty]$.*

Then every number $x \in \mathbb{RP}^1$ admits exactly one of the four representations $x = \pm\lfloor n_0, \dots \rfloor^{\pm 1}$ with all $n_j \in \mathbb{N}^*$, except for the rationals which have two, except again for the fixed points under the additive or multiplicative inverses which have four.

While the extended modular group $\mathrm{PGL}_2(\mathbb{Z})$ acts on $\mathbb{RP}^1$, its *extended euclidean submonoid* generated by $J\&R$ preserves $[0, \infty]$. If x_i denotes the i^{th} remainder of x, given by the tail $\lfloor n_i, \dots \rfloor$ of its continued fraction expansion, then $x_0 = (R^{n_0}J)x_1$. So the orbits of $x, y \in [0, \infty]$ under the extended euclidean monoid have non-empty intersection if and only if some tails x_i and y_j of their continued fractions coincide. Since $J\&R$ generate the group $\mathrm{PGL}_2(\mathbb{Z})$, we deduce that $x, y \in \mathbb{RP}^1$ belong to the same $\mathrm{PGL}_2(\mathbb{Z})$-orbit if and only if some tails of their continued fractions coincide.

Now consider the action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{RP}^1$ and of its *euclidean submonoid* generated by $L\&R$ on $[0, \infty]$. As $JRJ = L$ we have $R^{n_0}JR^{n_1}J = R^{n_0}L^{n_1}$, so assembling the $R^{n_k}J$ by pairs reveals the action of $L\&R$ as $x = R^{n_0}L^{n_1}x_2$. Hence the orbits of $x, y \in [0, \infty]$ under the euclidean monoid have non-empty intersection if and only if there exist even starting points i, j at which the tails x_i and y_j coincide. Since $L\&R$ generate the group $\mathrm{PSL}_2(\mathbb{Z})$, we deduce that $x, y \in \mathbb{RP}^1$ belong to the same $\mathrm{PSL}_2(\mathbb{Z})$-orbit if and only if there exist even starting points i, j at which the tails x_i and y_j coincide.

Remark 2.17. *We are interested in the action of the modular group* $\mathrm{PSL}_2(\mathbb{Z})$ *so we must keep an eye on the parity of the indices in the continued fractions.*

A real number is called *purely periodic* if its continued fraction expansion is periodic from n_0 onward ("immédiatement périodique" in [Gal29]). We call *periodic* a real number whose continued fraction expansion is periodic either from n_0 onward if $n_0 > 0$, or else from n_1 onward.

For us, the period will always have an even length, and we shall retain the earliest possible starting point (which may be even or odd). The positive and negative periodic irrationals are precisely attractive and repulsive fixed points of matrices which are products of L and R, with at least one occurrence of each.

Denote $\mathrm{SL}_2(\mathbb{N}) \subset \mathrm{SL}_2(\mathbb{Z})$ the submonoid of matrices with non-negative entries, which we identify with its image $\mathrm{PSL}_2(\mathbb{N})$ in $\mathrm{PSL}_2(\mathbb{Z})$. It contains the parabolic matrices L, R, and every element in the monoid that they generate.

Lemma 2.18. *The submonoid of* $\mathrm{PSL}_2(\mathbb{N})$ *generated by* L *and* R *is free, and it coincides with* $\mathrm{PSL}_2(\mathbb{N})$*.*

The orbit map $A \mapsto A(1)$ *defines a bijection* $\mathrm{PSL}_2(\mathbb{N}) \to]0, \infty[\cap \mathbb{Q}$ *which is increasing when* $\mathrm{PSL}_2(\mathbb{N})$ *is endowed with the lexicographic order extending* $L < R$*.*

Proof. The action of $\mathrm{PSL}_2(\mathbb{N})$ on $\mathbb{QP}^1$ preserves the interval $]0, \infty[$. The element R maps it into $]1, \infty[$ whereas L maps it into $]0, 1[$, so they generate a free submonoid.

The orbit map $A \mapsto A(1)$ defines a bijection $\mathrm{PSL}_2(\mathbb{N}) \to]0, \infty[\cap \mathbb{QP}^1$ since the reciprocal map consists in decomposing a fraction $\frac{p}{q} > 0$ as a Farey sum $\frac{a+b}{c+d}$, which is unique: it reduces to finding the minimal Bezout relations $aq - cp = 1 = pd - qb$.

A positive rational $\frac{p}{q}$ has a unique continued fraction expansion finishing by 1, and ignoring the last 1, this decomposes the corresponding matrix $A \in \mathrm{PSL}_2(\mathbb{N})$ as a product of positive powers of L and R, hence these generate $\mathrm{PSL}_2(\mathbb{N})$. □

In particular, $\mathrm{PSL}_2(\mathbb{N})$ has no elliptic elements, its parabolic elements are powers of L or R, and its hyperbolic elements are products containing both L and R.

The normal form for conjugacy classes in the amalgam $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$ provided in Theorem 2.11, and the previous Lemma 2.18 imply the following.

Corollary 2.19. *Suppose* $A \in \mathrm{PSL}_2(\mathbb{Z})$ *is not elliptic. Then the intersection of its conjugacy class with* $\mathrm{PSL}_2(\mathbb{N})$ *is non empty, and consists in all cyclic permutations of a same word in the L&R alphabet. We call them its Lyndon conjugates.*

Thus every non elliptic conjugacy class in $\mathrm{PSL}_2(\mathbb{Z})$ is represented by a unique maximal element in $\mathrm{PSL}_2(\mathbb{N})$ for the lexicographic order, which we call the *maximal Lyndon representative*.

We also deduce that those real irrationals with eventually periodic expansions are precisely the fixed points of hyperbolic matrices $A \in \mathrm{PSL}_2(\mathbb{Z})$, hence the roots of integral quadratic polynomials $lX^2 + mX + r$ with positive discriminant $m^2 - 4lr$, also known as real quadratic irrationalities.

Lemma 2.20. *Let $A \in \mathrm{PSL}_2(\mathbb{Z})$ be a hyperbolic matrix with attractive and repulsive fixed points α and α' in $\mathbb{RP}^1$.*

The fact that $A \in \mathrm{PSL}_2(\mathbb{N})$ is equivalent to each of the following:

$$A^{-1}(-1) < 0 < A(1) \iff \alpha' < 0 < \alpha \iff |\mathrm{cross}|(\alpha', \alpha, 0, \infty) = 1$$

Now supposing $A \in \mathrm{PSL}_2(\mathbb{N})$, the fact that $A \in R \cdot \mathrm{PSL}_2(\mathbb{N})$ is equivalent to:

$$A(1) > 1 \iff \alpha > 1 \iff |\mathrm{cross}|(\alpha', \alpha, 1, \infty) = 1$$

Still supposing $A \in \mathrm{PSL}_2(\mathbb{N})$, the fact that $A \in \mathrm{PSL}_2(\mathbb{N}) \cdot L$ is equivalent to:

$$A^{-1}(-1) > -1 \iff \alpha' > -1 \iff |\mathrm{cross}|(\alpha', \alpha, 0, -1) = 1$$

The set $\{L, R\}^{\mathbb{N}}$ of infinite binary sequences on the letters $L \& R$ is given the lexicographic order extending $L < R$. The monoid $\mathrm{PSL}_2(\mathbb{N})$ maps to $\{L, R\}^{\mathbb{N}}$ by sending a finite word A to its periodisation A^∞. This map is increasing, and injective in restriction to primitive elements.

To a primitive hyperbolic matrix $A \in \mathrm{PSL}_2(\mathbb{N})$ corresponds a real quadratic irrationality $\alpha \in]0, \infty[$ given by its attractive fixed point. Its continued fraction is periodic and corresponds to the binary sequence $A^\infty \in \{L, R\}^{\mathbb{N}}$, which uses both letters $L \& R$. Conversely, such a periodic sequence has a smallest period of even length and this yields a matrix A and a real quadratic surd α.

Corollary 2.21. *The maps between primitive hyperbolic matrices $A \in \mathrm{PSL}_2(\mathbb{N})$, periodic real quadratic irrationalities $\alpha > 0$, and periodic binary sequences using both letters $A^\infty \in \{L, R\}^{\mathbb{N}}$, are bijective and order preserving.*

Note that for all $A \in \mathrm{SL}_2$ we have ${}^tAS = SA^{-1}$. If $A \in \mathrm{PSL}_2(\mathbb{N})$ is hyperbolic with attractive fixed point $\alpha > 0$, then the attractive fixed point of A^{-1} is the Galois conjugate $\alpha' < 0$, so the attractive fixed point of ${}^tA$ is $S(\alpha') = -1/\alpha' > 0$. Thus the transposition involution $A \mapsto {}^tA$ which preserves $\mathrm{PSL}_2(\mathbb{N})$ corresponds to the involution $\alpha \mapsto -1/\alpha'$ on positive periodic quadratic irrationalities.

Remark 2.22 (Application to a theorem of Galois [Gal29])**.** *A real number $\alpha \neq 0$ is purely periodic, meaning it has a periodic continued fraction expansion as from n_0, if and only if it is a real quadratic surd such that $\alpha > 1$ and $-1/\alpha' > 1$. If so, then $-1/\alpha'$ is purely periodic with the mirror image period. This should now be clear.*

Symmetric classes of real quadratic irrationalities

Recall from Section 0.2 the dictionary between primitive hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$, quadratic irrationals $\alpha \in \mathbb{R}$, primitive indefinite $Q(x,y) = lx^2 + mxy + ry^2 \in \mathcal{Q}(\mathbb{Z})$, and primitive space-like $\mathfrak{a} = \frac{1}{2}\left(\begin{smallmatrix} -m & -2r \\ 2l & m \end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathbb{Z})^\vee$. In particular if $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then (l,m,r) is proportional to $(c, d-a, -b)$ by the factor $\operatorname{sign}(a+d)/\gcd(c, d-a, -b)$.

Recall from Section 1.1 the orthogonal decomposition $\mathfrak{gl}_2(\mathbb{Q}) = \mathbb{Q}\mathbf{1} \oplus \mathfrak{sl}_2(\mathbb{Q})$ with repesct to the scalar product $\langle M, N\rangle = \frac{1}{2}\operatorname{Tr}(MN^{\#})$.

Proposition 2.23. *Consider a primitive hyperbolic $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z})$, denote by $\gamma_A = (\alpha', \alpha)$ its geometric translation axis in $\mathbb{HP}$, and by (l,m,r) the coefficients of the corresponding primitive indefinite $Q \in \mathcal{Q}(\mathbb{Z})$ or primitive space-like $\mathfrak{a} \in \mathfrak{sl}_2(\mathbb{Z})^\vee$.*

The axis γ_A passes through $i \in \mathbb{HP}$ if and only if equivalently:

α: $1 + \alpha\alpha' = 0$

$\mathfrak{a}$: $r + l = 0$ *or equivalently* $\langle \mathfrak{a}, S\rangle = 0$

A: $b = c$ *or equivalently* $\operatorname{Tr}(AS) = \operatorname{Tr}(AS^{-1})$ *or equivalently* $A = {}^tA$

Q: $\alpha = x + \sqrt{1+x^2}$ *for some* $x \in \mathbb{Q}^*$ *such that* $1 + x^2 \notin (\mathbb{Q}^*)^2$

The axis γ_A passes through $j \in \mathbb{HP}$ if and only if equivalently:

α: $\frac{\alpha'+\alpha}{2} = 1 + \alpha\alpha'$

$\mathfrak{a}$: $m + 2l + 2r = 0$ *or equivalently* $\langle \mathfrak{a}, T\rangle = 0$

A: $a + 2b = 2c + d$ *or equivalently* $\operatorname{Tr}(AT) = \operatorname{Tr}(AT^{-1})$

Q: $\alpha = (1+x) + \sqrt{1+x+x^2}$ *for some* $x \in \mathbb{Q}^*$ *such that* $1 + x + x^2 \notin (\mathbb{Q}^*)^2$

Proof. The axis γ_A passes through $i \in \mathbb{HP}$ if and only if $\alpha = S\alpha' = -1/\alpha'$, which is equivalent to $A = SA^{-1}S^{-1} = {}^tA$ by identifying the attractive fixed points. In terms of the quadratic form Q the condition $\alpha\alpha' = -1$ rewrites as $r/l = -1$, but one may also use the condition ${}^t\mathfrak{a} = \mathfrak{a}$ to deduce that $r = -l$. In any case we have $\alpha = \frac{-m+\sqrt{m^2+4l^2}}{2l}$, and setting $x = -m/(2l)$ yields the last condition.

The axis γ_A passes through j if and only if the points $\alpha', \alpha, T\alpha, T^{-1}\alpha$ are conjugate to $(-1, 1, 0, \infty)$, that is when $\operatorname{bir}(T^{-1}\alpha, \alpha, T\alpha, \alpha') = \frac{1+\alpha\alpha'-\alpha}{1+\alpha\alpha'-\alpha'}$ is equal to -1. This equality can be rewritten $\alpha + \alpha' = 2 + 2\alpha\alpha'$, or in terms of Q as $m/l = 2 + 2r/l$. This amounts to the existence coprime $l, r \in \mathbb{Z}$ such that $\alpha = \frac{(l+r)+\sqrt{l^2+lr+r^2}}{l}$, and setting $x = r/l$ yields the last condition. ☐

Remark 2.24. *In particular, γ_A cannot pass through i and j, as this would imply $m = 0$ and $r = -l = \pm 1$, contradicting the fact that $\mathrm{disc}(Q) = 4$ is not a square.*

However can γ_A pass through points in the orbit of i and j ?

Remark 2.25. *If the geometric axis $(\alpha', \alpha) \subset \mathbb{HP}$ of $A \in \mathrm{PSL}_2(\mathbb{Z})$ passes through i, then there is a unique element in $\{A, SAS^{-1}\}$ which belongs to $\mathrm{PSL}_2(\mathbb{N})$.*

Conversely, given $A \in \mathrm{PSL}_2(\mathbb{N})$ it is an easy matter to read from is L&R-factorisation if it is symmetric.

If the geometric axis $(\alpha', \alpha) \subset \mathbb{HP}$ of $A \in \mathrm{PSL}_2(\mathbb{Z})$ passes through j, then there is a unique $B \in \{A, TAT^{-1}, T^{-1}AT\}$ which belongs to $\mathrm{PSL}_2(\mathbb{N})$. Furthermore, there is a unique $C \in \{TB^{-1}T^{-1}, TB^{-1}T^{-1}\}$ which belongs to $\mathrm{PSL}_2(\mathbb{N})$.

Conversely, given $B, C \in \mathrm{PSL}_2(\mathbb{N})$ related as such, can one read from their L&R-factorisation if their axes pass through j ?

Definition 2.26. *A matrix of $\mathrm{SL}_2(\mathbb{Z})$ is* symmetric *when it equals its transpose. This relation descends to $\mathrm{PSL}_2(\mathbb{Z})$. We define a conjugacy class in $\mathrm{PSL}_2(\mathbb{Z})$ to be* symmetric *when it is globally preserved by inversion or transposition.*

Lemma 2.27. *A symmetric hyperbolic conjugacy class admits a symmetric Lyndon representative $A \in \mathrm{PSL}_2(\mathbb{N})$, it may thus be written $A = {}^tBB$ for some $B \in \mathrm{PSL}_2(\mathbb{N})$.*

A primitive symmetric hyperbolic conjugacy class admits exactly two symmetric Lyndon representatives, they are of the form ${}^tBB$ and $B{}^tB$ for some $B \in \mathrm{PSL}_2(\mathbb{N})$.

If a hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$ is symmetric, then A or A^{-1} belongs to $\mathrm{PSL}_2(\mathbb{N})$, so up to inversion all symmetric representatives of a hyperbolic class are Lyndon.

Proof. Consider a symmetric hyperbolic conjugacy class. Transposition preserves the set of Lyndon representatives, which are all cyclic permutations of some word in L&R. Let A be one of them: its transpose is equal to a cyclic permutation, so one may write $A = UV$ and ${}^tA = VU$ for $U, V \in \mathrm{PSL}_2(\mathbb{N})$, which implies that U and V are symmetric. But transposition exchanges L and R, which freely generate the monoid $\mathrm{PSL}_2(\mathbb{N})$, so $U = {}^tXX$ and $V = {}^tYY$ for some $X, Y \in \mathrm{PSL}_2(\mathbb{N})$. Hence $A = ({}^tXX)({}^tYY)$ is conjugate to ${}^tBB$ for $B = X{}^tY$.

For a symmetric hyperbolic conjugacy class, consider two representatives written as ${}^tBB = UV$ and ${}^tCC = VU$ for $B, C, U, V \in \mathrm{PSL}_2(\mathbb{N})$, and suppose we may find U such that $0 < \operatorname{len} U < \operatorname{len} V$. Then ${}^tB = UW$ with $W \in \mathrm{PSL}_2(\mathbb{N})$ non trivial, so ${}^tBB = UW{}^tW{}^tU$ whence ${}^tCC = W{}^tW{}^tUU$. The latter also equals its transpose ${}^tCC = {}^tUUW{}^tW$ from which we deduce that $\overline{U} = {}^tUU$ and $\overline{W} = W{}^tW$ commute. Consequently, there is some non trivial $X \in \mathrm{PSL}_2(\mathbb{N})$ such that $\overline{U} = X^u$ and $\overline{W} = X^w$ with $u, w > 0$, from which we deduce that ${}^tCC = X^{u+w}$ is not primitive. By contraposition, if the symmetric conjugacy class were primitive, then we should have either $\operatorname{len} U = 0$ in which case $B = C$ or $\operatorname{len} U = \operatorname{len} V$ in which case $B = {}^t C$. $\square$

Intersection patterns of lines with simplicial complexes

We now relate the continued fraction expansions of real numbers to the combinatorics of the Lagrangian complex, the lotus and the spiderweb.

Recall that Proposition 2.5 defined an isomorphism $\mathbb{P}(\bar{\psi})$ from the geometric realisation of $\triangle_4$ in $\mathbb{R}^2$ to the geometric realisation of $\triangle_2$ in $\mathbb{RP}^2$, which is projective linear in restriction to each triangle, in particular it defines an isomorphism between their first barycentric subdivisions. (Beware that is reverses orientations.)

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{R}^2$, and the matrices sending the lotus $\triangle_1$ into itself form the submonoid $\mathrm{SL}_2(\mathbb{N})$. The group $\mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathbb{HP}$ and the matrices sending the right half of its ideal triangulation $\triangle_2$ into itself form the submonoid $\mathrm{PSL}_2(\mathbb{N})$. By proposition 2.5, these actions are isomorphic under $\mathbb{P}(\bar{\psi})$.

In this paragraph, the continued fraction expansion of a positive rational number is always chosen to be the one with even length, hence given by an $L\&R$-sequence as explained in the first paragraph of this Section 2.2.

Proposition 2.28. *The (even) continued fraction expansion of $\alpha \in]0, \infty[$ encodes the sequence of triangles in the following simplicial complexes intersected by lines.*

1. *The lotus $\triangle_1$ intersected by the line V_α of inclination $\alpha \in \mathbb{RP}^1$.*

2. *The ideal triangulation $\triangle_2$ of $\mathbb{HP}$ intersected by the hyperbolic geodesic from $i \in \mathbb{HP}$ to $\alpha \in \partial\mathbb{HP}$.*

3. *The Lagrangian complex $\triangle_2$ intersected by the combinatorial geodesic of $\mathcal{T}$ from the barycenter $i \in \mathcal{T}'$ of the base edge to the boundary point $\alpha \in \partial\mathcal{T}$.*

In the last two items we may replace i by any real $\alpha_- < 0$ and restrict attention to the intersection of the geodesic (α_-, α) with the right hand side of $\triangle_2$.

Proof. (1). The euclidean algorithm implies that the line $V_\alpha \subset \mathbb{R}^2$ of inclination $\alpha \in \mathbb{RP}^1$ intersects the triangles of $\triangle_1$ according to its continued fraction expansion.

(1 $\iff$ 2). The map $\psi \colon \mathbb{R}^2 \to \mathbb{X}$ sends $\triangle_4$ to a simplicial complex $\psi(\triangle_4)$ in $\mathbb{X}$, which intersects the line $\psi(V_\alpha)$ in the same way as V_α intersects $\triangle_1$. The plane of $\mathfrak{sl}_2(\mathbb{R})$ generated by the lines $\mathbb{R}.S$ and $\psi(V_\alpha)$ intersects the edges of the spiderweb according to the same pattern. Projectifying sends the spiderweb to the ideal triangulation $\triangle_2$ of $\mathbb{HP}$, the line $\mathbb{R}.S$ to $i \in \mathbb{HP}$ and the line $\psi(V_\alpha)$ to $\alpha \in \partial\mathbb{HP}$.

($2 \iff 3$). This follows from the $\mathrm{PSL}_2(\mathbb{Z})$-equivariant identification of the Lagrangian complex $\triangle_2$ with its geometric realisation $\triangle_2$ in $\mathbb{HP} \cup \mathbb{QP}^1$. Indeed, the combinatorial and hyperbolic geodesics from $i \in \mathcal{T}' \subset \mathbb{HP}$ to the same point $\alpha \in \partial\mathcal{T}' = \partial\mathbb{HP}$ stay a bounded distance from one another, and this implies that they intersect the same triangles because the triangulation of $\mathbb{HP}$ by $\triangle_2$ is ideal. □

Remark 2.29. *In the dual tree $\mathcal{T}$ of the complex $\triangle_2$, the geodesic from the base edge to the boundary point $\alpha \in \partial\mathcal{T}$ cuts a sequence of edges in $\triangle_2$ whose extremities in $\mathbb{QP}^1$ are the successive slow convergents to the continued fraction expansion of α.*

Remark 2.30. *Recall that S acts on $\mathbb{RP}^1$ by $S(\alpha) = -1/\alpha$, on $\triangle_4 \subset \mathbb{R}^2$ by linear $(\pi/4)$-rotation, and on $\mathcal{T} \subset \mathbb{HP}$ by hyperbolic $(\pi/2)$-rotation around i.*

So for $\alpha < 0$, to read the intersection patterns $V_\alpha \cap S\triangle_1$ or $(\alpha, 0) \cap \triangle_2$ coming from infinity, one must transpose the L&R-word of $-1/\alpha$, which means exchanging L&R and reversing the order of lecture so that it becomes infinite to the left.

Corollary 2.31. *For distinct $\alpha_-, \alpha_+ \in \mathbb{RP}^1$, the oriented bi-infinite geodesics (α_-, α_+) connecting those boundary points, combinatorial in $\mathcal{T}$ and hyperbolic in $\mathbb{HP}$, intersect $\triangle_2$ according to the same sequence of triangles (which is empty if it is an edge).*

If moreover $\alpha_- < 0 < \alpha_+$, they intersect the base edge $(0, \infty)$, and on each side the sequence of triangles in $\triangle_2$ is dictated by the continued fractions of $\frac{-1}{\alpha_-}$ and α_+, concatenating the transpose of the former with the latter.

Intersection of V_α with the first barycentric subdivision of $\triangle_1$

Every triangle of $\triangle_1$ is cut by the long legs of it medians into three triangles which we call *gliders*: in the first barycentric subdivision $\triangle_1'$ they are formed by unions of two triangles, adjacent along an edge connecting the barycenters of an edge and a face. They map by $\mathbb{P}(\bar{\psi})$ and $S\mathbb{P}(\bar{\psi})$ to similarly defined gliders in $\triangle_2'$, which are in bijection with the edges of $\mathcal{T}'$, and form a free transitive set under $\mathrm{PSL}_2(\mathbb{Z})$.

The line $V_\alpha \subset \mathbb{R}^2$ of irrational inclination $\alpha \in]0, \infty[$ avoids the vertices of $\triangle_1'$ and intersects it in a connected sequence of gliders. We encode this by the sequence of $\mathfrak{s}^{\pm1}$&$\mathfrak{t}^{\pm1}$ describing how to rotate the gliders around their bases or tips to pass from one to the next, with positive or negative exponents when V_α passes to the right or left of the center of rotation, as in figure 2.5.

Scholium 2.32. *The letters $\mathfrak{s}$ and $\mathfrak{t}$ are meant to evoke the actions of S and T on the gliders, or the edges of $\mathcal{T}'$. However we chose different symbols $\mathfrak{s}, \mathfrak{t}$ to avoid confusion with the elements $S, T \in \mathrm{SL}_2(\mathbb{Z})$ or their classes in $\mathrm{PSL}_2(\mathbb{Z})$. In particular we should not confuse the symbol $\mathfrak{s}^{-1}$ with the element $S^{-1} = -S \in \mathrm{SL}_2(\mathbb{Z})$.*

Besides, in Chapter 3, the letters $\mathfrak{s}, \mathfrak{t}$ will be identified with the generators of a free group $\mathbb{Z} * \mathbb{Z}$, the fundamental group the disc with two punctures obtained from the modular orbifold by removing its conical singularities. There again we will have to be careful about avoiding some confusions, and tracking orientation conventions.

Proposition 2.33. *The line $V_\alpha \subset \mathbb{R}^2$ of irrational inclination $\alpha \in]0, \infty[$ intersects a sequence of gliders in $\triangle_1'$ whose $\mathfrak{t}\&\mathfrak{s}$-encoding is deduced from the continued fraction expansion of α by the following translation rules:*

$$LL \rightsquigarrow (\mathfrak{t}^{-1}\mathfrak{s}^{-1})L \qquad RR \rightsquigarrow (\mathfrak{t}^{+1}\mathfrak{s}^{+1})R$$
$$LR \rightsquigarrow (\mathfrak{t}^{-1}\mathfrak{s}^{+1})R \qquad RL \rightsquigarrow (\mathfrak{t}^{+1}\mathfrak{s}^{-1})L$$

Up to a cyclic permutation (corresponding to a conjugacy by $\mathfrak{s}^{\pm 1}$), these translations amount to $L \rightsquigarrow \mathfrak{s}^{-1}\mathfrak{t}^{-1}$ and $R \rightsquigarrow \mathfrak{s}^{+1}\mathfrak{t}^{+1}$.

Remark 2.34. *The line $V_\alpha \subset \mathbb{R}^2$ passes through a vertex of $\triangle_1$ if and only if $\alpha \in \mathbb{Q}$, in which case it passes through exactly one vertex of each type in $\triangle_1'$.*

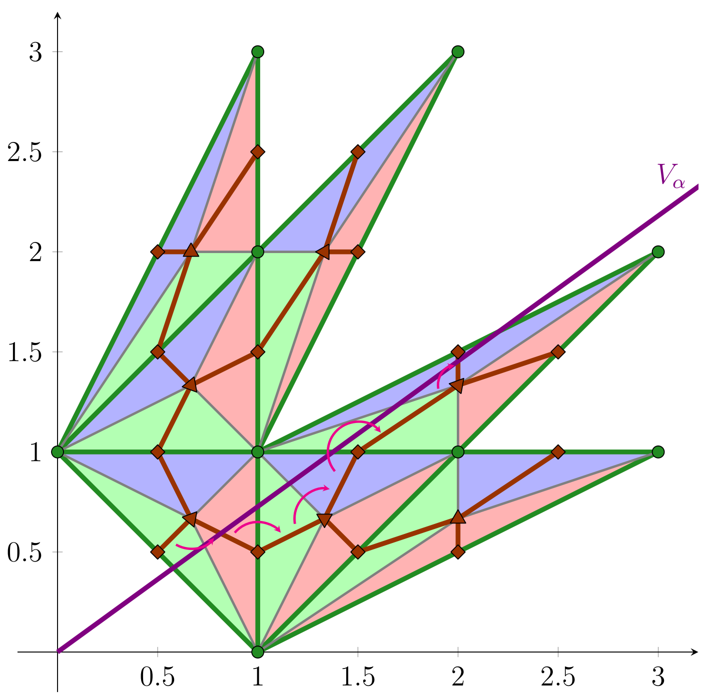


Figure 2.5: The line V_α intersects a sequence of gliders in $\triangle_1'$ encoded by $\mathfrak{s}^{\pm 1}\&\mathfrak{t}^{\pm 1}$.

Remark 2.35. *Remember that $\mathbb{P}\bar{\psi}$ sends the orientation of $\Delta_4 \subset \mathbb{R}^2$ to the opposite orientation of $\Delta_2 \subset \mathbb{P}(\mathbb{H}\cup\mathbb{X})$. The cone $\mathbb{X}$ is seen from beneath or above respectively.*

However it is the orientation of the cell complex Δ_4 or Δ_2 which determines the meaning of expressions like "a line passes to the right or to the left of a vertex", or "turning a glider around a vertex in the positive or negative direction".

Intersection of (α', α) with the first barycentric subdivision of $\triangle_2$

For distinct $\alpha_-, \alpha_+ \in \mathbb{RP}^1$, the geodesic $(\alpha_-, \alpha_+) \subset \mathbb{HP}$ intersects a sequence of gliders in $\triangle_2'$ encoded by a word alternating a letter in $\{\mathfrak{s}^{-1}, \mathfrak{s}^{\circ}, \mathfrak{s}^{+1}\}$ and a letter in $\{\mathfrak{t}^{-2}, \mathfrak{t}^{-1}, \mathfrak{t}^{\circ}, \mathfrak{t}^{+1}, \mathfrak{t}^{+1}\}$ describing how to rotate them around the bivalent and trivalent vertices of $\mathcal{T}'$. The exponents are positive, negative or $\circ$ when (α_-, α_+) passes to the left, to the right or through the vertex. The letters $\mathfrak{t}^{\pm 2}$ appear when a glider must be turned twice around its trivalent vertex in the direction given by the sign.

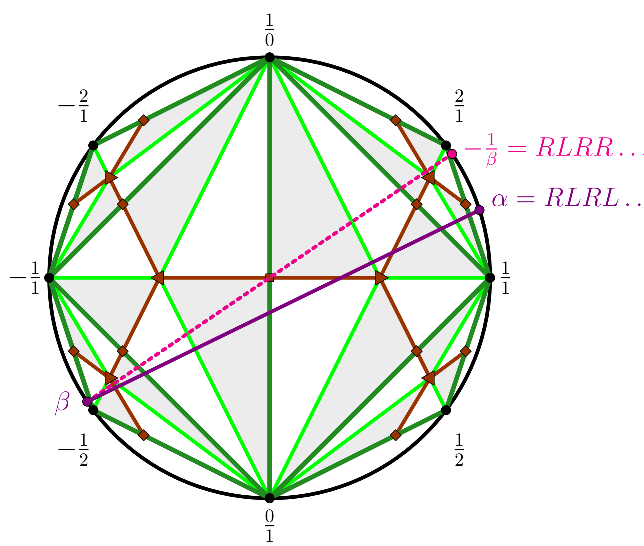


Figure 2.6: Geodesic of $\mathbb{P}(\mathbb{H})$ intersecting the gliders of $\triangle_2'$ according to a sequence $\ldots \mathfrak{t}^{-2}\mathfrak{s}^{-1}\mathfrak{t}^{-1}\mathfrak{s}^{-1} \mid \mathfrak{t}^{+1}\mathfrak{s}^{-1}\mathfrak{t}^{-1}\mathfrak{s}^{-1}\ldots$, the separation marks the intersection with the base glider.

Now suppose that $\alpha_-, \alpha_+ \in \mathbb{R}$ are such that $\alpha_- < 0 < \alpha_+$. Then (α_-, α_+) intersects positively the edge $(0, \infty)$, whence the glider containing the base edge (i, j) of $\mathcal{T}'$, so the $\mathfrak{s}$&$\mathfrak{t}$-word contains a subword of the form $\mathfrak{s}^{?}\mathfrak{t}^{?}$ characterising the intersection of (α_-, α_+) with the first barycentric subdivision ∇_2'. The signs of the exponents correspond to the position of (α_-, α_+) with respect to the points i and j.

The position of (α_-, α_+) with respect to i is obtained by comparing α_+ with $\frac{-1}{\alpha_-}$. The exponent of $\mathfrak{s}$ is given by $-\operatorname{sign}(1+\alpha_+\alpha_-)$. Thus (α_-, α_+) passes:

$+$ to the left of i if $1+\alpha_+\alpha_- < 0$, in which case we have $\mathfrak{s}^{+1}$

$\circ$ through i if $1+\alpha_+\alpha_- = 0$, in which case we have $\mathfrak{s}^{\circ}$

$-$ to the right of i if $1+\alpha_+\alpha_- > 0$, in which case we have $\mathfrak{s}^{-1}$

The position of (α_-, α_+) with respect to j is obtained by comparing the cross-ratio $\operatorname{bir}(\alpha_-, \alpha_+, T^{+1}\alpha_-, T^{-1}\alpha_-) = \frac{\alpha_+-\alpha_-}{1+\alpha_+\alpha_--\alpha_-}$ with 2, or equivalently $\frac{\alpha_++\alpha_-}{2}$ with $1+\alpha_+\alpha_-$. The exponent of $\mathfrak{t}$ has sign $\left(\frac{\alpha_++\alpha_-}{2} - (1+\alpha_+\alpha_-)\right)$. Thus (α_-, α_+) passes:

$+$ to the left of j if $1+\alpha_+\alpha_- < \frac{\alpha_++\alpha_-}{2}$, in which case we have $\mathfrak{t}^{+1}$ or $\mathfrak{t}^{+2}$

$\circ$ through j if $1+\alpha_+\alpha_- = \frac{\alpha_++\alpha_-}{2}$, in which case we have $\mathfrak{t}^{\circ}$

$-$ to the right of j if $1+\alpha_+\alpha_- > \frac{\alpha_++\alpha_-}{2}$, in which case we have $\mathfrak{t}^{-1}$ or $\mathfrak{t}^{-2}$

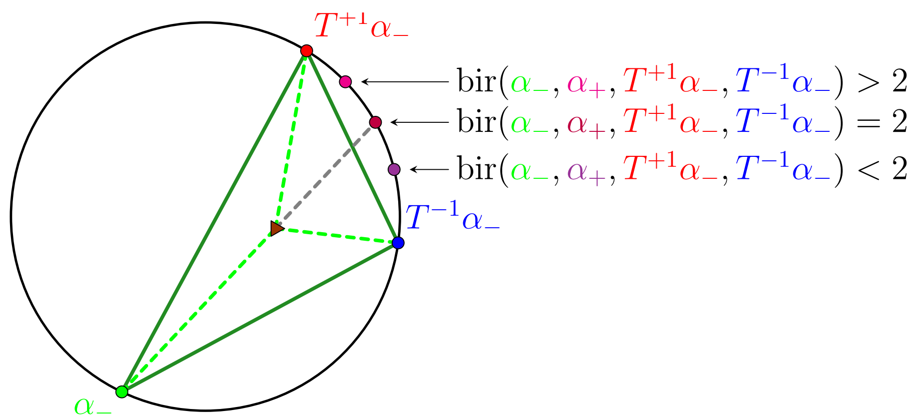


Figure 2.7: Find the exponent of $\mathfrak{t}$ by comparing $\operatorname{bir}(\alpha_+, \alpha_-, T^{-1}\alpha_-, T^{+1}\alpha_-)$ with 2.

We may recast this discussion in terms of the $L\&R$-words w_- and w_+ corresponding to the continued fractions of $S\alpha_-$ and α_+. Recall that w_- and w_+ describe the intersection patterns of $(i, S\alpha_-)$ and (i, α_+) with $\triangle_2$, and that $w = {}^t w_- w_+$ describes the intersection pattern of (α_-, α_+) with $\triangle_2$. After replacing the letters $L\&R$ by $\mathfrak{t}^? \mathfrak{s}^?$, we are focusing on the exponents of the last $\mathfrak{s}^?$ of ${}^t w_-$ and the first $\mathfrak{t}^?$ of w_+.

The exponent of $\mathfrak{s}^?$ is obtained by comparing w_- and w_+ for the lexicographic order on $L\&R$, and is given by the $\operatorname{sign}(w_+ - w_-)$. To compute the sign of the exponent of $\mathfrak{t}^?$, first note that the continued fractions of $T^{-1}\alpha_-$ and $T^{+1}\alpha_-$ are obtained by conjugating the continued fraction w_- of $S\alpha_- = S^{-1}\alpha_-$ by $T^{-1}S^{+1} = L$ and $T^{+1}S^{-1} = R$. When α_- is irrational, the word w_- is infinite and such conjugations amount to left multiplications by L and R. In that case the sign of the exponent of $\mathfrak{t}^?$ would be given by comparing the cross-ratio $\mathrm{bir}(w_+, Sw_-, Lw_-, Rw_-)$ with 2. However this cross-ratio is not well defined in purely combinatorial terms and one has to compute it in terms of the endpoints α_-, α_+ (or sufficiently good rational approximations obtained by truncating the words w_- and w_+ far enough).

Proposition 2.36. *Consider irrational real numbers α_-, α_+ such that $\alpha_- < 0 < \alpha_0$.*

The geodesic $(\alpha_-, \alpha_+) \subset \mathbb{HP}$ intersects $\triangle_2$ in a sequence of triangles encoded by a bi-infinite L&R-word w obtained by concatenating the transpose of the L&R-word of $\frac{-1}{\alpha_-}$ with the L&R-word of α_+.

The geodesic $(\alpha_-, \alpha_+) \subset \mathbb{HP}$ intersects $\triangle_2'$ in a sequence of gliders encoded by a bi-infinite $\mathfrak{s}$&$\mathfrak{t}$-word obtained from its L&R-word as follows.

First create a word by the translation rules $L \rightsquigarrow \mathfrak{t}^?\mathfrak{s}^?$ and $R \rightsquigarrow \mathfrak{t}^?\mathfrak{s}^?$, retaining for each $\mathfrak{s}^?$&$\mathfrak{t}^?$ the position of the L or R which gave rise to it. Every subword $\mathfrak{s}^?\mathfrak{t}^?$ corresponds to a factorisation $w = {}^t w_- w_+$ and the signs of those exponents $^?$ are obtained by comparing ${}^t w_-$ and w_+ as explained above. Besides, the exact translation rules must be chosen among:

$$L \rightsquigarrow \{\mathfrak{t}^{-1}\mathfrak{s}^?, \mathfrak{t}^{\circ}\mathfrak{s}^?, \mathfrak{t}^{+2}\mathfrak{s}^?\} \qquad R \rightsquigarrow \{\mathfrak{t}^{+1}\mathfrak{s}^?, \mathfrak{t}^{\circ}\mathfrak{s}^?, \mathfrak{t}^{-2}\mathfrak{s}^?\}$$

The sign computations together with these multi-valued translation rules for L&R determine a unique $\mathfrak{s}$&$\mathfrak{t}$-word.

Remark 2.37. *One may accelerate the translation process by noticing the that we must have the local translation rules $LL \rightsquigarrow \mathfrak{t}^?\mathfrak{s}^{-1}\mathfrak{t}^{-1}\mathfrak{s}^?$ and $RR \rightsquigarrow \mathfrak{t}^?\mathfrak{s}^{+1}\mathfrak{t}^{+1}\mathfrak{s}^?$, thus*

$$\forall m, n \in \mathbb{N}\colon \qquad L^m \rightsquigarrow \mathfrak{t}^?(\mathfrak{s}^{-1}\mathfrak{t}^{-1})^{m-1}\mathfrak{s}^? \quad R^n \rightsquigarrow \mathfrak{t}^?(\mathfrak{s}^{+1}\mathfrak{t}^{+1})^{n-1}\mathfrak{s}^?$$

After that it remains to determine the signs in exponents of the $\mathfrak{s}$ and $\mathfrak{t}$ in portions coming from alternations between L and R.

One may provide local translation rules for portions of the form $L^m(RL)^kR^n$ and $R^n(LR)^kL^m$ in terms of inequalities on the triples (k, m, n), but those will depend on the (unbounded) quantity k, so the translation process is ultimately non-local.

Remark 2.38. *We wish to lift the ambiguity on the exponents of the $\mathfrak{t}^{\circ}$ and $\mathfrak{s}^{\circ}$.*

Notice that every $\mathfrak{t}^{\circ}$ must be surrounded by equal powers of $\mathfrak{s}^{\pm 1}$. In particular we deduce a canonical way of replacing its exponent, to equal either ± 2 or ∓ 1, and we shall systematically choose the latter. This corresponds to a slight perturbation of the geodesic (α_-, α_+) in such a way as to avoid the orbit of j, as depicted in figure 2.8.

Notice that every $\mathfrak{s}^\circ$ is must be surrounded by opposite powers of $\mathfrak{t}^{\pm 1}$. In particular we deduce a canonical way of replacing its exponent, to equal that of the $\mathfrak{t}$ just before or just after, and we shall systematically choose the latter. Hence every subword of the form $\mathfrak{t}^{-1}\mathfrak{s}^\circ\mathfrak{t}^{+1}$ or $\mathfrak{t}^{+1}\mathfrak{s}^\circ\mathfrak{t}^{-1}$ gets replaced by $\mathfrak{t}^{-1}\mathfrak{s}^{+1}\mathfrak{t}^{+1}$ or $\mathfrak{t}^{+1}\mathfrak{s}^{-1}\mathfrak{t}^{-1}$ respectively. This corresponds to a slight perturbation of the geodesic (α_-, α_+) in such a way as to avoid the orbit of i, as depicted in figure 2.10.

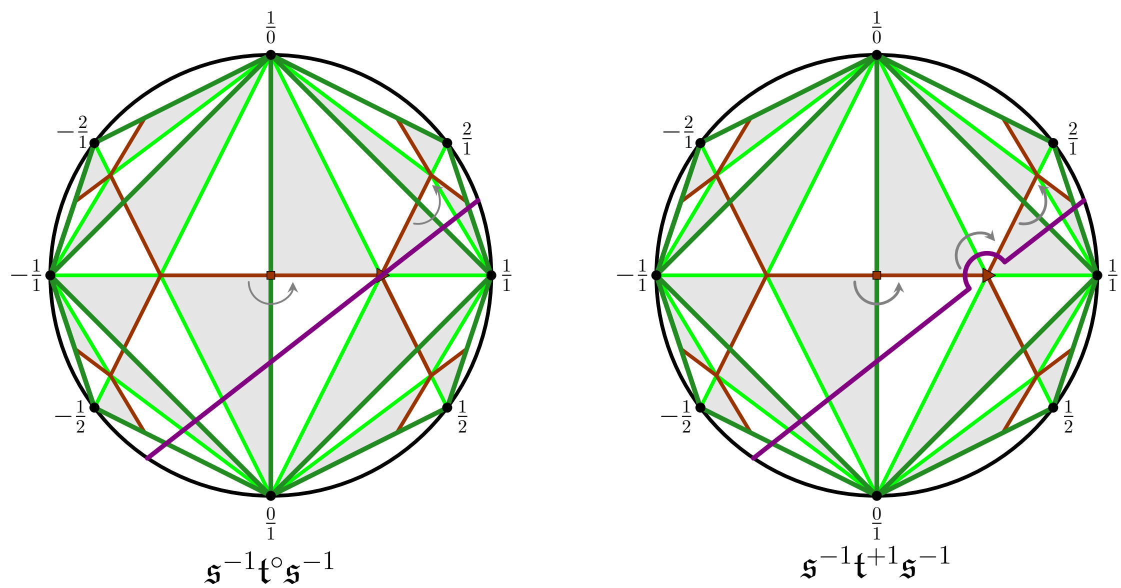


Figure 2.8: A $\mathfrak{t}^\circ$ must be surrounded by equal powers $\epsilon \neq \circ$ of $\mathfrak{s}$. The canonical perturbation changes $\mathfrak{t}^\circ$ in $\mathfrak{t}^{-\epsilon}$.

We now recast those statements in terms of a hyperbolic $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ with translation axis $\gamma_A = (\alpha_-, \alpha_+)$, which we assume to have $c > 0$ and thus $b > 0$. This follows from the relations $\alpha_+\alpha_- = -b/c$ and $\alpha_+ + \alpha_- = -(d-a)$.

The position of γ_A with respect to i is given by comparing $\mathrm{Tr}(AS^{-1})$ and $\mathrm{Tr}(AS)$, and the exponent of $\mathfrak{s}$ equals $\operatorname{sign}\mathrm{Tr}(A(S - S^{-1})) = \operatorname{sign}\mathrm{Tr}(AS)$. Thus γ_A passes:

$+$ to the left of i if $\mathrm{Tr}(AS) > 0$, in which case we have $\mathfrak{s}^{+1}$

$\circ$ through i if $\mathrm{Tr}(AS) = 0$, in which case we have $\mathfrak{s}^\circ$

$-$ to the right of i if $\mathrm{Tr}(AS) < 0$, in which case we have $\mathfrak{s}^{-1}$

The position of γ_A with respect to j is given by comparing $\mathrm{Tr}(AT)$ and $\mathrm{Tr}(AT^{-1})$, and the sign in the exponent of $\mathfrak{t}$ equals $\operatorname{sign}\mathrm{Tr}(A(T - T^{-1}))$. Thus γ_A passes:

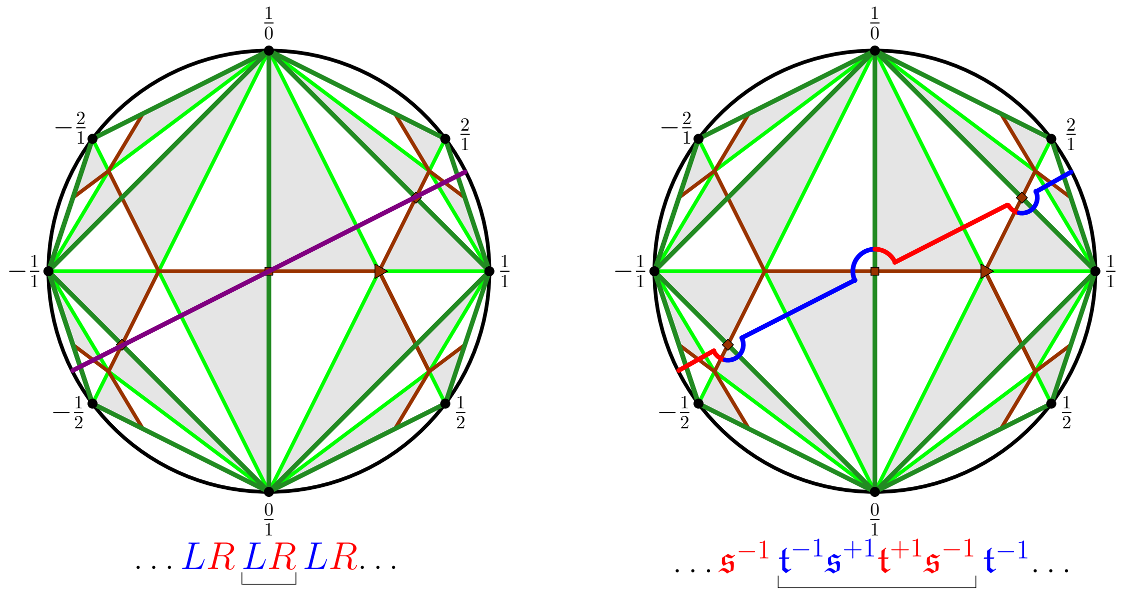


Figure 2.9: Geodesic axis of γ_{RL} and a canonical perturbation: the $\mathfrak{s}^{\circ}$ change to $\mathfrak{s}^{\pm 1}$.

- $+$ to the left of j if $\mathrm{Tr}(AT) > \mathrm{Tr}(AT^{-1})$, in which case we have $\mathfrak{t}^{+1}$ or $\mathfrak{t}^{+2}$
- $\circ$ through j if $\mathrm{Tr}(AT = \mathrm{Tr}(AT^{-1})$, in which case we have $\mathfrak{t}^{\circ}$
- $-$ to the right of j if $\mathrm{Tr}(AT) < \mathrm{Tr}(AT^{-1})$, in which case we have $\mathfrak{t}^{-1}$ or $\mathfrak{t}^{-2}$

Remark 2.39. *In view of Chapter 1, the signs in the exponents of* $\mathfrak{s}$ *and* $\mathfrak{t}$ *equal:*

$$\begin{aligned}\mathrm{Tr}(A(S - S^{-1})) &= -2\langle A, \mathrm{pr}\, S\rangle = -2\langle A, S\rangle\\ \mathrm{Tr}(A(T - T^{-1})) &= -2\langle A, \mathrm{pr}\, T\rangle = -2\langle A, S - \tfrac{1}{2}K\rangle\end{aligned}$$

We may also propose an equivariant formulation in terms of $A \in \mathrm{PSL}_2(\mathbb{R})$*, that is independent of its lift* $A \in \mathrm{SL}_2(\mathbb{R})$*, by noticing that* $c + b = \mathrm{Tr}(AJ) = -2\langle A, J\rangle$*. Hence the signs in the exponents of* $\mathfrak{s}$ *and* $\mathfrak{t}$ *are respectively equal to the signs of:*

$$\begin{aligned}\mathrm{Tr}(AJ)\cdot\mathrm{Tr}(A\,\mathrm{pr}(S)) &= 4\cdot\langle A, J\rangle\cdot\langle A, S\rangle\\ \mathrm{Tr}(AJ)\cdot\mathrm{Tr}(A\,\mathrm{pr}(T)) &= 4\cdot\langle A, J\rangle\cdot\langle A, S - \tfrac{1}{2}K\rangle\end{aligned}$$

Proposition 2.40. *Consider a primitive hyperbolic element* $A \in \mathrm{PSL}_2(\mathbb{N})$ *and let* $A_k = \sigma^k A \in \mathrm{PSL}_2(\mathbb{N})$ *be its Lyndon conjugates. The translation axis* $(\alpha', \alpha) \subset \mathbb{HP}$

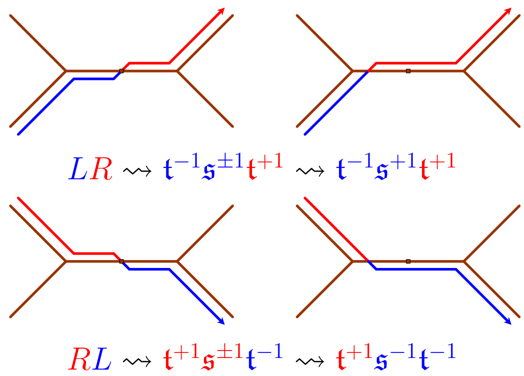


Figure 2.10: Canonical choice for perturbing the axes passing through i.

of A intersects $\triangle_2'$ according to a sequence of gliders encoded by a bi-finite periodic $\mathfrak{s}$&$\mathfrak{t}$-word determined from the L&R-factorisation of A as follows.

First apply the replacement rules $L \rightsquigarrow \mathfrak{t}^{\{-1,\circ,2\}}\mathfrak{s}^{\{1,\circ,-1\}}$ and $R \rightsquigarrow \mathfrak{t}^{\{1,\circ,-2\}}\mathfrak{s}^{\{1,\circ,-1\}}$. Then for every $k \in [0, \mathrm{len}(A)[$, the $\mathfrak{s}^?$ at the end of A_k has exponent $\mathrm{sign}\,\mathrm{Tr}(AS)$, and the $\mathfrak{t}^?$ at the beginning of A_k has exponent of the $\mathrm{sign}\,\mathrm{Tr}(A(T - T^{-1}))$.

Example 2.41. *The translation axis of RL intersects $\triangle_2'$ in the sequence of gliders encoded by the word w^∞ where $w = (\mathfrak{t}^{+1}\mathfrak{s}^\circ)(\mathfrak{t}^{-1}\mathfrak{s}^\circ)$.*

The translation axis of RLL intersects $\triangle_2'$ in the sequence of gliders encoded by the word w^∞ where $w = (\mathfrak{t}^\circ\mathfrak{s}^{-1})(\mathfrak{t}^{-1}\mathfrak{s}^{-1})(\mathfrak{t}^{-1}\mathfrak{s}^{-1})$.

The translation axis of $RLLL$ intersects $\triangle_2'$ in the sequence of gliders encoded by the word w^∞ where $w = (\mathfrak{t}^{-2}\mathfrak{s}^{-1})(\mathfrak{t}^{-1}\mathfrak{s}^{-1})(\mathfrak{t}^{-1}\mathfrak{s}^{-1})(\mathfrak{t}^{-1}\mathfrak{s}^{-1})$.

It may be amusing to find the $\mathfrak{s}$&$\mathfrak{t}$ translations of L&R-sequences encoding Markov irrationals [Ser85a].

2.3 Invariants of two conjugacy classes in $\mathrm{PSL}_2(\mathbb{Z})$

The previous section described conjugacy classes of hyperbolic matrices in $\mathrm{PSL}_2(\mathbb{Z})$ in terms of their intersection with $\mathrm{PSL}_2(\mathbb{N})$ and their combinatorial axes in $\mathcal{T}$.

We now turn to the conjugacy classes of pairs of matrices in $\mathrm{PSL}_2(\mathbb{Z})$. At the end of this section, we shall average conjugacy invariants for pairs of matrices to obtain functions of pairs of conjugacy classes.

Combinatorics of the crossing and cosign functions

The main idea is to consider the relative position of the oriented tree-axes, and there are two features to take into account: if they share an oriented edge, and if their endpoints are linked on the boundary.

Recall the discussion following Corollary 2.8 which introduced geodesics in $\mathcal{T}$. The tree property implies that two geodesics of $\mathcal{T}$ intersect along a geodesic: it may be empty, but otherwise it has positive length so one my compare their orientations.

Definition 2.42. *For oriented geodesics g_a and g_b in $\mathcal{T}$, let $\mathrm{sinc}(g_a, g_b) \in \mathbb{Z} \cup \{\pm\infty\}$ be the length of their intersection, whose sign given by $\mathrm{cosign}(g_a, g_b) \in \{-1, 0, +1\}$ compares their orientations along their intersection when it is not empty.*

These functions are symmetric, invariant under the action of $C \in \mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{T}$:

$$\mathrm{sinc}(C \cdot g_a, C \cdot g_b) = \mathrm{sinc}(g_a, g_b) \qquad \mathrm{cosign}(C \cdot g_a, C \cdot g_b) = \mathrm{cosign}(g_a, g_b)$$

and inverting the orientation of one of their arguments results in a change of sign.

The previous definition holds for any two oriented geodesics: they could be finite, half-infinite or bi-infinite; and periodic or aperiodic. However, we shall only consider those in the set $\mathcal{G}$ of oriented bi-infinite geodesics. The translation axes g_A of infinite order elements $A \in \mathrm{PSL}_2(\mathbb{Z})$ are precisely the bi-infinite periodic geodesics in $\mathcal{G}$.

For infinite order elements $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ we denote $\mathrm{sinc}(A, B) = \mathrm{sinc}(g_A, g_B)$ and $\mathrm{cosign}(A, B) = \mathrm{cosign}(g_A, g_B)$. Now we have invariance under the conjugacy action of $C \in \mathrm{PSL}_2(\mathbb{Z})$ on its subset of infinite order elements:

$$\mathrm{sinc}(CAC^{-1}, CBC^{-1}) = \mathrm{sinc}(A, B) \qquad \mathrm{cosign}(CAC^{-1}, CBC^{-1}) = \mathrm{cosign}(A, B)$$

Lemma 2.43. *Two infinite order elements $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ can be simultaneously conjugated in $\mathrm{PSL}_2(\mathbb{N})$ if and only if their combinatorial axes share an oriented edge, that is when $\mathrm{cosign}(A, B) = 1$.*

More precisely the set of edges belonging to the intersection of their oriented $\mathcal{T}$-axes corresponds to the set $\{C \in \mathrm{PSL}_2(\mathbb{Z}) \mid CAC^{-1}, CBC^{-1} \in \mathrm{PSL}_2(\mathbb{N})\}$, whose cardinal equals the positive part of $\mathrm{sinc}(A, B)$.

Proof. Recall that the automorphism group $\mathrm{PSL}_2(\mathbb{Z})$ of the cyclically oriented $\mathcal{T}$ acts freely transitively on its oriented edges: the base edge provides an identification between the two sets. Moreover, a matrix belongs to $\mathrm{PSL}_2(\mathbb{N})$ if and only if its combinatorial translation axis passes through the base edge. Hence the matrices C which simultaneously conjugate A and B in $\mathrm{PSL}_2(\mathbb{N})$ are in bijection with the oriented edges belonging to the intersection of g_A and g_B. □

Let us represent the different configurations for pairs of axes g_A, g_B and provide the values for the invariants cross and cosign. Note that cosign and cross can take their values independently, except for the implication cosign $= 0 \implies$ cross $= 0$.

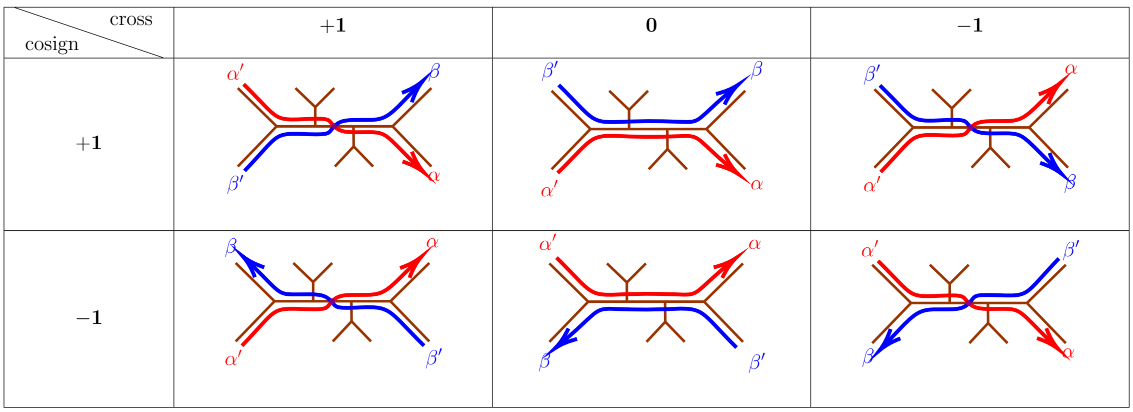


Figure 2.11: Configurations of two geodesics: values of cross & cosign.

For $A \in \mathrm{PSL}_2(\mathbb{Z})$, denote $\mathrm{len}(A) \in \mathbb{N}$ the minimum displacement length $d(e, A \cdot e)$ of an edge $e \in \mathcal{T}$ (this quantity was denoted l_A in the proof of Proposition 2.15). When A has infinite order, it is the $L\&R$-length of a Lyndon conjugate, and when A has finite order it is zero.

Proposition 2.44. *For hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ such that $g_A \cap g_B \neq \emptyset$, we have:*

$$\mathrm{cosign}(A, B) = \mathrm{sign}\left(\mathrm{len}(AB) - \mathrm{len}(AB^{-1})\right).$$

Moreover, if $n, m \in \mathbb{N}$ satisfy $\min\{n\,\mathrm{len}(A), m\,\mathrm{len}(B)\} > |\mathrm{sinc}(A, B)|$, we have:

$$\mathrm{sinc}(A^m, B^n) = \mathrm{len}(A^m B^n) - \mathrm{len}(A^m B^{-n}).$$

Proof. These identities follow from [CP20], but let us prove the first one to explain an idea which will serve in 2.48. We will not use the second, but the first will play a crucial role in Theorem 5.24.

Since both sides of the formula change of sign under inversion of A or B, we may suppose that $\mathrm{cosign}(A, B) > 0$.

We have $\mathrm{len}(AB) = \mathrm{len}(A) + \mathrm{len}(B)$ because if $e \in g_A \cap g_B$ then AB sends $B^{-1} \cdot e$ to $A \cdot e$, but the geodesic $(B^{-1}(e), A(e))$ consists in a sequence of $\mathrm{len}(A) + \mathrm{len}(B)$ oriented edges of $\mathcal{T}$ following one another so AB is the a hyperbolic translation along an axis which contains that segment as fundamental domain.

To see why $\mathrm{len}(AB^{-1}) < \mathrm{len}(A) + \mathrm{len}(B)$, notice that AB sends $B(e)$ to a $A(e)$ and for $e \in g_A \cap g_B$ the geodesic $(B(e), A(e))$ has length at most $\mathrm{len}(A) + \mathrm{len}(B) - 1$. □

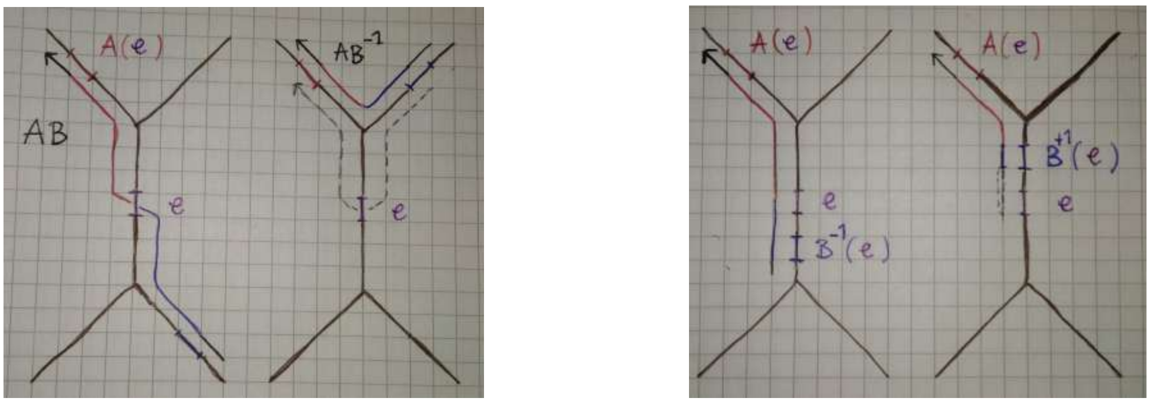


Figure 2.12: AB sends $B^{-1}(e)$ to $A(e)$ separated by $\mathrm{len}(A) + \mathrm{len}(B)$.

Remark 2.45. *Compare the formula* $\mathrm{cosign}(A, B) = \mathrm{sign}\,(\mathrm{len}(AB) - \mathrm{len}(AB^{-1}))$ *with the one from Proposition 1.89:* $\mathrm{sign}\cos(A, B) = \mathrm{sign}\,(\mathrm{tr}(AB)^2 - \mathrm{tr}(AB^{-1})^2)$. *In Chapter 5, we shall recover the former as a limit of the latter by deforming the representation* $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R})$.

Commutation or Coprimality: equal or different axes

We now discuss in more detail the case when the unoriented axes are equal.

We begin with a lemma concerning elements in a free monoid, whose generators called *letters* form an alphabet, its elements being called *words*. The alphabet could be of any cardinality (even empty, in which case the monoid is reduced to the neutral element, that is the empty word) and the lemma will hold for any elements (including the neutral ones).

The *primitive root* of a word w is the smallest word admitting a power equal to w. The *boundary* of the monoid is the set of words which extend infinitely to the right. The monoid acts on its boundary by left multiplication, and the unique fixed point of w is $w^\infty = ww\cdots$.

Lemma 2.46. *For u et v in a free monoid, the following are equivalent:*

1. *Commutation: $uv = vu$.*
2. *Primitive roots: there is a word w and $k, l \in \mathbb{N}$ such that $u = w^k$ and $v = w^l$.*
3. *Common power: $u^{\mathrm{len}(v)} = v^{\mathrm{len}(u)}$.*
4. *Fixed points: $u^\infty = v^\infty$.*

Proof. We index the letters of a word w by $\mathbb{Z}/\mathrm{len}(w) = \{0, \ldots, \mathrm{len}(w) - 1\}$, denote $w[: k[$ its prefix of length k, and $w[k: [$ the word obtained by removing this prefix.

Let us first show $1 \implies 2$: if u & v commute, then they are powers of a same w. Suppose $\mathrm{len}(u) \geq \mathrm{len}(v)$ and reason by induction on $\mathrm{len}(u) - \mathrm{len}(v)$.

If u and v have the same length then $u = v$ so $u = w = v$ convenes. Otherwize suppose $\mathrm{len}(u) > \mathrm{len}(v)$ so that $vu = vmv = uv$ with $u[: l] = m = u[-l :]$ and $l = \mathrm{len}(u) - \mathrm{len}(v)$ We have $mv = u = vm$ and $\mathrm{len}(v) - \mathrm{len}(u) = \mathrm{len}(m)$ which is strictly larger than $\mathrm{len}(v) - \mathrm{len}(m)$ so the induction hypothesis applied to m, v says they are both powers of a same word w, and we deduce the same for u.

The implications $2 \implies 1$ and $2 \implies 3 \implies 4$ are clear, we are left with $4 \implies 1$: let us show that if u^∞ and v^∞ have the same prefix of length $\mathrm{len}(u)+\mathrm{len}(v)$, then u and v have a common primitive root w.

We may suppose $\mathrm{len}(u) \geq \mathrm{len}(v)$, and denoting $k, s, t \in \mathbb{N}$ satisfying $s = \mathrm{len}(v)$ and $k.s+t = \mathrm{len}(u)+s$ we have $u.u[: s] = v^k v[: t]$. Identifying the suffix of length s in this equality shows that $u[: s]$ is a cyclic permutation of v, precisely $u[: s] = v[t :]v[: t]$.

The previous equality can also be written $u[: s].u[s :].u[: s] = v^k v[: t]$ and now identifying the prefix of length s shows that $v[t :]v[: t] = v$. In other terms v is equal to its t-cyclic permutation so $v = w^l$ where $w = v[: t]$ and $lt = \mathrm{len}(v)$.

Consequently we have found that $u[:s] = v[t:]v[:t]$ which is equal to $w^l = v$. Finally $u.u[:s] = v^k v[:t]$ rewrites as $uw^l = w^{kl+1}$ so $u = w^{(k-1)l+1}$. □

Remark 2.47. *Since u and v have a common primitive root r if and only if they have common powers $u^{\mathrm{len}(v)} = v^{\mathrm{len}(u)}$, we may say that in a free monoid it is equivalent to have a greatest common divisor w and to have a smallest common multiple m.*

Conversely, two elements in a free monoid generate a free submonoid of rank two if and only if they do not belong to a submonoid generated by one element, if and only if they generate submonoids with intersection reduced to the neutral element.

The previous lemma applies to the free monoid $\mathrm{PSL}_2(\mathbb{N})$ generated by $L\&R$, and we use it to describe commuting elements in the group $\mathrm{PSL}_2(\mathbb{Z})$.

Recall that the automorphism group of the cyclically oriented trivalent tree $\mathcal{T}$ is $\mathrm{PSL}_2(\mathbb{Z})$, and that its bi-infinite periodic geodesics are precisely the translation axes of infinite order elements.

Proposition 2.48. *For infinite order $A, B \in \mathrm{PSL}_2(\mathbb{Z})$, the following are equivalent:*

1. *Commutation: $AB = BA$.*
2. *Primitive root: $\exists W \in \mathrm{PSL}_2(\mathbb{Z}),\ \exists k, l \in \mathbb{Z} :\ A = W^k,\ B = W^l$.*
3. *Common power: $\exists k, l \in \mathbb{Z} :\ A^l = B^k$.*
4. *Fixed points: $\{\alpha', \alpha\} = \{\beta', \beta\}$.*
5. *The translations axes of A and B in $\mathcal{T}$ are equal or mutually inverse.*

We say that A and B are coprime *when these conditions are not satisfied: they generate a free group of rank two.*

Proof. The implications $2 \implies 3 \implies 4$ are easy and $4 \iff 5$ follows from Corollary 2.31. To show that $5 \implies 2$ consider a minimal period of the translation axes, this yields (up to inversion) the primitive root W of A and B.

Now clearly $2 \implies 1$ and we shall conclude with $1 \implies 5$. Assume that A and B commute. After replacing A or B by its inverse we may also assume that $\mathrm{bir}(A, B) > 0$, which roughly means that g_A and g_B go in the same direction.

Suppose by contradiction that they are disjoint: there is a unique geodesic (e_A, e_B) from g_A to g_B and it has positive length $l \in \mathbb{N}^*$. Then it is not hard to see, as on figure 2.13, that the commutator $[A, B]$ sends the edge $BA(e_A)$ to an edge $AB(e_A)$ separated by $2l_A + 3d + l_B$ edges, so it cannot equal the identity.

The contradiction shows that g_A and g_B intersect along at least one edge, which we may assume to be the base edge after conjugating everything by $C \in \mathrm{PSL}_2(\mathbb{Z})$. Thus $A, B \in \mathrm{PSL}_2(\mathbb{N})$ by Lemma 2.43, and we may conclude with Lemma 2.46. □

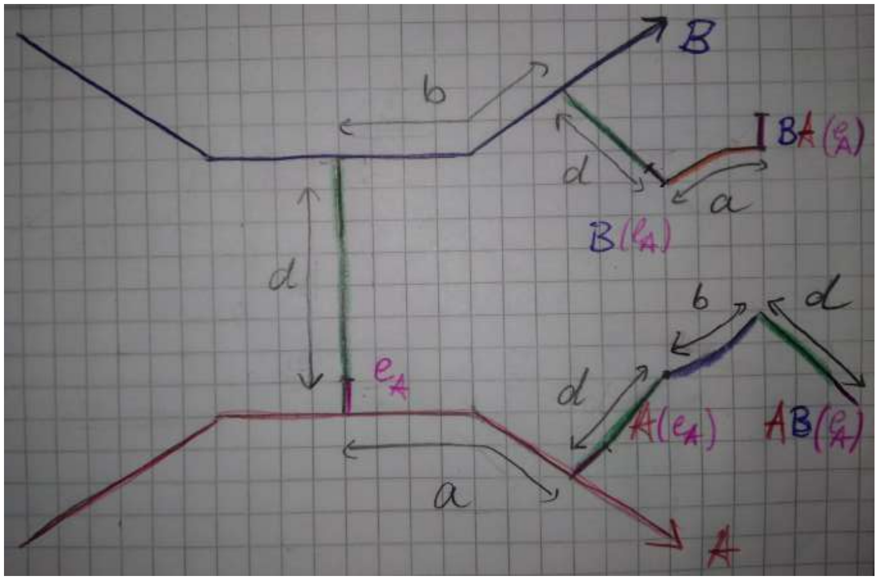


Figure 2.13: $[A, B]$ sends $BA(e_A)$ to $AB(e_A)$ separated by $2l_A + 3d + l_B$.

Corollary 2.49. *If $A \in PSL_2(\mathbb{Z})$ has infinite order, then its centraliser is the infinite cyclic subgroup generated by its primitive root.*

Proof. We just that saw that the infinite order elements which commute with A form the subgroup generated by its primitive root.

All we need to show is that A cannot commute with an element of finite order, that is a conjugate of S or T by Theorem 2.11. If A is conjugate to itself by CSC^{-1} or CTC^{-1} for some $C \in PSL_2(\mathbb{Z})$ then $B = C^{-1}AC$ is conjugate to itself by S or T, which is impossible from the normal form of elements in $\mathbb{Z}/2 * \mathbb{Z}/3$.

Alternatively, the actions of $PSL_2(\mathbb{Z})$ by conjugacy on its primitive infinite order elements and on the periodic axes of $\mathcal{T}$ are equivalent by the map $W \mapsto g_W$. Hence the centraliser of A is the stabiliser of its axis g_A which is reduced to the infinite cyclic subgroup generated by its primitive root. ∎

Functions of two conjugacy classes: sum over double cosets

Consider a group Γ acting on a space Σ and a function f defined on $\Sigma \times \Sigma$ with values in a commutative group Λ which is invariant under the diagonal action of Γ:

$$f : \Sigma \times \Sigma \to \Lambda \qquad \forall W \in \Gamma,\ \forall a, b \in \Sigma :\ f(a, b) = f(W \cdot a, W \cdot b)$$

We define an invariant F for pairs of Γ-orbits $[a], [b]$ by summing f over all pairs of representatives of the orbits considered modulo the diagonal action of Γ.

The pairs of representatives for the orbits are parametrized by the $(U \cdot a, V \cdot b)$ for $(U, V) \in \Gamma/(\mathrm{Stab}\, a) \times \Gamma/(\mathrm{Stab}\, b)$, and the quotient of this set by the diagonal action of Γ by left translations is denoted $\Gamma/(\mathrm{Stab}\, a) \times_\Gamma \Gamma/(\mathrm{Stab}\, b)$.

Consequently, the sum indexed by $(U, V) \in (\Gamma/\mathrm{Stab}\, a) \times_\Gamma (\Gamma/\mathrm{Stab}\, b)$ defines our desired invariant:

$$F([a], [b]) := \sum_{(U,V)} f(U \cdot A, V \cdot b)$$

This can also be written as the sum over double cosets $W \in (\mathrm{Stab}\, a)\backslash\Gamma/(\mathrm{Stab}\, b)$:

$$F([a], [b]) = \sum_{W} f(a, W \cdot b)$$

because the map $(\Gamma/\mathrm{Stab}\, a) \times (\Gamma/\mathrm{Stab}\, b) \to (\mathrm{Stab}\, a)\backslash\Gamma/(\mathrm{Stab}\, b)$ sending (U, V) to $W = U^{-1}V$ is surjective, and its fibers are the orbits under the diagonal action of Γ by left translations.

Remark 2.50. *To ensure that the sum is well defined, it must have finite support or converge in a completion of Λ for an appropriate norm, and that depends on the behaviour of f. (We can always bypass this problem by adding formal variables in the sum at the expense of finding an invariant F which is too rich to be computable.)*

We shall apply this discussion to the actions of $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$: on the group $\mathrm{PSL}_2(\mathbb{R})$ or its lattice $\mathrm{PSL}_2(\mathbb{Z})$; on the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, its lattice $\mathfrak{sl}_2(\mathbb{Z})$, or the symmetric space $\mathbb{H} \cup \mathbb{H}'$; on the trivalent tree $\mathcal{T}$ or its infinite oriented geodesics $\mathcal{G}$.

We have seen that several of these actions are closely related. For instance to an element $A \in \mathrm{PSL}_2(\mathbb{R})$ one may associate its projection $\mathrm{pr}(A) \in \mathfrak{sl}_2(\mathbb{R})$, and its projectivization $\mathbb{P}(\mathrm{pr}\, A) \in \mathbb{P}(\mathfrak{sl}_2(\mathbb{R}))$. One can make a similar correspondence between elements in $\mathrm{PSL}_2(\mathbb{Z})$ and their (weighted) fixed points in $\mathcal{T} \cup \partial\mathcal{T} \cup \mathcal{G}$.

The function $f(a, b)$ could be obtained from geometrical invariants such as the scalar product $\langle a, b\rangle$, like the cross-ratio $\mathrm{bir}(\alpha', \alpha, \beta', \beta)$, as well as combinatorial invariants related to $\mathrm{sinc}(g_a, g_b)$ and $\mathrm{cross}(g_a, g_b)$, like $\mathrm{cosign}(g_A, g_B) \times \mathrm{cross}(g_a, g_b)$.

Of course when $\mathrm{PSL}_2(\mathbb{Z})$ acts on itself by conjugacy, we obtain invariants for pairs of conjugacy classes, and when they are hyperbolic the stabilisers have been described in Corollary 2.49.

Functions of conjugacy classes: sum over Lyndon cycles

Consider a function f defined on pairs of coprime primitive infinite order elements $A, B \in \mathrm{PSL}_2(\mathbb{Z})$, which is invariant under the diagonal action of $\mathrm{PSL}_2(\mathbb{Z})$ on itself by left conjugacy. This amounts to a function defined on pairs of geodesics $g_A, g_B \subset \mathcal{T}$ with distinct fixed points, which is invariant under the diagonal action of $\mathrm{PSL}_2(\mathbb{Z})$.

In order to compute the sum defining $F([A],[B])$, we may group the terms $f(UAU^{-1}, VBV^{-1})$ according to the $\mathrm{cosign}(UAU^{-1}, VBV^{-1}) \in \{-1, 0, 1\}$ to obtain:

$$F = F_- + F_0 + F_+$$

The sum F_+ has finite support, contained in the set of pairs $(\sigma^i A, \sigma^j B)$ of Lyndon representatives for the conjugacy classes of $A, B \in \mathrm{PSL}_2(\mathbb{N})$, thus:

$$F_+([A],[B]) = \sum_{i=1}^{\mathrm{len}(A)} \sum_{j=1}^{\mathrm{len}(B)} f\left(\sigma^i A, \sigma^j B\right)$$

Similarly the sum F_- has finite support, which we may also index by pairs of Lyndon representatives $(\sigma^i A, \sigma^j B)$ for the conjugacy classes of $A, B \in \mathrm{PSL}_2(\mathbb{N})$ using the fact that $\mathrm{cosign}(A, B) = -\mathrm{cosign}(A, SBS^{-1})$, thus:

$$F_-([A],[B]) = \sum_{i=1}^{\mathrm{len}(A)} \sum_{j=1}^{\mathrm{len}(B)} f\left(\sigma^i A, S(\sigma^j B)S^{-1}\right)$$

These sums are easy to compute, either by hand or with the help of a computer, given Lyndon representatives $A, B \in \mathrm{PSL}_2(\mathbb{N})$ and an expression for f.

Remark 2.51. *One may similarly decompose the sum F_0 in two parts depending on the relative orientations of the axes, which are interchanged by the action of S on one of the components of f. However their index sets are infinite.*

Now suppose that for all A, B we have $\mathrm{cosign}(A, B) = 0 \implies f(A, B) = 0$ and $f(A, B^{-1}) = \epsilon f(A, B)$ with $\epsilon \in \{\pm 1\}$. This holds for cross and cosign with $\epsilon = -1$, and for their product or their absolute values with $\epsilon = 1$. Then $F_0 = 0$ and using $SCS^{-1} = {}^tC^{-1}$ we find that $F_-([A],[B]) = \epsilon \cdot F_+([A],[{}^tB])$. Thus we may compute F from the expression of f on pairs of Lyndon representatives:

$$F([A],[B]) = F_+([A],[B]) + \epsilon \cdot F_+([A],[{}^tB])$$

Scholium 2.52. *In the second part of this thesis we will compute intersection numbers of modular geodesics and linking numbers of modular knots, and recover the sums F or F_+ associated to functions f involving* cross & cosign.

Part II

Linking numbers of modular knots

Chapter 3

Modular geodesics

Outline of the chapter

This chapter is divided in two sections, both of which will serve in the next chapter.

Geometry of modular geodesics

The first section concerns the topology and combinatorics of modular geodesics.

We first define the modular orbifold $\mathbb{M} = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{HP}$ which has two conical singularities $i\&j$ of order 2&3 and a hyperbolic metric. We also define a "projective linear structure" on $\mathbb{M}$ using the map $\mathbb{P}\bar{\psi}$ from proposition 2.5.

Then we discuss the topology of its closed hyperbolic geodesics. Proposition 2.23 characterises those which pass through i or j in terms of their Lyndon representatives, and we use 2.38 to define their canonical perturbations. This is the content of Corollary 3.4. Let us mention that Sarnak estimated in [Sar07] the asymptotic number of geodesics containing i in order to investigate their distribution.

The (perturbed) modular geodesics lift in the thrice punctured sphere $\mathbb{M} \setminus \{i, j\}$ whose fundamental group is freely generated by $\mathfrak{s}\&\mathfrak{t}$. We describe their homotopy classes in terms of the corresponding cycles of $\mathfrak{s}^{\pm 1}\&\mathfrak{t}^{\pm 1}$. This relies on Proposition 2.40 and Remark 2.38 and forms the content of Corollary 3.8.

Then we express in Proposition 3.10 the intersection numbers between hyperbolic geodesics using the formalism developed in the last paragraphs of chapter 2. We also show that the lifts of (perturbed) modular geodesics in $\mathbb{M} \setminus \{i, j\}$ are taut, that is minimally intersecting in their homotopy class (the same holds for finite collections). We mention that Birman and Series [BS84] have an algorithm for computing self-intersection numbers of taut loops in a smooth surface of negative euler characteristic.

Finally, we define in 3.17 the *linear representative* of hyperbolic conjugacy class in $\pi_1(\mathbb{M})$ as a loop in the projective linear model of $\mathbb{M}$. It avoids the singularities and we describe the homotopy class of its lift in $\mathbb{M} \setminus \{i, j\}$ using Proposition 2.33. We also provide practical algorithms to draw those linear representatives. The hyperbolic and linear representatives are homotopic in the orbifold $\mathbb{M}$ but they may have non-homotopic lifts in $\mathbb{M} \setminus \{i, j\}$, and different self-intersection numbers. We explain why (finite collections of) linear representatives in $\mathbb{M} \setminus \{i, j\}$ are taut, so that one may compute their self-intersection numbers using the algorithm of Birman-Series.

Lifting geodesics in Galois covers

In the second section we first describe the covers $\mathbb{M}_0$ of the modular orbifold $\mathbb{M}$ through their Galois correspondence with the subgroups Γ_0 of its fundamental group $\Gamma = \mathbb{Z}/2 * \mathbb{Z}/3$. They lead to the "dessins d'enfants" introduced by A. Grothendieck in [Gro97], to which [SV90, LZ04] may serve as nice introductions. Those are embedded graphs in the cover $\mathbb{M}_0 \to \mathbb{M}$ obtained as pull back of the geodesic arc $(i, j) \subset \mathbb{M}$.

In particular, we focus on the Galois cover $\mathbb{T}^* \to \mathbb{M}$ by the punctured torus which corresponds to the first derived subgroup Γ', or equivalently to the abelianisation morphism $\mathbb{Z}/2 * \mathbb{Z}/3 \to \mathbb{Z}/2 \times \mathbb{Z}/3$. The Galois action of a generator for the quotient $\Gamma/\Gamma' = \mathbb{Z}/6$ on the cover $\mathbb{T}^*$ is by rotation of order 6. This is well known (see [Ghy17] for instance), and forms the content of Proposition 3.26. Then we consider the universal abelian cover of this punctured torus $\mathbb{T}^*$, corresponding to the second derived subgroup Γ''. The Galois action of the quotient $\Gamma'/\Gamma'' = \mathbb{Z}^2$ is by translation. Putting those constructions together we find that Γ/Γ'' is canonically isomorphic to the semi-direct product $\Gamma/\Gamma' \rtimes \Gamma'/\Gamma'' = \mathbb{Z}/6 \rtimes \mathbb{Z}^2$, realised as the isometry group of a hexagonal lattice $\Gamma'/\Gamma'' = \mathbb{Z}^2$ in the plane. These (presumably new) results are contained in Proposition 3.29 and its Corollary 3.31, but they can be considered as excursions.

Finally we consider the lifts of modular geodesics in Galois covers, in order to "simplify" them and define several topological quantities. We show in Proposition 3.36 that the lifts of homotopic and linear representatives in finite Galois covers with no orbifold singularities are connected by isotopies and Reidemeister triangle moves. Then we focus on the lifts of combinatorial axes in the dessin d'enfant, especially in that corresponding to the second derived subgroup Γ'' which is a hexagonal graph embedded in the punctured plane $\mathbb{R}^2 \setminus \mathbb{Z}^2$ forming the universal abelian cover of $\mathbb{T}^*$. In particular, we recover the Rademacher invariant as an asymptotic winding number of combinatorial paths in this hexagonal graph.

3.1 Loops in the modular orbifold

The modular orbifold

We denote $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Recall Proposition 2.2 describing the action of Γ on $\triangle_2$ and Proposition 2.5 providing an ideal triangulation of $\mathbb{HP}$ by the interior of $\triangle_2$. We deduce that the subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ acts properly discontinuously on $\mathbb{HP}$, with fundamental domain the triangle $(\infty, 0, j)$ formed by the fixed points of L and R on the boundary and of T inside. We let $\mathbb{M} = \triangle_2/\Gamma$ and $\mathbb{M} = \mathbb{HP}/\Gamma$.

This fundamental domain $(\infty, 0, j)$ can be cut along the geodesic arc joining i and j to obtain a pair of isometric triangles (i, j, ∞) and $(i, j, 0)$: identifying them along their isometric edges yields the quotient $\mathbb{M}$. It is a hyperbolic two-dimensional orbifold, with two conical points of order 2 and 3 respectively associated to the fixed points i and j for the elliptic elements S and T, and a cusp associated to the fixed point ∞ for the parabolic element R. It is homotopically equivalent to a three-holed sphere with discs attached along two boundary components by homeomorphisms of the circle having degrees 2 and 3. See [Hae90] for homotopies in the orbifold category.

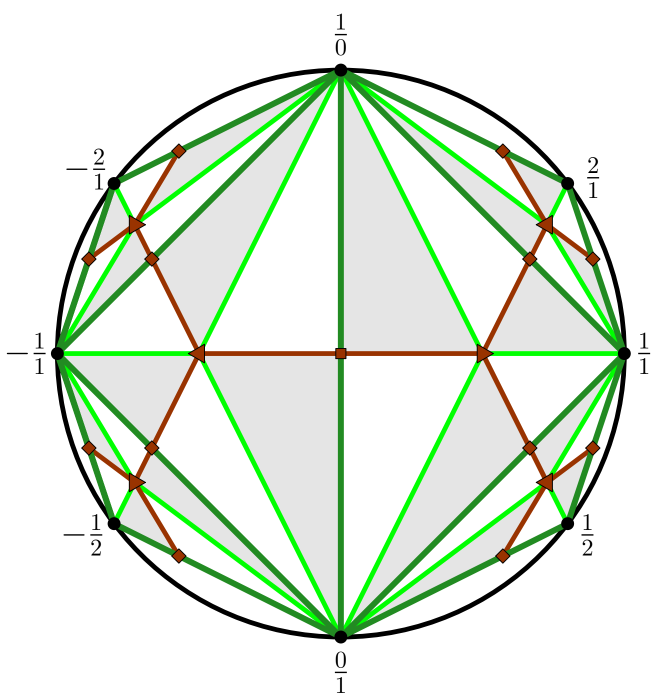


Figure 3.1: The hyperbolic plane $\mathbb{HP}$ is triangulated by $\triangle_2'$. The union of a grey triangle and a white triangle forms a fundamental domain under the action of Γ.

The hyperbolic metric descending on $\mathbb{M}$ has finite area $\pi/3$ since three Γ-translates of the fundamental triangle pave an ideal triangle (whose area equals π by the Gauss-Bonnet theorem, because its angles add up to zero). This shows that Γ is a lattice in $\mathrm{PSL}_2(\mathbb{R})$, that is a discrete subgroup with finite covolume.

Remark 3.1. *A finitely generated discrete subgroup of* $\mathrm{PSL}_2(\mathbb{R})$ *with infinite covolume acts on* $\mathbb{HP}$ *with quotient an orbifold having ends of infinite area: they are trumpets (whose local model is the quotient of* $\mathbb{H}$ *by a hyperbolic cyclic group) instead of cusps (whose local model is the quotient of* $\mathbb{HP}$ *by a parabolic cyclic group).*

Projective model for the modular orbifold

Recall that Proposition 2.5 defined an isomorphism $\mathbb{P}(\bar{\psi})$ from the geometric realisation of $\triangle_4$ in $\mathbb{R}^2$ to the geometric realisation of $\triangle_2$ in $\mathbb{RP}^2$ which reverses their orientation. It sends the base triangle $\triangledown_1 = (v_0, v_1, v_\infty)$ of the lotus to the base triangle $\triangledown_2 = (0, 1, \infty)$ in the ideal triangulation of the hyperbolic plane. It also conjugates the actions of the monoid $\mathrm{PSL}_2(\mathbb{N})$ on the lotus and its image.

The map $\mathbb{P}(\bar{\psi})$ is projective linear in restriction to each triangle, so it matches their first barycentric subdivisions: the center of gravity of $\triangledown_1$ is sent to the incenter of $\triangledown_2$. Hence $\triangledown_1$ is cut into three triangles by the long legs of its medians, each one mapping to a fundamental domain for the action of Γ on $\triangle_2$.

At the target, the triangle $\triangledown_2$ quotients to $\mathbb{M}$ under the action of T by hyperbolic rotation of order three around its incenter, and after the identification of the edge $(0, \infty)$ with $(0, 1)$ and $(1, \infty)$ by L and R. Let us pull these back on $\triangledown_1$ to obtain (after removing its vertices) a projective model of the modular orbifold $\mathbb{M}$.

The action of T pulls back on $\triangledown_1$ to the unique projective transformation which cyclically permutes the vertices (v_∞, v_1, v_0) and fixes the barycenter. The actions of L and R identify hypothenuse of $\triangledown_1$ with its vertical and horizontal edges. Notice that the transformation T restricted to the segments of the base glider joining j to ∞ corresponds to the euclidean reflection across the diagonal. Moreover, in restriction to the hypothenuse of $\triangledown_1$, we may compose the transformations $TL = S = R^{-1}T$ and find the symmetry across its barycenter. For the euclidean metric on $\triangledown_1$, the pairs of equidistant points on the edges $(v_\infty, i)\&(i, v_0)$ and $(v_\infty, j)\&(j, v_0)$ are identified. This metric descends on $\mathbb{M}$ to a flat metric with conical singularities at i and j.

Remark 3.2. *Recall that* $\mathbb{P}(\bar{\psi}) \colon \triangle_4 \subset \mathbb{R}^2 \to \triangle_2 \subset \mathbb{P}(\mathbb{H} \cup \mathbb{X})$ *is orientation reversing, and* $\triangledown_2 \to \mathbb{M}$ *is orientation preserving, thus* $\triangledown_1 \to \mathbb{M}$ *is orientation reversing.*

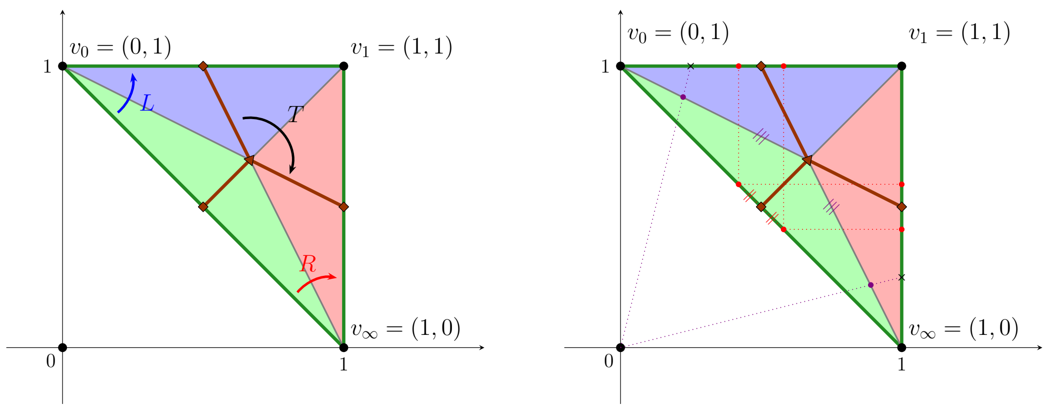


Figure 3.2: Identifications of ∇_1 yielding the projective model for $\mathbb{M}$.

Modular geodesics: homotopy classes of their lifts in $\mathbb{M} \setminus \{i, j\}$

A geodesic in $\mathbb{M}$ lifts to a geodesic in $\mathbb{HP}$ which is the translation axis of a unique one parameter subgroup of hyperbolic matrices in $\mathrm{PSL}_2(\mathbb{R})$, and two geodesics in $\mathbb{HP}$ project to the same geodesic in $\mathbb{M}$ if and only if the corresponding one parameter subgroups generated by those matrices are conjugate by an element of $\mathrm{PSL}_2(\mathbb{Z})$.

We are exclusively interested in closed geodesics: they correspond to conjugacy classes of hyperbolic matrices in $\mathrm{PSL}_2(\mathbb{Z})$. Taking a power of a closed geodesic, that is winding several times around it (and inverting its direction if the exponent is negative), has the same effect on (any representative of) the conjugacy class. The length λ_A of the closed geodesic γ_A associated to $A \in \mathrm{PSL}_2(\mathbb{Z})$ is the translation length of A given by $\mathrm{Tr}(A)/2 = \cosh(\lambda_A/2)$.

One may ask how many closed geodesics there are of length at most λ, or how are they distributed in the orbifold. In this chapter we investigate how to draw them, what are their intersection numbers, and how do they lift in Galois covers.

We first describe their relative position with respect to the conical singularities.

Remark 3.3. *A neighbourhood of $i \in \mathbb{M}$ is the quotient of a neighbourhood of $i \in \mathbb{HP}$ by the action of S, so an arc through $i \in \mathbb{HP}$ projects to an arc which reaches i in $\mathbb{M}$, and leaves with an angle $\pm\pi \equiv 0 \bmod \pi$, in the opposite direction.*

Similarly, a neighbourhood of $j \in \mathbb{M}$ is the quotient of a neighbourhood of $j \in \mathbb{HP}$ by the action of J, so an arc through $j \in \mathbb{HP}$ projects to an arc which reaches j in $\mathbb{M}$, and leaves with an angle $\pm\pi \equiv \pm\pi/3 \bmod 2\pi/3$.

Corollary 3.4. *A closed geodesic in $\mathbb{M}$ contains i if and only if the corresponding conjugacy class in $\mathrm{PSL}_2(\mathbb{Z})$ is symmetric (2.26). If it is furthermore primitive then it reaches and leaves i twice, making an angle $2(\arctan(\alpha) - \arctan(1/\alpha')) \bmod \pi$.*

A closed geodesic in $\mathbb{M}$ *contains* j *if and only if it admits a Lyndon representative* $A \in \mathrm{PSL}_2(\mathbb{N})$ *such that* $\mathrm{Tr}(AT) = \mathrm{Tr}(AT^{-1})$.

Proof. Consider a primitive closed geodesic $\gamma\colon \mathbb{S}^1 \to \mathbb{M}$.

The moments γ reaches $i \in \mathbb{M}$ are defined as the preimages $\gamma^{-1}(i) \subset \mathbb{S}^1$. They are in bijection with the lifts of γ passing through $i \in \mathbb{HP}$, modulo the action of the stabiliser of $i \in \mathbb{HP}$, which is generated by S. By Proposition 2.23, these lifts are the translation axes of the symmetric representatives in the conjugacy class. By Lemma 2.27 such symmetric representatives belong to $\mathrm{PSL}_2(\mathbb{Z})$ up to conjugacy by S, an there are two of them: they are of the form ${}^tBB$ and $B{}^tB$ for some $B \in \mathrm{PSL}_2(\mathbb{N})$. The angle in $\mathbb{R}/2\pi$ made by two geodesics of $\mathbb{HP}$ intersecting at i is deduced from their attractive fixed points by stereographic projection.

The moments γ reaches $j \in \mathbb{M}$ are described similarly using Proposition 2.23. □

Definition 3.5. *Consider a modular geodesic in* $\mathbb{M}$*, and let us define its* privileged perturbation *in* $\mathbb{M} \setminus \{i, j\}$.

If the geodesic avoids i *and* j*, then it equals its privileged perturbation. Otherwise, choose a lift* $\gamma \subset \mathbb{HP}$ *and perturb it slightly off each point in the orbits of* i *and* j *as prescribed by the canonical choices of Remark 2.38, and project it down to* $\mathbb{M} \setminus \{i, j\}$.

Remark 3.6. *The canonical choices of 2.38 imply that the axis of a symmetric* $A \in \mathrm{PSL}_2(\mathbb{N})$ *is perturbed in the neighbourhood of* i *so that it surrounds it clockwize if* A *starts with an* L*, and couter-clockwize if* A *starts with an* R*. The opposite choice would lead to a perturbed loop in* $\mathbb{M} \setminus \{i, j\}$ *with the inverse homotopy class.*

Definition 3.7. *A* multiloop *in* $\mathbb{M}$ *with* $k \in \mathbb{N}$ *components is a smooth map from the disjoint union of* k *oriented circles whose components are labelled, considered up to individual reparametrizations. A* loop *is a multiloop with one component.*

We base the orbifold fundamental group of $\mathbb{M}$ at a point very close to i on the geodesic arc towards j. It is isomorphic to Γ.

One should be carefull with homotopies in orbifolds. For instance a loop which circles n times around a cyclic singularity of order n is contractible in the orbifold. More generally, various representatives of the same homotopy class in $\mathbb{M}$ may lift to different homotopy classes in $\mathbb{M} \setminus \{i, j\}$. The inclusion of this three-holed sphere in the modular orbifold yields a surjection $\pi_1(\mathbb{M} \setminus \{i, j\}) \to \pi_1(\mathbb{M})$ and a single conjugacy class of the target may lift to several conjugacy classes of the source.

Still, the algebraic simplifications in the modular group mirror the loop simplifications in the modular orbifold, hence conjugacy classes in Γ correspond to free homotopy classes of loops in $\mathbb{M}$. Accordingly a closed loop has the same attributes

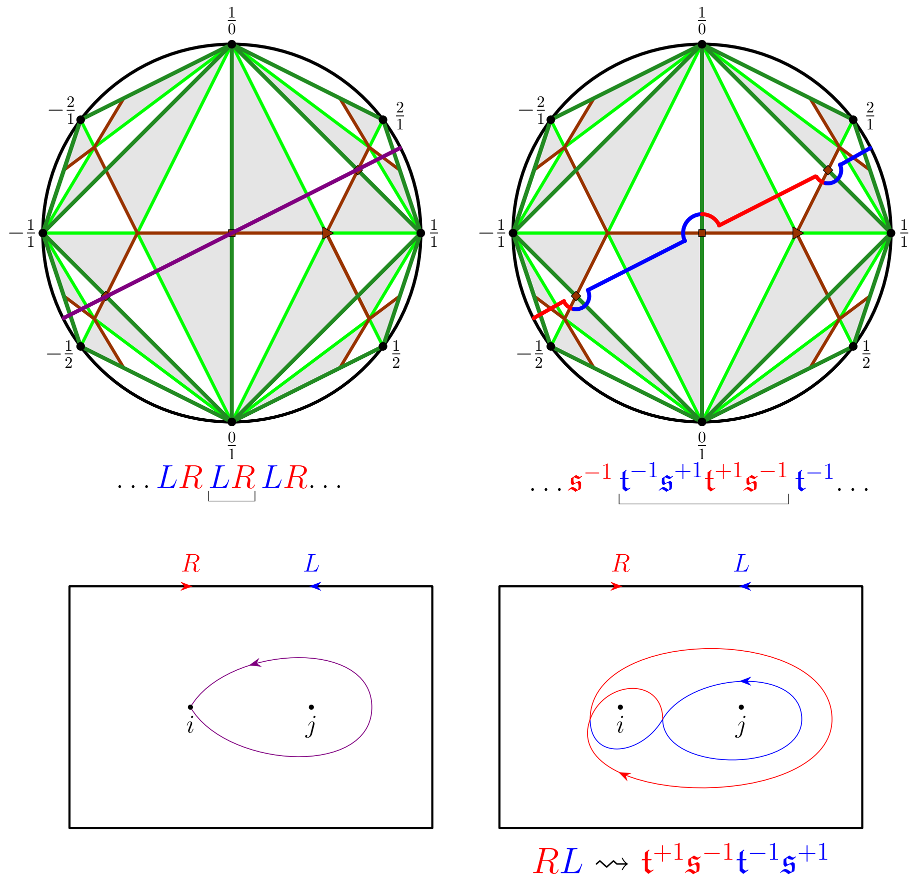


Figure 3.3: The geodesic axis γ_{RL} and its canonical perturbation. Their respective projections in $\mathbb{M}$ and in $\mathbb{M} \setminus \{i, j\}$.

as a conjugacy class. It is trivial if it bounds a disc with no singularities, elliptic if it circles around a conical singularity, parabolic if it circles around the cusp, and hyperbolic otherwize. Moreover, a non elliptic loop is primitive unless it is homotopic to a loop wrapping several times around the same path.

To summarize, the hyperbolic conjugacy classes in the modular group parametrize the homotopy classes of hyperbolic loops as well as the closed geodesics in $\mathbb{M}$. In other terms, every free homotopy class of hyperbolic loop contains a unique geodesic.

Consider the free homotopy class of a hyperbolic loop in $\mathbb{M}$. Its unique geodesic representative has a privileged perturbation in $\mathbb{M}\setminus\{i, j\}$: what is its homotopy class?

Recall that Theorem 2.11 used the presentation of $\pi_1(\mathbb{M}) = \mathbb{Z}/2 * \mathbb{Z}/3$ as the free amalgam of its subgroups generated by S&T to derive a normal form for its

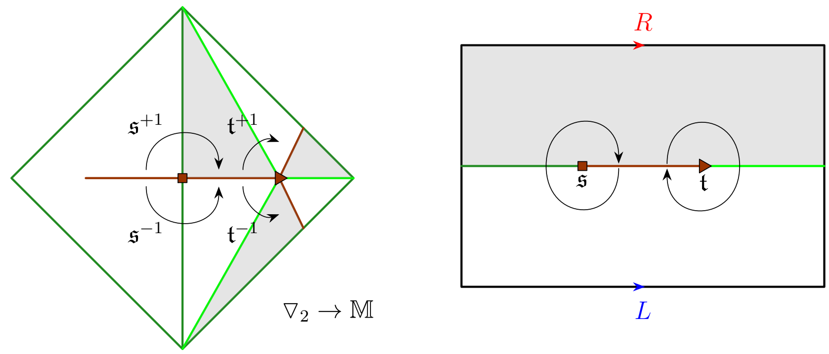


Figure 3.4: The loops $\mathfrak{s}, \mathfrak{t} \subset \mathbb{M} \setminus \{i, j\}$ freely generate $\pi_1(\mathbb{M} \setminus \{i, j\})$.

conjugacy classes. By Corollary 2.19, the submonoid $\mathrm{PSL}_2(\mathbb{N}) \subset \mathrm{PSL}_2(\mathbb{Z})$ which is freely generated by $L = T^{-1}S$ and $R = TS^{-1}$, intersects every hyperbolic class along its Lyndon representatives, which are cyclic permutation of a single $L\&R$-word.

The group $\pi_1(\mathbb{M} \setminus \{i, j\}) = \mathbb{Z} * \mathbb{Z}$ is freely generated by the loops $\mathfrak{s}\&\mathfrak{t}$ surrounding the removed punctures $i\&j$, with the orientation induced by $\mathbb{M}$, as traced in figure 3.4. A homotopy class of loops in $\mathbb{M} \setminus \{i, j\}$ corresponds to a conjugacy class in its fundamental group hence to a unique word in $\mathfrak{s}\&\mathfrak{t}$ up to cyclic permutation.

The inclusion $\mathbb{M} \setminus \{i, j\} \subset \mathbb{M}$ induces the morphism of fundamental groups $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z}/2 * \mathbb{Z}/3$ defined by $\mathfrak{s} \mapsto S$ and $\mathfrak{t} \mapsto T$. The question amounts to finding a "geodesic section" of this morphism at the level of conjugacy classes.

Corollary 3.8. *A closed geodesic in $\mathbb{M}$ encoded by an $L\&R$-cycle lifts in $\mathbb{M} \setminus \{i, j\}$ to a homotopy class encoded by an $\mathfrak{s}\&\mathfrak{t}$-cycle computed using the translation algorithm of Proposition 2.40 and applying the canonical perturbations of Remark 2.38.*

Scholium 3.9. *The algorithm of Proposition 2.40 is rather cumbersome because it does not provide local translation rules $L\&R \rightsquigarrow \mathfrak{s}\&\mathfrak{t}$, as explained in Remark 2.37.*

In a later subsection, we will use the projective model of the orbifold to define a loop which is homotopic to our modular geodesic, and which lifts in $\mathbb{M} \setminus \{i, j\}$ according to the simpler algorithm of 2.33.

Proof. Consider a closed geodesic in $\mathbb{M}$ corresponding to a conjugacy class in $\pi_1(\mathbb{M})$. Among its lifts in $\mathbb{HP}$, those which intersect $(0, \infty)$ positively are the translation axes (α'_k, α_k) of its Lyndon representatives $A_k \in \mathrm{PSL}_2(\mathbb{N})$. The translation axis (α'_k, α_k) intersects the base triangle $\nabla_2 \subset \triangle_2$ along a segment, which equals the inverse image by an element in $\mathrm{PSL}_2(\mathbb{N})$ of the intersection between (α'_0, α_0) and the k-th triangle of $\triangle_2$ from the base edge. We obtain a union of n segments in ∇_2 which intersect its first barycentric subdivision ∇_2' as described by the algorithm of Proposition 2.40 translating $L\&R$-words into $\mathfrak{s}\&\mathfrak{t}$-words (which tells in particular whether each segment passes to the left or right of the points i and j). (We some axes pass through any of $\{i, Ti, T^{-1}i\}$, the same holds for their canonical perturbations.)

These (perturbed) segments project to the (perturbed) modular geodesic under the action of T by rotation of order 3 which cyclically permutes the three gliders in ∇_2', and the identifications of the edges by $L\&R$. Consequently, this (perturbed) geodesic lifts in $\mathbb{M} \setminus \{i, j\}$ to a loop whose homotopy class is described by an $\mathfrak{s}\&\mathfrak{t}$ cycle, obtained by translating the $L\&R$-cycle according to Proposition 2.40. □

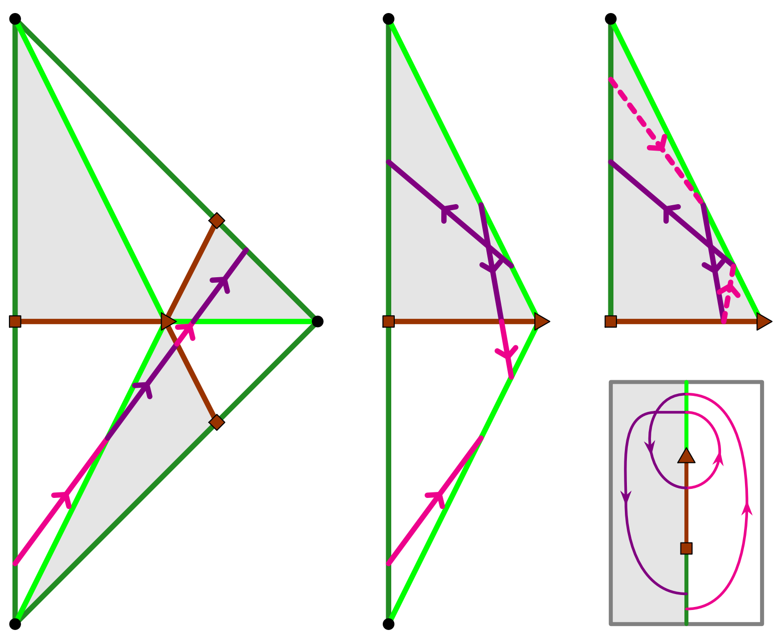


Figure 3.5: Quotient $\nabla_2 \to \mathbb{M}$ with the portion of an axis encoded by $S^{-1}T^{-2}S^{-1}$.

Intersection numbers of modular geodesics

Recall we learnt at the end of Section 2.3 how to construct invariants of two conjugacy classes in the modular group. One of them, well defined for primitive hyperbolic classes, is the intersection number between the corresponding modular geodesics. Of course, this intersection number extends to non primitive hyperbolic conjugacy classes provided we count intersection numbers with the appropriate multiplicities.

Proposition 3.10. *The geometric intersection of the closed geodesics in $\mathbb{M}$ associated to the hyperbolic elements $A, B \in \Gamma = \pi_1(\mathbb{M})$ is equal to the finite sum:*

$$I([A],[B]) = \sum_{(U,V)} |\mathrm{cross}|(UAU^{-1}, VBV^{-1})$$

indexed by the $(U,V) \in (\Gamma/\operatorname{Stab} A) \times_\Gamma (\Gamma/\operatorname{Stab} B)$ with $\operatorname{Stab}(A)$ and $\operatorname{Stab}(B)$ the cyclic subgroups generated by the primitive roots of A and B.

This sum has finite support, and can be computed as :

$$I([A],[B]) = I_+([A],[B]) + I_+([A],[{}^tB])$$

where we define $I_+([A],[B])$ as the same sum restricted over the pairs $(\sigma^i A, \sigma^j B)$ of Lyndon representatives for the conjugacy classes of $A, B \in \mathrm{PSL}_2(\mathbb{N})$, that is:

$$I_+([A],[B]) = \sum_{i=1}^{\mathrm{len}(A)} \sum_{j=1}^{\mathrm{len}(B)} |\mathrm{cross}| \left(\sigma^i A, \sigma^j B\right)$$

Remark 3.11. *The topological space underlying the modular orbifold has trivial homology, so the algebraic intersection numbers between modular geodesics are trivial. Hence for all hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$, the* cross *function satisfies the identity:*

$$\sum_{(U,V)} \mathrm{cross}(UAU^{-1}, VBV^{-1}) = 0.$$

Remark 3.12. *One may also compute the intersection numbers between (perturbed) modular geodesics using the $\mathfrak{s}\&\mathfrak{t}$-cycles of their lifts in $\mathbb{M} \setminus \{i, j\}$.*

This relies on the algorithm of Birman-Series [BS84] which computes the minimal intersection numbers between homotopy classes of loops in a compact surface of negative euler characteristic with non-empty boundary. The fundamental group of such a surface is free of finite type. The algorithm takes as input a free cyclically ordered set of generators, and the description of loops as cyclic words in those generators.

Example 3.13. *The formulae of Proposition 3.10 are valid even when A, B are equal or opposite, in which case they count the intersection number between two parallel copies of a same loop, which is twice its self-intersection. More generally they are valid when A, B are not coprime, that is equal to powers of a same element.*

For instance the loop corresponding to RLL has self-intersection 3 since:

$$I([RLL],[RLL]) = I_+([RLL],[RLL]) + I_+([RLL],[RRL]) = 4 + 2 = 2 \times 3.$$

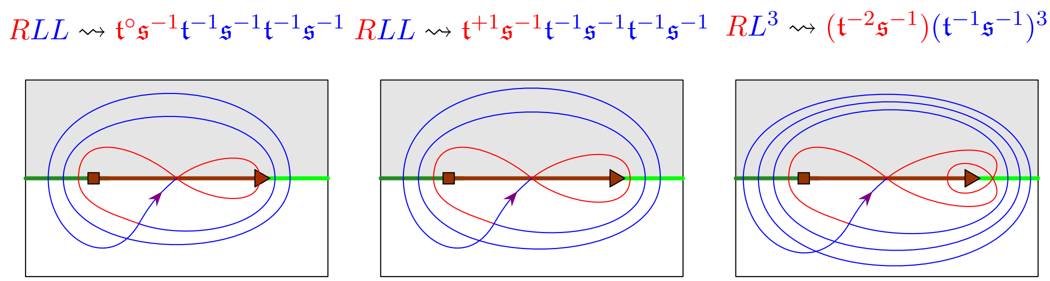


Figure 3.6: Minimally intersecting representatives in $\mathbb{M} \setminus \{i, j\}$ for the homotopy class of $(\mathfrak{t}^{+1}\mathfrak{s}^{-1})(\mathfrak{t}^{-1}\mathfrak{s}^{-1})(\mathfrak{t}^{+1}\mathfrak{s}^{-1})$ and $(\mathfrak{t}^{-2}\mathfrak{s}^{-1})(\mathfrak{t}^{-1}\mathfrak{s}^{-1})(\mathfrak{t}^{-1}\mathfrak{s}^{-1})(\mathfrak{t}^{-1}\mathfrak{s}^{-1})$.

By Corollary 3.8 the modular geodesic lifts in $\mathbb{M} \setminus \{i, j\}$ *to the homotopy class of* $(\mathfrak{t}^{+1}\mathfrak{s}^{-1})(\mathfrak{t}^{-1}\mathfrak{s}^{-1})(\mathfrak{t}^{+1}\mathfrak{s}^{-1})$*, of which a representative is depicted in Figure 3.6.*

For another example, the loop corresponding to RLLL has self-intersection 4 *as:*

$$I([RLLL],[RLLL]) = I_+([RLLL],[RLLL]) + I_+([RLLL],[RRRL]) = 6 + 2 = 2 \times 4.$$

By Corollary 3.8 the modular geodesic lifts in $\mathbb{M} \setminus \{i, j\}$ *to the homotopy class of* $(\mathfrak{t}^{-2}\mathfrak{s}^{-1})(\mathfrak{t}^{-1}\mathfrak{s}^{-1})(\mathfrak{t}^{-1}\mathfrak{s}^{-1})(\mathfrak{t}^{-1}\mathfrak{s}^{-1})$ *of which a representative is depicted in Figure 3.6.*

Definition 3.14. *A multiloop in an orientable orbifold is called* taut *when it is minimally self-intersecting in its homotopy class. It is called* essential *when none of its components can be homotoped to a point whether smooth or singular, into a boundary component, or into a cusp.*

Freedman-Hass-Scott have shown [FHS82] that in a compact orientable riemannian surface, a finite collection of essential geodesics which remain in the interior defines a taut multiloop. (If the boundary is convex, and in particular if it is geodesic, then any essential geodesic must remain in the interior).

Proposition 3.15. *Any collection of (perturbed) modular geodesics defines a multiloop in* $\mathbb{M} \setminus \{i, j\}$ *which is taut.*

Proof. The perturbed modular geodesic remains in the complement of two disjoint disc neighbourhoods $\mathbb{D}_i, \mathbb{D}_j$ of i, j in $\mathbb{M}$. Consider the restriction of the hyperbolic metric of $\mathbb{M}$ to $\mathbb{M} \setminus (\mathbb{D}_i \cup \mathbb{D}_j)$. If the modular geodesic avoids $i \& j$, then we are done. Otherwize, provided its perturbation and the disc neighbourhoods $\mathbb{D}_i, \mathbb{D}_j$ are chosen small enough, one may slightly increase the metric in the collar neighbourhoods of the boundaries $\partial\mathbb{D}_i$ and $\partial\mathbb{D}_j$ so that the unique geodesic of $\mathbb{M} \setminus (\mathbb{D}_i \cup \mathbb{D}_j)$ in the homotopy class of the perturbed geodesic lies inside $\mathbb{M} \setminus (\mathbb{D}_i \cup \mathbb{D}_j)$ and remains isotopic to the perturbed modular geodesic. □

Question 3.16. *We conjecture that modular geodesics are minimally intersecting in their orbifold homotopy class (which is a stronger statement than Proposition 3.15).*

In other terms, we believe that among all homotopy classes in the surface $\mathbb{M}\setminus\{i,j\}$ *which map to a given hyperbolic homotopy class in the orbifold* $\mathbb{M}$*, the one defined by the lift of the (perturbed) modular geodesic has minimal self-intersection.*

Linear representatives for hyperbolic conjugacy classes in $\pi_1(\mathbb{M})$

Let us now define the *linear representative* of a hyperbolic conjugacy class in the modular group. It is a loop in the projective model of the modular orbifold described at the beginning of this section. In particular it will have a well defined isotopy class.

Definition 3.17. *Fix a conjugacy class in* $\mathrm{PSL}_2(\mathbb{Z})$ *and let* $A_k = \sigma^k A \in \mathrm{PSL}_2(\mathbb{N})$ *be its* n *Lyndon representatives: their stable eigen-directions intersect the base triangle* $\triangledown_1$ *of the lotus* $\triangle_1$ *in* n *disjoint segments avoiding all the vertices of* $\triangledown_1'$*.*

After quotienting $\triangledown_1$ *by the* $L\&R\&T$*-identifications yielding the projective model of* $\mathbb{M}$*, we obtain a loop which we call the* linear representative *of the conjugacy class.*

This linear representative lifts in $\mathbb{M}\setminus\{i,j\}$ *to a homotopy class whose* $\mathfrak{s}\&\mathfrak{t}$*-cycle is obtained from the* $L\&R$*-cycle of the* A_j *by the translation rules of Proposition 2.33:*

$$LL \rightsquigarrow (\mathfrak{t}^{-1}\mathfrak{s}^{-1})L \qquad RR \rightsquigarrow (\mathfrak{t}^{+1}\mathfrak{s}^{+1})R$$
$$LR \rightsquigarrow (\mathfrak{t}^{-1}\mathfrak{s}^{+1})R \qquad RL \rightsquigarrow (\mathfrak{t}^{+1}\mathfrak{s}^{-1})L$$

which up to a cyclic permutation corresponding to a conjugacy by $\mathfrak{s}^{+1}$ *or* $\mathfrak{s}^{-1}$ *amount to* $L \rightsquigarrow \mathfrak{s}^{-1}\mathfrak{t}^{-1}$ *and* $R \rightsquigarrow \mathfrak{s}^{+1}\mathfrak{t}^{+1}$*.*

Remark 3.18. *Since* $\triangledown_1 \to \mathbb{M}$ *is orientation reversing, the rules* $L \rightsquigarrow \mathfrak{s}^{-1}\mathfrak{t}^{-1}$ *and* $R \rightsquigarrow \mathfrak{s}^{+1}\mathfrak{t}^{+1}$ *tell us how the axis of an* $L\&R$*-word in* $\triangle_1$ *projects to* $\mathbb{M}$ *seen from beneath. From this perspective, the loops* $\mathfrak{s},\mathfrak{t} \subset \mathbb{M}\setminus\{i,j\}$ *appear to turn anti-clockwize around the singularities, as in Figure 3.7 (contrary to 3.4).*

Let us propose practical algorithms to draw from a primitive $L\&R$-cycle of length n, the stable eigendirections of its Lyndon representatives A_k intersected with $\triangledown_1'$, with enough precision. The required precision will be such that the translates under the projective action of T provides a faithful picture of its isotopy class in the base glider , and we shall explain how to report all segments in the base glider just after.

Arithmetic algorithm. Recall the increasing bijections $A_k \leftrightarrow A_k(1) \leftrightarrow \alpha_k = A_k^{\infty}(1)$, and note that the lines of inclination α_k and $A_k^2(1)$ intersect the gliders in the first n triangles in the lotus according to the same pattern.

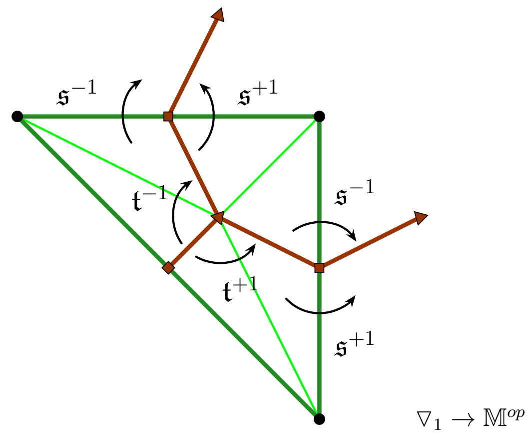


Figure 3.7: The loops $\mathfrak{s}, \mathfrak{t}$ in $\mathbb{M}$ seen from beneath.

This yields the first algorithm: for each k, intersect the base triangle of the lotus with the line of inclination $A_k^2(1)$ to obtain a segment, and label its initial and endpoint by k and $k+1$.

Combinatorial algorithm. The k-th segment goes from the hypothenuse (v_∞, v_0) to either (v_1, v_0) or (v_∞, v_1) depending on whether A_k starts with an L or an R, the relative position of its endpoint with respect to the middle of the edges depends on the second letter. This determines the individual positions of the segments with respect to the first barycentric subdivision $\triangle_1'$, and we know they are disjoint.

Let n_w be the number of cyclic permutations of the $L\&R$-word with prefix w. Place n points on the base edge, partitioned as $n_L + n_R$ on the left & right of its midpoint, and label them by the $L\&R$-factorisations of the A_k in lexicographic order.

Trace a segment from each of the n_R points on the left to a point on the horizontal edge, with n_{RR} landing to the left and n_{RL} landing to the right of its barycenter. Trace a segment from each of the n_L points on the right to a point on the vertical edge, with n_{LR} landing to the left and n_{LL} landing to the right of its barycenter.

Hybrid algorithm. In the linear model, the endpoint of one segment determines the starting point of the next as suggested in figure 3.9, whence a third algorithm.

Find an approximation of the first point on (v_0, v_∞) by intersecting with the rational line through $A_0^2(1)$. Then follow the radial flow and return to (v_0, v_∞) each time leaving the exiting the base triangle using the $L\&R$ identifications. Stop when the $L\&R$-factorisation of A_0 has been read: the loop almost closes up.

Drawing the loop after projection $\nabla_1 \to \mathbb{M}$. Now that we can draw the linear representative in ∇_1, we may report all segments in the base glider using the identifications of its edges with those of the other two gliders provided by $L\&R\&T$.

In fact, it is enough to consider the identifications given by $L\&R$, and of the pairs of equidistant points on the edges $(v_\infty, i)\&(i, v_0)$ as well as $(v_\infty, j)\&(j, v_0)$. This is because the lines under consideration will never cross the edge (j, v_1).

We may thus report all segments in the base glider and obtain a configuration of segments providing a faithful picture for the isotopy class of the linear representative.

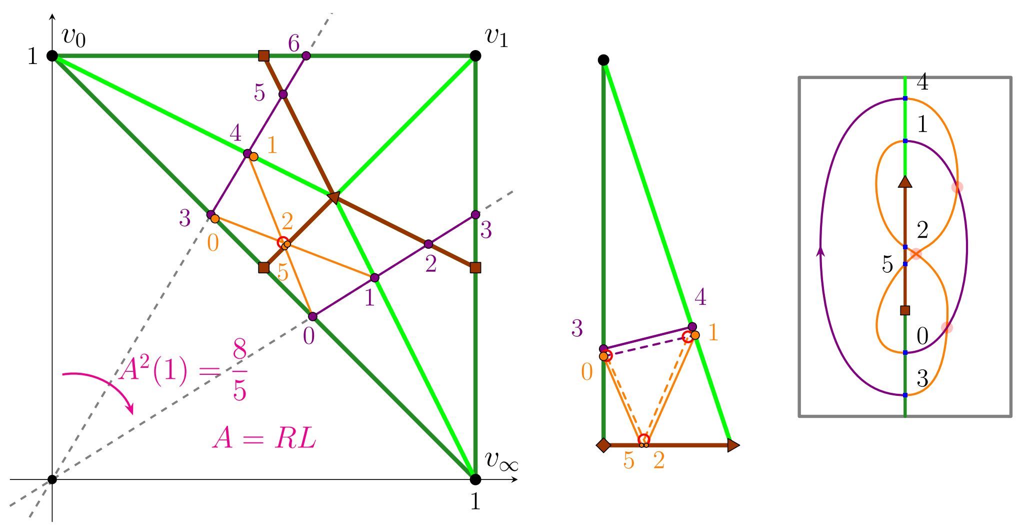


Figure 3.8: Quotienting the axes of RL & LR under $\nabla_1 \to \mathbb{M}$. Compare Figure 3.3.

Figure 3.9: Apply the hybrid algorithm to RLL, report the segments in the base glider and project by $\nabla_1 \to \mathbb{M}$ to recover the homotopy class of $\mathfrak{s}^{+1}\mathfrak{t}^{+1}\mathfrak{s}^{-1}\mathfrak{t}^{-1}\mathfrak{s}^{-1}\mathfrak{t}^{-1}$. Compare Figure 3.6.

Remark 3.19. *The linear representatives for a finite set of primitive hyperbolic conjugacy classes of* $\pi_1(\mathbb{M})$ *defines a multiloop in* $\mathbb{M}$.

In particular, it has a well defined isotopy class which we may draw using the previous algorithms provided we adjust some precision issues to take into account the relative positions between the various components.

Remark 3.20. *Using these drawing algorithms, we may observe on figure 3.9 that the linear representative of RLL has self-intersection* 4*. However, we showed in Example 3.13 that the corresponding modular geodesic has self-intersection* 3*.*

Consequently, the linear representatives are not always minimally intersecting in their orbifold homotopy class.

Proposition 3.21. *The linear representative of a primitive hyperbolic conjugacy class in $\pi_1(\mathbb{M})$ defines a loop in $\mathbb{M} \setminus \{i, j\}$ which is minimally intersecting in its homotopy class. In $\mathbb{M} \setminus \{i, j\}$, any two linear representatives realise the minimal intersection number between loops in their homotopy classes.*

Proof. The euclidean metric on the base triangle ∇_1 of the lotus minus its barycenters (of face, edges and vertices) descends to a flat metric on $\mathbb{M} \setminus \{i, j\}$, for which the linear representatives are closed geodesics. We conclude as in Proposition 3.15 that they must be minimally intersecting. □

Remark 3.22. *The intersection numbers between linear representatives may thus be computed by applying the algorithm of Birman-Series [BS84] to the* $\mathfrak{s}$&$\mathfrak{t}$*-cycles encoding their homotopy classes in $\mathbb{M} \setminus \{i, j\}$.*

For example, the linear representative of RLL has homotopy class in $\mathbb{M} \setminus \{i, j\}$ given by $\mathfrak{t}^{+1}\mathfrak{s}^{-1}\mathfrak{t}^{-1}\mathfrak{s}^{-1}\mathfrak{t}^{-1}\mathfrak{s}^{+1}$, and the Birman-Series algorithm confirms that it has self-intersection 4*.*

Scholium 3.23. *The linear representative and the perturbed hyperbolic representative of a same hyperbolic conjugacy class in $\pi_1(\mathbb{M})$ are homotopic loops in the orbifold $\mathbb{M}$, but they often lift to non homotopic loops in the surface $\mathbb{M} \setminus \{i, j\}$. In algorithms terms, the Propositions 2.33 and 2.40 translate L&R-cycles into* $\mathfrak{s}$&$\mathfrak{t}$*-cycles which have the same exponents of $\mathfrak{s}$ modulo* 2 *and of $\mathfrak{t}$ modulo* 3*, but may well be different.*

Can we improve the orbifold-homotopy to a stronger equivalence relation, for instance corresponding to homotopy of the lifted loops in the unit tangent bundle ? We will address such questions in Section 4.1.

3.2 Lifting geodesics to Galois covers

We wish to lift closed geodesics from the modular orbifold to Galois covers, in order to simplify their topology, construct tractable invariants and reveal hidden structures.

Covers and Galois covers

We mark the hyperbolic plane $\mathbb{HP}$ at a point infinitely close to the fixed point i of S and on the geodesic towards the fixed point j of T. The quotient orbifold $\mathbb{M} = \mathbb{H}/\operatorname{PSL}_2(\mathbb{Z})$ is thus pointed accordingly, and this identifies its loop fundamental group $\pi_1(\mathbb{M})$ with the Galois group $\operatorname{PSL}_2(\mathbb{Z})$ of its universal cover $\mathbb{HP} \to \mathbb{M}$.

Every cover $\mathbb{M}_0$ of $\mathbb{M}$ is obtained by quotienting the universal cover $\mathbb{HP}$ of $\mathbb{M}$ by the action of a subgroup of its fundamental group $\pi_1(\mathbb{M}) = \operatorname{Gal}(\mathbb{HP}/\mathbb{M})$, that is the fundamental group $\pi_1(\mathbb{M}_0) = \operatorname{Gal}(\mathbb{HP}/\mathbb{M})$.

The cover is Galois when the subgroup is normal, so the quotient is a group. Thus a Galois cover $\mathbb{M}_0 \to \mathbb{M}$ corresponds to a morphism from the fundamental group of the base $\pi_1(\mathbb{M}) = \operatorname{Gal}(\mathbb{HP}/\mathbb{M})$ onto the symmetry group of the cover $\operatorname{Gal}(\mathbb{M}_0/\mathbb{M})$, and the kernel is the fundamental group of the total space $\pi_1(\mathbb{M}_0) = \operatorname{Gal}(\mathbb{HP}/\mathbb{M}_0)$.

Dessin d'enfants in general covers

Now fix a cover $\mathbb{M}_0 \to \mathbb{M}$ and denote $\Gamma_0 \to \Gamma$ the corresponding inclusion of fundamental groups. The *cusps* of $\mathbb{M}_0$ lie above the cusp of $\mathbb{M}$ and are parametrized by the orbits under Γ_0 of the ideal vertices $\mathbb{QP}^1 \subset \partial\mathbb{HP}$ for the triangulation $\triangle_2 \subset \mathbb{HP}$.

Consider the graph $\mathcal{T}'/\Gamma$ in $\mathbb{M} = \mathbb{HP}/\Gamma$ whose vertices are the orbifold singularities and whose only edge is the geodesic arc between them. Its preimage in $\mathbb{M}_0 = \mathbb{HP}/\Gamma_0$ is a graph $\mathcal{T}'/\Gamma_0$ homotopically embedded in the topological space underlying the orbifold structure of $\mathbb{M}_0$. Every connected component of the complement contains a unique cusp of $\mathbb{M}_0$. The graph is bipartite since every edge connects one vertex covering i to another covering j, and their degrees divide 2 and 3 respectively. Note that the edges of $\mathcal{T}'/\Gamma_0$ are in bijection with the cosets of Γ/Γ_0.

The vertices of degree one in $\mathcal{T}'/\Gamma_0$ are (all) the orbifold singularities of $\mathbb{M}_0$, they correspond to the conjugacy classes of torsion subgroups in Γ_0, which are necessarily isomorphic to $\mathbb{Z}/2$ or $\mathbb{Z}/3$. Indeed, an orbifold singularity of $\mathbb{M}_0$ is the projections of a point $p \in \mathbb{HP}$ with non-trivial stabiliser $\operatorname{Stab}(p) \subset \Gamma_0$, and the singularity $p \bmod \Gamma_0$ corresponds to the conjugacy class of $\operatorname{Stab}(p)$ in Γ_0. Since $\Gamma_0 \subset \Gamma$, the point $p \in \mathbb{HP}$ must be a vertex of $\mathcal{T}'$, and its stabiliser must be either $\mathbb{Z}/2$ or $\mathbb{Z}/3$. Accordingly, it projects to either i or j under the cover $\mathbb{M}_0 \to \mathbb{M}$.

In particular Γ_0 is torsion-free if and only if $\mathbb{M}_0$ is a smooth Riemann surface. In that case every edge of $\mathcal{T}'/\Gamma_0$ has an extremity of degree 2 and an extremity of degree 3, and this quotient graph is the first barycentric subdivision of a trivalent graph $\mathcal{T}/\Gamma_0$. The surface $\mathbb{M}_0$ retracts by deformation on this embedded trivalent graph, so its fundamental group Γ_0 is free.

Remark 3.24. *The previous discussion implies that if a subgroup $\Gamma_0 \subset \Gamma$ is torsion-free then it is free. If moreover it has finite index, then the graph $\mathcal{T}/\Gamma_0$ has twice that amount of edges and is thus finite, so its euler characteristic equals that of $\mathbb{M}_0$. Hence the rank of Γ_0 equals the abelian rank of $H_1(\mathbb{M}_0;\mathbb{Z})$ which is one minus the number of vertices plus the number of edges in $\mathcal{T}/\Gamma_0$.*

Corollary 3.25. *The modular group has exactly two free subgroups of rank 2, or equivalently with index 6. They correspond to covers of the modular orbifold by a surface homeomorphic to a sphere with three punctures and a torus with one puncture.*

Since these surfaces are not homeomorphic, these two subgroups are not conjugate, hence they are normal by unicity, and the covers are Galoisian.

The Galois covers with conical singularities

Now suppose the covering $\mathbb{M}_0 \to \mathbb{M}$ is Galois and denote $1 \to \Gamma_0 \to \Gamma \to \Pi \to 1$ the corresponding short exact sequence of Galois groups. The Galois action of Π is freely transitive on the edges of $\mathcal{T}'/\Gamma_0$ since the quotient is $\mathcal{T}'/\Gamma$.

In particular the vertices above i all have degree 1 or 2 and those above j all have degree 1 or 3. Since the graph is connected, there is at most one vertex of degree 1, so there are only two Galois covers of $\mathbb{M}$ which have singularities, pictured in 3.10.

The first has two singularities of order 3, it corresponds to the kernel of the morphism $\mathbb{Z}/2*\mathbb{Z}/3 \to \mathbb{Z}/2$ counting the number of S modulo 2. The second has three singularities of order 2, it corresponds to the kernel of the morphism $\mathbb{Z}/2*\mathbb{Z}/3 \to \mathbb{Z}/3$ counting the number of T modulo 3.

The derived subgroup and the punctured torus

We now consider the Galois cover of $\mathbb{M}$ corresponding to the derived subgroup $\Gamma' = [\Gamma, \Gamma]$, or to the abelianisation morphism $\Gamma \to \Gamma/\Gamma'$.

Proposition 3.26. *The modular group Γ has abelianisation $\mathbb{Z}/2*\mathbb{Z}/3 \to \mathbb{Z}/2\times\mathbb{Z}/3$. The kernel Γ' is freely generated by $RL = [T, S]$ and $LR = [T^{-1}, S]$.*

The corresponding galois cover of the modular orbifold is a punctured torus $\mathbb{T}^*$, the action of the Galois group $\mathbb{Z}/6$ is represented in figure 3.11.

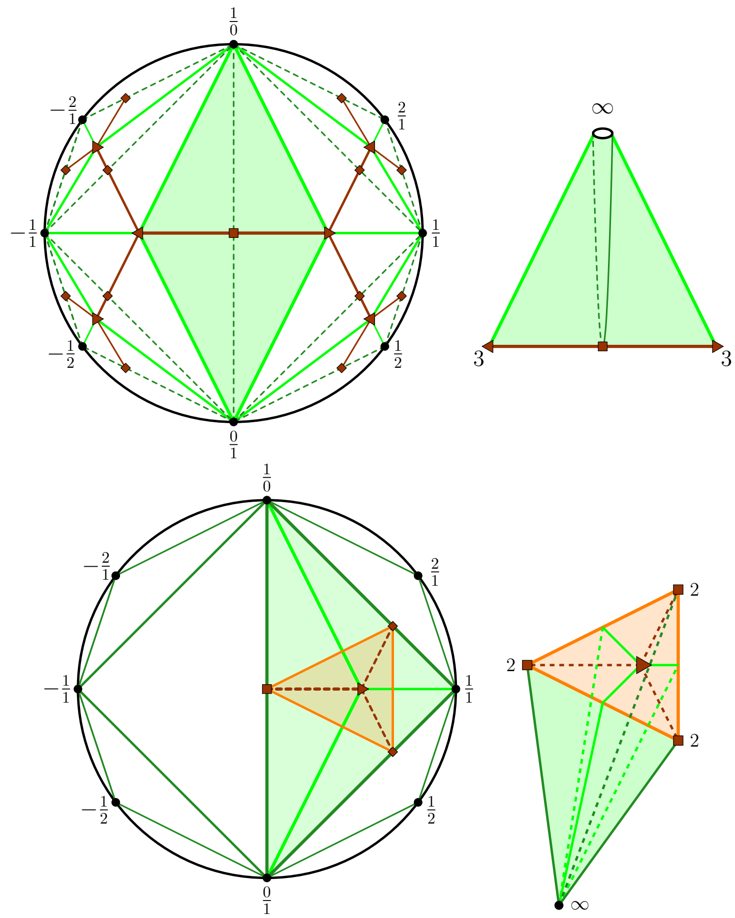


Figure 3.10: Tilings of $\mathbb{HP}$ under the kernels of $\Gamma \to \mathbb{Z}/2$ and $\Gamma \to \mathbb{Z}/3$. The quotient orbifolds are the only two covers of $\mathbb{M}$ with conical singularities.

Proof. The abelianisation functor from the category of groups to the category of abelian groups (sending a group to its largest abelian quotient) is defined by a universal initial property, so it maps the sum in the category of groups (free amalgam) to the sum in the category of abelian groups (equal to the product).

The derived subgroup of Γ, defined as the kernel Γ' of its abelianisation map, is generated by all its commutators $[A, B] = ABA^{-1}B^{-1}$ for $A, B \in \Gamma$.

The elements $RL = [T, S^{-1}]$ and $LR = [T^{-1}, S]$ generate a subgroup of Γ' acting on $\triangle_2$ with fundamental domain a quadrilateral obtained as the union of the two adjacent triangles $(\infty, 0, 1)$ and $(0, \infty, -1)$.

In the first barycentric subdivision $\triangle_2'$, this quadrilateral is covered by 6 copies

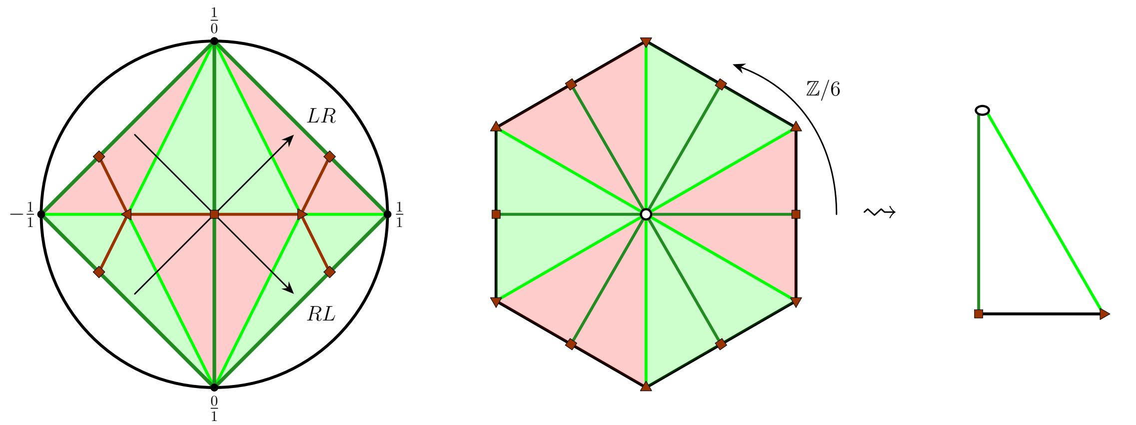


Figure 3.11: The group Γ' acts on (the inside of) $\triangle_2$ with quotient a (punctured) torus. The Galois action of $\Gamma/\Gamma' = \mathbb{Z}/6$ on the torus covering the modular orbifold.

of a fundamental domain for Γ, so the group generated by RL and LR has index 6 in Γ, and therefore equals Γ'.

As RL and LR match the opposite edges of this quadrilateral, the quotient $\mathbb{HP}/\Gamma'$ is a punctured torus $\mathbb{T}^*$ whose fundamental group is free on those two generators. □

Remark 3.27. *The derived subgroup Γ' is the intersection of the kernels of the maps $\Gamma \to \mathbb{Z}/2$ and $\Gamma \to \mathbb{Z}/3$. These count the (signed) number of occurrences of the letters S and T modulo 2 and 3 respectively.*

Remark 3.28. *The vertices $\mathbb{P}(\mathbb{Z}^2)$ of $\triangle_2$ correspond to the horocycles $\Gamma/\langle R\rangle$ of $\mathcal{T}$. They can also be visualised in $\mathbb{HP}$ as the components of the complement $\mathbb{HP} \setminus \mathcal{T}$. What we just saw implies that Γ' acts transitively on $\mathbb{P}(\mathbb{Z}^2)$, and the unique orbit corresponds to the puncture of the torus, so Γ' acts on $\triangle_2$ with quotient $\mathbb{T}$.*

The second derived subgroup and the hexagonal graph

Finally we study the Galois cover of $\mathbb{M}$ corresponding to the second derived subgroup $\Gamma'' = [\Gamma', \Gamma']$ of Γ, or the metabelianisation morphism $\Gamma \to \Gamma/\Gamma''$.

We have already covered $\mathbb{M}$ by $\mathbb{T}^*$ using the abelianisation morphism $\Gamma \to \Gamma/\Gamma'$, and we now investigate the Galois cover of $\mathbb{T}^*$ corresponding to the abelianisation morphism $\Gamma' \to \Gamma'/\Gamma''$ (whose compactification yields the universal cover of $\mathbb{T}$).

Proposition 3.29. *The abelianitation $\Gamma' \to \Gamma'/\Gamma''$ corresponds to the Hurwicz map $\pi_1(\mathbb{T}^*) \to H_1(\mathbb{T}^*; \mathbb{Z})$ of the punctured torus, and is thus given by $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$.*

It corresponds to a Galois covering of the punctured torus $\mathbb{T}^*$ by a punctured plane along a lattice $\mathbb{R}^2 \setminus \mathbb{Z}^2$, identified (by the Jacobian map) with $H_1(\mathbb{T}^*;\mathbb{R}) \setminus H_1(\mathbb{T}^*;\mathbb{Z})$.

Its kernel $\Gamma'' = \pi_1(\mathbb{R}^2 \setminus \mathbb{Z}^2)$ is freely generated by an infinite set of elements (whose conjugacy classes in Γ' are) indexed by $H_1(\mathbb{T}^*;\mathbb{Z}) = \Gamma'/\Gamma''$. For example, the conjugates of $[RL, LR]$ by all $(RL)^m(LR)^n \in \Gamma'$ with $m, n \in \mathbb{Z}^2$ freely generate Γ''.

Remark 3.30. *The vertices $\mathbb{P}(\mathbb{Z}^2)$ of $\triangle_2$ parametrize the horocycles $\Gamma/\langle R\rangle$ of $\mathcal{T}$. They can also be visualised in $\mathbb{HP}$ as the components of the complement $\mathbb{HP} \setminus \mathcal{T}$.*

Proposition 3.29 implies that the group Γ'' acts on $\mathbb{P}(\mathbb{Z}^2)$ with quotient a lattice $\mathbb{Z}^2$ corresponding to the punctures in $\mathbb{R}^2 \setminus \mathbb{Z}^2$. In particular it acts on $\triangle_2$ with quotient the universal cover $\mathbb{R}^2$ of $\mathbb{T}$.

We now deduce the Galois action of Γ/Γ'' on $\mathbb{R}^2 \setminus \mathbb{Z}^2 \to \mathbb{M}$ by combining the action of Γ/Γ' on $\mathbb{R}^2 \setminus \mathbb{Z}^2 \to \mathbb{T}^*$ with the action of Γ/Γ' on $\mathbb{T}^* \to \mathbb{M}$.

The group Γ'' acts on the trivalent tree $\mathcal{T} \subset \triangle_2 \setminus \mathbb{QP}^1$ with quotient a trivalent graph $\mathcal{H} \subset \mathbb{R}^2 \setminus \mathbb{Z}^2$. The latter inclusion a homotopy equivalence, and the regions in the complement $(\mathbb{R}^2 \setminus \mathbb{Z}^2) \setminus \mathcal{H}$ are punctured hexagons (for the hyperbolic metric).

Corollary 3.31. *The group Γ/Γ'' acts freely transitively on the oriented edges of $\mathcal{H}$, or equivalently on the edges of its first barycentric subdivision $\mathcal{H}'$. The subgroup Γ'/Γ'' acts by translation of $\mathcal{H} \subset \mathbb{R}^2 \setminus \mathbb{Z}^2$ with fundamental domain a tripod formed by 3 incident edges of $\mathcal{H}$ as in figure 3.12.*

This represents Γ/Γ'' as the affine isometry group of a hexagonal lattice in the oriented euclidean plane. More precisely $\Gamma/\Gamma'' = \Gamma'/\Gamma'' \rtimes \Gamma'/\Gamma$ where the action of $\Gamma/\Gamma' = \mathbb{Z}/2 \times \mathbb{Z}/3$ on $\Gamma'/\Gamma'' = \mathbb{Z}^2$ is generated by the rotations S and T of order 2 and 3 around the 2-valent and 3-valent vertices of $\mathcal{H}$ as in figure 3.12.

Remark 3.32. *Notice that Γ'/Γ'' identifies canonically with the edges of $\mathcal{H}$, which are parallel to the base edge. On the other hand the set $\mathbb{QP}^1/\Gamma''$ identifies canonically with the regions of $\mathbb{R}^2 \setminus \mathcal{H}$.*

The group Γ'/Γ'' acts freely transitively on the set $\mathbb{QP}^1/\Gamma''$, and one may fix a bijection sending an edge of $\mathcal{H}$ to the region of $\mathbb{R}^2 \setminus \mathcal{H}$ placed immediately on its right (this is not quite canonical as one may have chosen the region above or under).

Corollary 3.33. *The set $\mathbb{QP}^1/\operatorname{PSL}_2(\mathbb{Z})''$ is a free module of rank one over the ring of integers $\mathbb{Z}[j]$.*

The addition of two (classes of) rational numbers is obtained by concatenating their even continued fraction expansions.

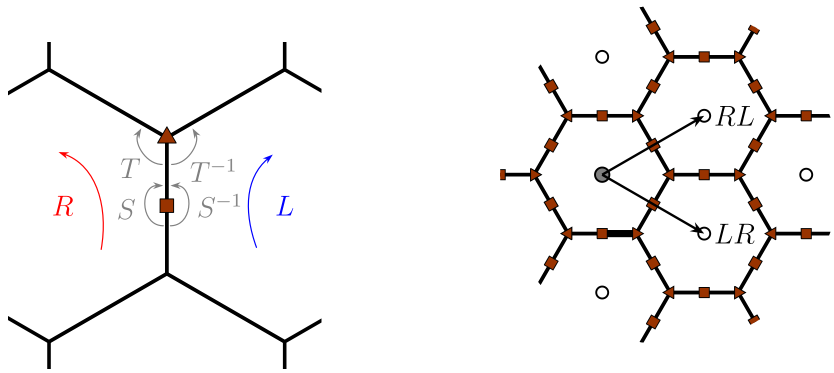


Figure 3.12: Action of $\Gamma/\Gamma'' = \Gamma'/\Gamma'' \rtimes \Gamma/\Gamma'$ on the hexagonal graph $\mathcal{H}$.

The metabelian cover $\mathbb{HP}/\Gamma''$ is naturally endowed with the hyperbolic metric, but also with the unique euclidean metric which is invariant by the automorphism action of Γ/Γ'', for which the hexagonal lattice $\mathcal{H} \subset \mathbb{R}^2 \setminus \mathbb{Z}^2$ is regular.

Question 3.34. *The discovery of this hexagonal lattice structure on $\mathbb{QP}^1/\Gamma''$ with automorphism group Γ/Γ'' is very intriguing.*

In particular, many questions arise concerning the group structure of $\mathbb{QP}^1/\Gamma'$, namely the arithmetics underlying the addition of such classes of fractions...

For instance can we complete this addition to Γ''-classes of quadratic irrationals, or to $\mathbb{RP}^1/\Gamma''$? How to interpret alignment in Γ'/Γ'' in terms of $\mathbb{QP}^1/\Gamma''$?

Lifting geodesics in Galois covers

Consider first a finite Galois covering $\mathbb{M}_0 \to \mathbb{M}$ whose total space is non singular, and denote $\Gamma_0 \to \Gamma$ the corresponding finite index normal subgroup which is free.

The action of Γ by inner automorphisms restricts to an action by automorphisms on Γ_0, which corresponds to projecting a loop $\gamma_0 \in \pi_1(\mathbb{M}_0)$ into $\pi_1(\mathbb{M})$, conjugating it by $\gamma \in \pi_1(\mathbb{M})$ and lifting it back into $\pi_1(\mathbb{M}_0)$. On the one hand this restricts to the inner automorphism action of Γ_0, which corresponds to conjugating by those loops in $\pi_1(\mathbb{M})$ which lift to loops in $\pi_1(\mathbb{M}_0)$; on the other hand it quotients to the action of Γ/Γ_0 by outer automorphism on Γ_0, that is by monodromy on the fiber of $\mathbb{M}_0$ above the base point of $\mathbb{M}$, or by Galois transformations of $\mathbb{M}_0 \to \mathbb{M}$.

For $\gamma \in \Gamma$ and $n \in \mathbb{Z}$, the conjugacy class of γ^n is either disjoint or contained in the normal subgroup Γ_0, and the latter happens when n belongs to the ideal $m\mathbb{Z}$ generated by the order of $\gamma \bmod \Gamma_0$. Topologically, the loop $\gamma^n \in \pi_1(\mathbb{M})$ lifts to an arc which is a loop in $\pi_1(\mathbb{M}_0)$ when $n \in m\mathbb{Z}$. The Γ-conjugacy class γ^m splits modulo Γ_0-conjugacy into the disjoint union of k classes $\alpha_1, \ldots, \alpha_k$. The conjugacy action of Γ/Γ_0 on the set $\{\alpha_1, \ldots, \alpha_k\}$ is transitive, and the stabiliser of α_1 is the cyclic subgroup of order m generated by $\gamma \bmod \Gamma_0$.

To recast this dicussion geometrically, let $\gamma \in \Gamma$ be a hyperbolic conjugacy class. A closed geodesic γ of length l in the base $\mathbb{M}$ has preimage in $\mathbb{M}_0$ a finite union of closed geodesics $\alpha_1 \cup \cdots \cup \alpha_k$ which all have the same length ml with $mk = \operatorname{Card} \Gamma/\Gamma_0$. Indeed, the Galois group Γ/Γ_0 acts transitively on the α_j and the stabiliser of each α_j is a cyclic group of order m. The integer m is minimal such that γ^m lifts to a closed loop in $\mathbb{M}_0$, and together with the degree $\operatorname{Card} \Pi$ of the covering, it determines the number of components k in the preimage.

Remark 3.35. *Most of this discussion can be adapted to infinite Galois covers, but now the preimages of loops in $\mathbb{M}$ may by infinite arcs in $\mathbb{M}_0$, and one must be careful to consider the right notions of homotopy adapted to the context (either equivariant with respect to the cover, with compact support, or relative to the ends).*

Recall from Section 3.1 that a hyperbolic conjugacy class in $\pi_1(\mathbb{M})$ encodes the homotopy class of two loops in the orbifold $\mathbb{M}$, namely its hyperbolic representative and its linear representative. More generally, a finite collection of hyperbolic conjugacy classes in $\pi_1(\mathbb{M})$ has hyperbolic and linear representatives, which are multiloops in $\mathbb{M}$ whose isotopy classes are well defined.

Recall from Example 3.13 that the (perturbed) hyperbolic and linear representatives may have distinct homotopy classes in $\mathbb{M} \setminus \{i, j\}$. However, they are homotopic in the orbifold $\mathbb{M}$, and so are their preimages in any Galois cover $\mathbb{M}_0$. If we assume that the Galois cover has no orbifold singularities, then one can say more.

Proposition 3.36. *Consider a normal subgroup $\Gamma_0 \subset \Gamma$ of finite index which is torsion free, and let $\mathbb{M}_0 \to \mathbb{M}$ be the corresponding finite Galois cover.*

Consider the preimage in $\mathbb{M}_0$ of the hyperbolic and linear representatives associated to a finite collection of hyperbolic conjugacy classes in Γ.

Both of these multiloops are taut, and they are connected by a sequence of isotopies and Reidemeister triangle moves RIII.

Proof. First we lift the hyperbolic and flat metrics from $\mathbb{M}$ to $\mathbb{M}_0$. The preimages of the hyperbolic and linear representatives are collections of geodesics for those lifted metrics. We deduce, using [FHS82] as in Propositions 3.15 and 3.21, that these multiloops be taut. Hass and Scott [HS95] proved that in a smooth surface with negative euler characteristic, all minimally intersecting multiloops in a same homotopy class are connected by a sequence of isotopies and triangle moves. □

Remark 3.37. *Since a sub-multiloop of a taut multiloop is taut, the previous result applies equally well to sub-multiloops of the preimage in $\mathbb{M}_0$ of the hyperbolic and linear representatives, labelled by the same subsets of conjugacy classes in Γ_0.*

Scholium 3.38. *We may apply this proposition to the case of the finite Galois cover $\mathbb{T}^* \to \mathbb{M}$ associated to the derived subgroup $[\Gamma, \Gamma] \subset \Gamma$.*

We will do this in Section 4.1 to deduce that the hyperbolic and linear representatives have isotopic lifts in the unit tangent bundle of $\mathbb{M}$.

Question 3.39. *We may now ask: which loops in $\mathbb{M}$ pull-back in $\mathbb{M}_0$ to simple multiloops, that is with disjoint and simple components? More generally, what are the intersection numbers between these components?*

Lifting combinatorial paths and geodesics

Let us define edge-paths and local geodesics in $\mathcal{T}_0' = \mathcal{T}'/\Gamma_0$ as we did in $\mathcal{T}'$ between statements 2.8 and 2.11. An *edge-path* in $\mathcal{T}_0'$ is a sequence of edges indexed by an interval of $\mathbb{Z}$ whose length may be finite or infinite, such that two successive elements share an extremity. This yields a sequence of S and $T^{\pm 1}$ indicating how to turn the edges around their bivalent and trivalent vertices to get from one to the next (which is unique if $\Gamma_0 \neq \Gamma$). An edge-path is *locally geodesic* when this sequence of S and $T^{\pm 1}$ alternates between both letters (so we are never turning around vertices).

The projection $\mathcal{T}' \to \mathcal{T}_0'$ between cyclically oriented based graphs sends local geodesics of $\mathcal{T}'$ to local geodesics in $\mathcal{T}_0'$. Conversely, if Γ_0 is torsion free, then every local geodesic in $\mathcal{T}_0'$ lifts to a unique local geodesic in $\mathcal{T}'$ once we have lifted one edge. Note that in $\mathcal{T}_0'$ the locally geodesic edge-paths may not be injective, and that two edges may be connected by several global geodesics.

Lifting geodesics in the hexagonal graph

We now specialise to the discussion of the previous paragraph to case $\Gamma_0 = \Gamma''$ so that $\mathcal{T}'_0 = \mathcal{H}'$. (The primes denoting the derived subgroups and the primes denoting the first barycentric subdivisions are unrelated, but no confusion should arise.) The projection $\mathcal{T}' \to \mathcal{H}'$ between cyclically oriented based graphs sends local geodesics of $\mathcal{T}'$ to local geodesics in $\mathcal{H}'$. Conversely, every local geodesic in $\mathcal{H}'$ lifts to a unique local geodesic in $\mathcal{T}'$ once we have lifted one edge.

In particular, an element of $A \in \Gamma$ corresponds to a finite local geodesic segment starting from the base edge of $\mathcal{H}'$. Notice how the last edge $e_A \bmod \Gamma''$ encodes the metabelianisation $A \bmod \Gamma''$ as an automorphism of $\mathcal{H}'$. In particular we may read the abelianisation $A \bmod \Gamma'$ by comparing the angle made between the last edge and the first edge which lies in $2\pi\mathbb{Z}/6$, and if $A \in \Gamma'$ we may read the metabelianisation $A \bmod \Gamma''$ from the difference between the last and first edge which is a vector in $\mathbb{Z}^2$.

Moreover, the translation axis in $\mathcal{T}'$ of a primitive hyperbolic $A \in \Gamma$ projects to a local geodesic $g_A \bmod \Gamma''$ in $\mathcal{H}'$ which uniquely determines A. Indeed this amounts to choosing a local geodesic from the base edge to that local geodesic, and extracting from there its minimal period (being careful that the first and last edge of the period may not be aligned, this depends on the abelianisation of A in $\mathbb{Z}/6$).

Conversely, two primitive hyperbolic elements are conjugate in Γ if and only if the projections of their translation axes in $\mathcal{H}'$ belong to the same orbit under the automorphism action of Γ/Γ''. We may thus define invariants of a primitive hyperbolic conjugacy class $A \in \Gamma$ using the shape of the periodic (bi-infinite or closed) local geodesic $g_A \bmod \Gamma''$ in $\mathcal{H}'$. Of course, we know how to extract the period of $g_A \bmod \Gamma''$ which encodes everything (such as the conjugacy class of $A \bmod \Gamma''$), but let us show how to read off simpler invariants from its geometry.

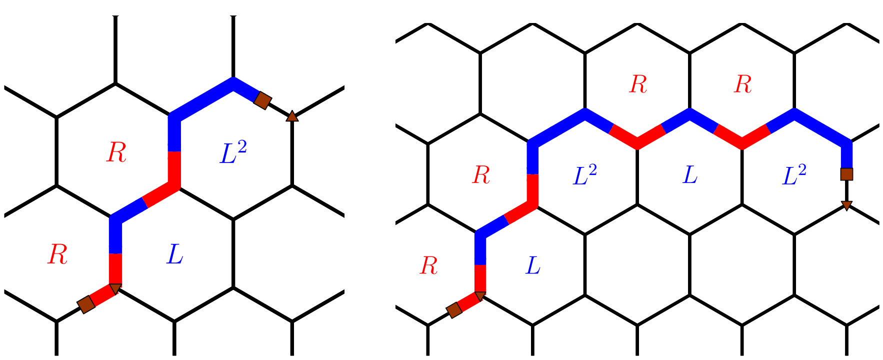


Figure 3.13: Paths encoded by $RLRLL$ & $(RLRLL)^2$ in the hexagonal graph $\mathcal{H}$.

Proposition 3.40. *For a hyperbolic $A \in \Gamma$, the hyperbolic geodesic $\gamma_A \bmod \Gamma''$ in $\mathbb{HP}/\Gamma''$ and the combinatorial geodesic $g_A \bmod \Gamma''$ in $\mathcal{T}/\Gamma''$ avoid the vertices $\mathbb{P}(\mathbb{Z}^2)/\Gamma''$ and intersect the same sequence of triangles in $\triangle_2/\Gamma''$; in particular they are homotopic in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ relative to the complement of large euclidean balls.*

Proof. The proof follows from Proposition 2.28 and Corollary 2.31 which identify the sequences of triangles in $\triangle_2$ intersected by γ_A and g_A. □

Remark 3.41. *In chapter 5, we shall deform the inclusion $\mathrm{PSL}_2(\mathbb{Z}) \subset \mathrm{PSL}_2(\mathbb{R})$ to a one parameter family of discrete and faithful representations $\rho_q \colon \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ such that the quotient $\rho_q(\Gamma)\backslash\mathbb{HP}$ corresponds to opening the cusp of $\mathbb{M}$.*

In $\mathbb{HP}/\rho_q(\Gamma'')$ this yields a homotopy from the hyperbolic geodesic $\gamma_A \bmod \Gamma''$ when $q = 0$ to the combinatorial geodesic $g_A \bmod \Gamma''$ when $q \to \infty$.

Until the end of this subsection, for $A \in \Gamma$, let a be its order in Γ/Γ', and $A' = A^a$ its smallest power which belongs to Γ'. The (geometric or combinatorial) translation axes of A and A' coincide in $\mathcal{T}$, hence so do their projections in $\mathcal{H}$.

Note that A' acts by translation on $\mathcal{H} \subset \mathbb{HP}/\Gamma''$ and preserves the projected axes $g_A \bmod \Gamma''$ or $\gamma_A \bmod \Gamma''$. (When $A' \in \Gamma''$ the translation is trivial and these projected axes are closed.) A fundamental domain for these projected axes under A' consists in na periods where na is the primitivity exponent of $A' \in \Gamma$, thus n is the primitivity exponent of $A \in \Gamma$.

Length and distance. For an infinite order primitive $A \in \Gamma$, let $\mathrm{len}(A) \in \mathbb{N}^*$ be the period length of g_A in $\mathcal{T}$, and set $\mathrm{len}(A^n) = n\,\mathrm{len}(A)$. If $A \in \mathrm{PSL}_2(\mathbb{N})$ then $\mathrm{len}(A) = \#L + \#R$ counts the sum of the numbers of $L\&R$ in its factorisation. Since len is invariant by conjugacy we can always compute on those Lyndon representatives. For any $A \in \Gamma$ with infinite order, $\mathrm{len}(A)$ is read off a fundamental domain for $g_A \bmod \Gamma''$ under the translation action of $A' \bmod \Gamma''$ as half the length of its $S\&T^{\pm 1}$ factorisation divided by the order a of $A \in \Gamma/\Gamma'$.

Recall that if λ_A denotes the hyperbolic translation length of A, that is the hyperbolic length of a fundamental domain for $\gamma_A \bmod A$ then $\mathrm{Tr}(A)/2 = \cosh \lambda_A/2$ where λ_A is the hyperbolic length. One may ask for the relationship between the integers $l = \mathrm{len}(A)$ and $t = \mathrm{Tr}(A)$. In terms of bounds we have $l + 1 \leq t \leq Fib(l)$ where the left term is attained for $A = RL^l$ and the right one is attained for $A = (RL)^l$, and we expect the normalised statistics of Tr given len to approach a Gaussian. In terms of deformations of representations $\rho_q \colon \Gamma \to \mathrm{PSL}_2(\mathbb{R})$, we shall see that the hyperbolic length λ_{A_q} converges to the combinatorial length $\mathrm{len}(A)$ as $q \to \infty$.

Another invariant of the conjugacy class of A mod Γ'' is the euclidean distance in $\triangle_2/\Gamma''$ between the cusps indexed by $\imath(1)$ and $A(1)$. This is easy to read off the geometry of g_A mod Γ'' provided we know $\operatorname{len}(A)$.

We may also define the breadth of A as the length of the smallest $B \in \Gamma$ such that g_A mod Γ'' and $g_{BAB^{-1}}$ mod Γ'' are disjoint. Alternatively we may consider the norm of the smallest translation $B \in \Gamma'/\Gamma''$ such that g_A mod Γ'' and $B \cdot (g_A \bmod \Gamma'')$ are disjoint.

Rademacher cocycle and the index function. For an infinite order $A \in \Gamma$, consider a fundamental domain for g_A mod Γ'' under the translation action of A', and let $a\,\operatorname{Rad}(A) \in \mathbb{Z}$ be its winding number counted in multiples of $2\pi/6$. Of course $\operatorname{Rad}(A)$ is invariant by conjugacy and $\operatorname{Rad}(A^n) = n\operatorname{Rad}(A)$. We may also define $\operatorname{Rad}(S^{\pm 1}) = \pm 3$ and $\operatorname{Rad}(T^{\pm 1}) = \pm 2$.

Note that if $A \in \mathrm{PSL}_2(\mathbb{N})$ then $\operatorname{Rad}(A) = \#R - \#L$ counts the difference between the numbers of $L\&R$ in its factorisation, so the restriction $\operatorname{Rad}\colon \mathrm{PSL}_2(\mathbb{N}) \to \mathbb{Z}$ is a morphism of monoids.

Now for a infinite order element $A \in \Gamma$ we define its asymptotic index function

$$\operatorname{Ind}_A \colon \mathbb{QP}^1/\Gamma'' \to \mathbb{Z}$$

which to a cusp of $\mathbb{HP}/\Gamma''$ associates the index of the hyperbolic geodesic γ_A mod Γ'', or equivalently of the combinatorial geodesic g_A mod Γ''. If $A \in \Gamma''$ then the projected geodesics are closed, and we must travel around them once.

The index function of A^n is equal to that of A, so we might as well suppose A primitive. The index function of a conjugacy class is well defined only modulo the action of Γ/Γ'' at the source, because $\operatorname{Ind}_{CAC^{-1}} = \operatorname{Ind}_A \circ C$.

This defines a "frieze pattern" with integers decorating the points of the hexagonal lattice which is periodic under the action of A' mod Γ''. Then $\operatorname{Rad}(A')$ is equal to the eulerian integral of Ind_A along a period.

Chapter 4

Modular knots

Outline of the chapter

Let us recall a few facts (contained in Proposition 1.82) concerning the action of $\mathrm{PSL}_2(\mathbb{R})$ on the hyperbolic plane $\mathbb{HP}$, its unit tangent bundle, and its boundary $\mathbb{RP}^1$. The group $\mathrm{PSL}_2(\mathbb{R})$ acts freely transitively on the unit tangent bundle of $\mathbb{HP}$, with which it therefore identifies once we have chosen a base point such as the unit vector based at the fixed point i of S and pointing towards the fixed point j of T.

As $\mathrm{PSL}_2(\mathbb{R})$ is the unit tangent bundle to the hyperbolic plane $\mathrm{PSL}_2(\mathbb{R})/\,\mathrm{PSO}_2(\mathbb{R})$, the quotient $\mathbb{U} := \mathrm{PSL}_2(\mathbb{Z})\backslash\,\mathrm{PSL}_2(\mathbb{R})$ is the unit tangent bundle to the modular orbifold $\mathbb{M} = \mathrm{PSL}_2(\mathbb{Z})\backslash\,\mathrm{PSL}_2(\mathbb{R})/\,\mathrm{PSO}_2(\mathbb{R})$. The left action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathrm{PSL}_2(\mathbb{R})$ is free, so the quotient $\mathbb{U}$ is a 3-manifold. It is filled by the circular orbits for the right action of $\mathrm{PSO}_2(\mathbb{R})$, and the space of orbits is the two-dimensional orbifold $\mathbb{M}$.

The primitive closed hyperbolic geodesics in $\mathbb{M}$, also known as modular geodesics, lift in $\mathbb{U}$ to the primitive periodic orbits for the geodesic flow, called modular knots. Let us recall from Section 2.2 that these correspond to the primitive hyperbolic conjugacy classes in the modular group. Such a conjugacy class intersects the monoid $\mathrm{PSL}_2(\mathbb{N})$ freely generated by $L\&R$ along its Lyndon representatives: those are the cyclic permutations of a primitive $L\&R$-word in which both letters occur.

This chapter is divided in two sections. The first describes the topology of the Seifert fibration $\mathbb{U} \to \mathbb{M}$ and the homotopy classes of modular knots and does not contain many new results. The second expresses the linking numbers between modular knots by various formulae with combinatorial, dynamical or group theoretical flavour.

The unit tangent bundle of the modular orbifold

The first section opens with a short discussion concerning orientation matters (which one may skip as it mostly serves to disentangle the canonical choices and conventions on which depend some chirality issues encountered along the way).

To describe the topology of the Seifert fibration $\mathbb{U} \to \mathbb{M}$ with generic fibers $\mathbb{S}^1$, we first study the corresponding exact sequence of fundamental groups, which is a central extension $1 \to \pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{U}) \to \pi_1(\mathbb{M}) \to 1$. Here we find it instructive to treat it as a special case of unit tangent bundle to a Fuchsian orbifold (the quotient of $\mathbb{HP}$ by a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$).

Then we describe the Seifert parameters of the fibration, following [Sco83] and [Mon87]. We recover the fact that $\mathbb{U}$ is homeomorphic to the complement of a trefoil knot in the sphere. This had been shown in [Mil71, ğ10] and more precisely concerning compactification matters in [PH79]. We also refer to [Pin14] and [Klo16]. Its chirality (which seems to have never been considered), depends on the orientation matters discussed at the beginning. In any case, this homeomorphism yields an isomorphism $\pi_1(\mathbb{U}) \simeq \mathcal{B}_3$ with the braid group on 3 strands.

Recall from Chapter 3 that a collection of hyperbolic conjugacy classes in $\pi_1(\mathbb{M})$ has a hyperbolic representative and a linear representative, which are two immersed multiloops with no self-tangencies in $\mathbb{M}$. We claim in Theorem 4.13 that their tangent lifts in $\mathbb{U}$ are isotopic links and provide an incomplete proof relying on Proposition 3.36. An alternative proof, referring to [Ghy07, ğ3.4], will be given in Theorem 4.24.

The homotopy classes of modular knots correspond to conjugacy classes in $\mathcal{B}_3$. To describe which ones, we first explain why a conjugacy class in $\mathcal{B}_3$ is uniquely characterised by that of its projection in $\mathrm{PSL}_2(\mathbb{Z})$ and of its abelianisation in $\mathbb{Z}$. Then we show that the modular knot associated to a hyperbolic conjugacy class in $\mathrm{PSL}_2(\mathbb{Z})$ has abelianisation given by the Rademacher invariant $\mathrm{Rad}(A) = \#R - \#L$.

The proof relies on the homotopy equivalence of modular knots and Lorenz knots implied by Theorem 4.13. Indeed, unlike the hyperbolic representatives, the linear representatives have a straightforward description in terms of the $L\&R$-cycle encoding the $\mathrm{PSL}_2(\mathbb{Z})$-conjugacy class. The tangent lift of linear representatives as Lorenz knots can be realised algebraically as a partial section of the projection $\mathcal{B}_3 \to \mathrm{PSL}_2(\mathbb{Z})$, which is a morphism of monoids $\mathrm{PSL}_2(\mathbb{N}) \to \mathcal{B}_3$ sending $L\&R$ to $\sigma_1^{-1}\&\sigma_2$ where σ_j are the Artin generators of $\mathcal{B}_3$.

Since the abelianisation of $\pi_1(\mathbb{U})$ is given by the linking number with the trefoil, we recover the fact that the Rademacher invariant of a hyperbolic conjugacy class is the linking number with the trefoil knot. This was discovered by É. Ghys in [Ghy07] where he gave three proofs relying on various other interpretations of the Rademacher invariant.

Finally we describe a fibration $\mathbb{U} \to \mathbb{S}^1$, whose pages are punctured tori $\mathbb{T}^*$ transverse to the fibers $\mathbb{S}^1$ of the Seifert fibration $\mathbb{U} \to \mathbb{M}$. We take an algebraic standpoint, examining the corresponding diagrams between fundamental groups. This investigation could be generalised to unit tangent bundles of Fuchsian orbifolds with one cusp (their fundamental groups are free amalgams of cyclic groups), but we have not pushed the study in that direction. We refer to [Ghy17, Chapter on The cusp and the trefoil] and [Mil68] for a geometric point of view on those structures.

Linking numbers of modular knots

In this section we provide several formulae for computing the linking numbers of modular knots.

We first recall [Ghy07, ğ3.4] which isotopes the master modular braid consisting of all modular knots into a branched surface $\mathbb{Y} \subset \mathbb{U}$ called the Lorenz template. This yields a diagram for modular links, in which all crossings are positive. We deduce an algorithmic formula (Proposition 4.27) for computing linking numbers between modular knots, which was used by P. Dehornoy in [Deh11]. This is a finite sum on the set of all pairs of Lyndon representatives for the hyperbolic conjugacy classes.

As an excursion, we show a variation on this formula (in Proposition 4.34) involving an infinite sum, which amounts to reordering the crossings. The idea consists in cutting the Lorenz template to identify the regions containing the crossings between modular knots with the occurrence of certain patterns in their corresponding $L\&R$-cycles. We reformulate this as a factorisation of the infinite square matrix representing the linking form into a product of two infinite rectangular matrices counting the occurrences of patterns in $L\&R$-words. This opens a door onto the Hilbertian and statistical analysis of the linking form, although we do not pursue those directions here.

The linking pairing is a function defined on pairs of conjugacy classes in the modular group: we wish to express it in terms of the intrinsic algebra of $\mathrm{PSL}_2(\mathbb{Z})$. This is why we introduced the general formalism developed at the end of Section 2.3 to construct functions of pairs of conjugacy classes by averaging conjugacy invariants of pairs of elements. Indeed, we are able to recast the algorithmic formula in this formalism, using Lemma 2.43 which says when two matrices $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ can be simultaneously conjugated into $\mathrm{PSL}_2(\mathbb{N})$, and the quantities cross & cosign which describe the relative position of the combinatorial axes $g_A, g_B \subset \mathcal{T}$ as in Figure 2.11. Thus we arrive to the main Theorem 4.44 in this section, which expresses the linking number $\mathrm{lk}(A, B)$ between primitive hyperbolic matrices as the sum over double cosets $\langle A\rangle\backslash\Gamma/\langle B\rangle$ of $(1 + \mathrm{cross})(1 + \mathrm{cosign})$.

4.1 Unit tangent bundle of the modular orbifold

As $\mathrm{PSL}_2(\mathbb{R})$ is the unit tangent bundle to the hyperbolic plane $\mathrm{PSL}_2(\mathbb{R})/\mathrm{PSO}_2(\mathbb{R})$, the quotient $\mathbb{U} = \mathrm{PSL}_2(\mathbb{Z})\backslash \mathrm{PSL}_2(\mathbb{R})$ is the unit tangent bundle to the modular orbifold $\mathbb{M} = \mathrm{PSL}_2(\mathbb{Z})\backslash \mathrm{PSL}_2(\mathbb{R})/\mathrm{PSO}_2(\mathbb{R})$.

We can also write $\mathbb{HP} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ and $\mathbb{U} = \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})$ because the center $\{\pm 1\} \subset \mathrm{SL}_2(\mathbb{R})$ belongs both to $\mathrm{SO}_2(\mathbb{R})$ and to $\mathrm{SL}_2(\mathbb{Z})$, each of which implies that $\mathbb{M} = \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$.

Since the left action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathrm{PSL}_2(\mathbb{R})$ is free, the left quotient $\mathbb{U}$ is a three-manifold. It is filled by the circular orbits for the right action of $\mathrm{PSO}_2(\mathbb{R})$, and the space of orbits is the two-dimensional orbifold $\mathbb{M}$.

In this section we intend to describe the topology of this fibration $\mathbb{U} \to \mathbb{M}$, introduce a transverse fibration $\mathbb{T}^* \to \mathbb{U} \to \mathbb{S}^1$, and understand the position of their fibers with respect to periodic orbits for the geodesic flow in $\mathbb{U}$.

Orientation of $\mathbb{U}$

It is equivalent to orient $\mathrm{PSL}_2(\mathbb{R})$, its tangent space $\mathfrak{sl}_2(\mathbb{R})$ at the identity, or its quotient $\mathbb{U}$ under the left action of its lattice $\mathrm{PSL}_2(\mathbb{Z})$.

Note that the left action is isomorphic to the right action by the inversion map, which reverses the orientation of $\mathrm{PSL}_2(\mathbb{R})$, because its differential acts like minus the identity on the tangent space $\mathfrak{sl}_2(\mathbb{R})$ at $\mathbb{1} \in \mathrm{PSL}_2(\mathbb{R})$, whose dimension is odd. We have chosen to let $\mathrm{PSL}_2(\mathbb{Z})$ act on the left to stay coherent with the left action on the boundary of the hyperbolic plane and on continued fractions.

The real semi-simple Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is canonically oriented by the sign of a volume form associated to the non-degenerate Killing form. In Chapter 1 we worked with a metric which is $-1/8$ times the Killing form. Moreover, we defined our volume form in Corollary 1.20 so that $[K, J, S] = \langle \{K, J\}, S\rangle = \det(S) = 1$ and (K, J, S) is a positive basis, whereas it is negative for the Killing form.

The Killing convention defines a preferred orientation on $\mathrm{PSL}_2(\mathbb{R})$ whence on $\mathbb{U}$. Let us now provide a differential viewpoint on that matter.

Remark 4.1. *Let M be a differentiable manifold, and T its tangent bundle.*

A Riemannian (or Finslerian) metric on M defines a unit tubular neighbourhood of the trivial section $M \subset T$, whose boundary is the unit tangent bundle U for that metric. Hence the homeomorphism type of the unit tangent bundle does not depend on the metric. Intrinsically, one can define $U = (T \setminus M)/\sim$ as the complement of the trivial section modulo the equivalence relation $(p, v) \sim (p, \lambda v)$ for $\lambda \in \mathbb{R}_+^*$.

Now the tangent space at a point $(p, v) \in T$ *in the total space contains the tangent space to the fiber* T_pM*, which is identified with* $T_0(T_pM) = T_pM$*, and the quotient is isomorphic to the tangent space* T_pM *of the base at the projected point. Hence the canonical short exact sequence of vector spaces* $0 \to T_pM \to T_{(p,v)}TM \to T_pM \to 0$.

Consequently, T *is globally oriented by concatenating the orientations given by the base followed by the fiber, independently on the initial choice for the orientation of* M*. The orientation of* U*, thought as the boundary of a tubular neighbourhood of* $M \subset T$*, is chosen so that when preceded by the outwards normal vectors, one recovers the orientation of the total space* T*. This convention is employed to enhance the signs in the formulation of Stokes theorem, which can thus be written* $\int_N d\omega = \int_{\partial N} \omega$.

If M *has odd dimension then it is important to respect the convention "base followed by fiber" to orient* T*, and if* M *has even dimension it is important to respect the convention "outwards normal first" to orient* U*.*

The upshot of the previous remark is that Stokes convention yields a preferred orientation on $\mathrm{PSL}_2(\mathbb{R})$ as unit tangent bundle of $\mathbb{HP} = \mathrm{PSL}_2(\mathbb{R}) / \mathrm{PSO}_2(\mathbb{R})$. This amounts to declaring that the right adjoint action of the one parameter subgroup $\mathrm{PSO}_2(\mathbb{R})$ generated by S acts positively on the tangent plane $T_S\mathbb{H} = S^\perp = \mathrm{Span}(J, K)$. One may recall Section 1.6, especially Propositions 1.82 and 1.87 for such discussions. As $\{K, S\} = J$ and $\{J, S\} = -K$, we recover the orientation of $\mathfrak{sl}_2(\mathbb{R})$ defined by the basis (K, J, S), which coincides with the one derived from minus the Killing form.

Remark 4.2. *If we had defined* $\mathbb{HP}$ *as the left quotient by the maximal compact subgroup, then Stokes condition would have asked for the left adjoint action of* $\mathrm{PSO}_2(\mathbb{R})$ *to act positively on the plane* $S^\perp$*, and we would have obtained the opposite orientation on* $\mathrm{PSL}_2(\mathbb{R})$*, that is the one derived from the Killing form.*

Remark 4.3. *Note the distinction between the canonical structures which enable to compare different choices, and the conventions which suggest preferred choices.*

It is the sign conventions employed in the definition of the Killing form and its associated volume which orient a real semi-simple Lie algebra.

It is the sign conventions involved in the graded commutativity of alternate products which suggest using Stokes convention to orient boundaries.

To sum up, the orientation of $\mathfrak{sl}_2(\mathbb{R})$ given by the opposite of the Killing volume matches the orientation of $\mathrm{PSL}_2(\mathbb{R})$ arising from Stokes convention when it is identified with the unit tangent bundle to the right quotient $\mathbb{HP} = \mathrm{PSL}_2(\mathbb{R}) / \mathrm{PSO}_2(\mathbb{R})$.

We chose to define $\mathbb{U}$ as the left quotient of the Lie group $\mathrm{PSL}_2(\mathbb{R})$ by the lattice $\mathrm{PSL}_2(\mathbb{Z})$, and its orientation is equivalent to the previous. This discussion about the orientation of $\mathbb{U}$ will have implications on the signs of linking numbers and the chirality of certain knots.

The fundamental group of $\mathbb{U}$

Proposition 4.4. *The group* $\mathrm{PSL}_2(\mathbb{R})$ *retracts by deformation on* $\mathrm{PSO}_2(\mathbb{R})$ *which is a maximal compact subgroup, in particular its fundamental group is* $\mathbb{Z}$*.*

Proof. $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on its symmetric space $\mathbb{HP}$ with stabiliser $\mathrm{PSO}_2(\mathbb{R})$, so we have a locally trivial fibration $\mathrm{PSL}_2(\mathbb{R}) \to \mathbb{HP}$, which is globally trivial because the base is contractible, whence the homeomorphism $\mathrm{PSL}_2(\mathbb{R}) \simeq \mathbb{HP} \times \mathrm{PSO}_2(\mathbb{R})$. □

Remark 4.5. *More generally, every connected linear Lie group* G *has a unique maximal compact subgroup* K *up to conjugacy, on which it retracts by deformation.*

Proposition 4.6. *In a connected Lie group, a discrete normal subgroup is central. Thus a cover in the category of connected Lie groups is a central extension.*

In particular the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ *yields a central extension:*

$$1 \to \mathbb{Z} \to \widetilde{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R}) \to 1.$$

Proof. Denote G the connected Lie group, and C its discrete normal subgroup. Fix $(g, c) \in G \times C$ and let us show that $c = gcg^{-1}$. For this consider a continuous path $\gamma\colon [0,1] \to G$ connecting 1 to g. Since C is normal, the path $s \mapsto \gamma(s).c.\gamma(s)^{-1}$ remains in C. Since C is discrete this continuous path is constant. □

Corollary 4.7. *Let* G *be a connected Lie group and* Γ *a discrete subgroup. The fundamental group of* $\Gamma\backslash G$ *is the preimage* $\tilde{\Gamma}$ *of* Γ *in the universal cover* $\tilde{G} \to G$*. The central extension* $C \to \tilde{G} \to G$ *pulls-back to a central extension* $C \to \tilde{\Gamma} \to \Gamma$*.*

In particular the fundamental group $\widetilde{PSL}_2(\mathbb{Z})$ *of* $\mathbb{U}$ *fits in a central extension:*

$$1 \to \mathbb{Z} \to \widetilde{\mathrm{PSL}}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}) \to 1$$

Proof. The group $\tilde{\Gamma}$ fits in the following commutative diagram of covering maps (as quotients by discrete groups), in which the first two columns are central extensions:

$$\begin{array}{ccccc}
C & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{\Gamma} & \longrightarrow & \tilde{G} & \longrightarrow & \tilde{\Gamma}\backslash\tilde{G} \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma & \longrightarrow & G & \longrightarrow & \Gamma\backslash G
\end{array}$$

The last column shows that $\tilde{\Gamma}\backslash\tilde{G} \to \Gamma\backslash G$ is a covering with trivial group, hence a homeomorphism, so $\tilde{\Gamma} = \pi_1(\tilde{\Gamma}\backslash\tilde{G}) = \pi_1(\Gamma\backslash G)$. □

Remark 4.8. *Let Γ be a discrete subgroup of a connected semi-simple Lie group G. Let K be a maximal compact subgroup of G and $S = G/K = \tilde{G}/\tilde{K}$ the symmetric space. Denote C the fundamental group $\pi_1(K) = \pi_1(G)$, which is central in $\tilde{K} \subset \tilde{G}$. Finally denote $U = \Gamma\backslash G = \tilde{\Gamma}\backslash\tilde{G}$ and $M = \Gamma\backslash G/K = \tilde{\Gamma}\backslash\tilde{G}/\tilde{K}$.*

Then we have the following diagram of quotient of spaces by group actions.

$$\begin{array}{ccccc}
C & \longrightarrow & \tilde{K} & \longrightarrow & K \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{\Gamma} & \longrightarrow & \tilde{G} & \longrightarrow & U \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma & \longrightarrow & S & \longrightarrow & M
\end{array}$$

Observe the columns: the first is a short exact sequences of groups, the second is a trivial fibration between contractible spaces, and the last is a non-trivial fibration.

The lines are all universal covers, and the first one is a short exact sequence of groups.

The Lie group $G = \mathrm{PSL}_2(\mathbb{R})$ has maximal compact subgroup $K = \mathrm{PSO}_2(\mathbb{R})$ which is homeomorphic to a circle, so they have fundamental group $C = \mathbb{Z}$.

The symmetric space is $S = \mathbb{HP}$ and the fibration $\mathrm{PSO}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R}) \to \mathbb{HP}$ corresponds to its unit tangent bundle.

The lattice $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ acts on the symmetric space $S = \mathbb{HP}$ with quotient $\mathbb{M}$. Its premiage $\widetilde{\mathrm{PSL}}_2(\mathbb{Z})$ acts on the left of $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ with quotient $U = \mathbb{U}$ the unit tangent bundle of $\mathbb{M}$.

We find the diagram of fibrations and covers:

$$\begin{array}{ccccc}
\mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{S}^1 \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{\mathrm{PSL}}_2(\mathbb{Z}) & \longrightarrow & \widetilde{\mathrm{PSL}}_2(\mathbb{R}) & \longrightarrow & \mathbb{U} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{PSL}_2(\mathbb{Z}) & \longrightarrow & \mathbb{HP} & \longrightarrow & \mathbb{M}
\end{array}$$

We recover the fibration $\mathbb{S}^1 \to \mathbb{U} \to \mathbb{M}$, which corresponds to the central extension of fundamental groups $1 \to \mathbb{Z} \to \widetilde{\mathrm{PSL}}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}) \to 1$.

Topology of the Seifert fibered space

In the Lie group $\mathrm{PSL}_2(\mathbb{R})$, the discrete subgroup $\mathrm{PSL}_2(\mathbb{Z})$ acts freely and properly discontinuously by left translation, preserving the right $\mathrm{PSO}_2(\mathbb{R})$-cosets which form the fibration $\mathbb{S}^1 \to \mathrm{PSL}_2(\mathbb{R}) \to \mathbb{HP}$.

The fibers have trivial stabilisers, except for those above (a point in the orbit of) i or j with stabilisers isomorphic to $\mathbb{Z}/2$ or $\mathbb{Z}/3$ and generated by (a conjugate of) S or T. In a saturated tubular neighbourhood of such a fiber, homeomorphic to a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$, the elements S and T acts by diagonal-rotation of $2\pi/2$ and $2\pi/3$.

Hence the circle fibration quotients to a Seifert fibration $\mathbb{U} \to \mathbb{M}$. Our goal is to describe its topology and in particular the homotopy type of the pair $(\mathbb{U}, \mathbb{S}_i \cup \mathbb{S}_j)$.

Recall that we work in the category of smooth oriented orbifolds, and that a loop is an smooth map from the oriented circle. As explained in the paragraph following Definition 4.12, a loop in the total space of a unit tangent bundle is homotopic to a unit vector field carried by a loop in the base, and even to the tangent unit vector field along a an immersed loop in the base. Two vector fields along a loop in an oriented surface differ up to homotopy by multiple of the circular fiber, called their *relative index*.

Trivialising the unit tangent bundle of $\mathbb{M} \setminus \{i, j\}$.

Let $\mathbb{D}_i, \mathbb{D}_j$ be disjoint closed disc neighbourhoods of i, j in $\mathbb{M}$. In their complement, consider two based simple loops $\mathfrak{s}, \mathfrak{t}$ which are homotopic to the oriented boundaries $-\partial\mathbb{D}_i, -\partial\mathbb{D}_j$: they freely generate the fundamental group of $\mathbb{M} \setminus (\mathbb{D}_i \cup \mathbb{D}_j)$.

Denote $\mathbb{U}_i, \mathbb{U}_j$ the solid tori of $\mathbb{U}$ consisting in the union of fibers above $\mathbb{D}_i, \mathbb{D}_j$. Since the unit tangent bundle $\mathbb{U} \setminus (\mathbb{U}_i \cup \mathbb{U}_j)$ of the surface $\mathbb{M} \setminus (\mathbb{D}_i \cup \mathbb{D}_j)$ is trivial, we may choose a section such as the horizontal vector field depicted on figure 4.1. Restricting this section to a loop in the base $\mathbb{M} \setminus (\mathbb{D}_i \cup \mathbb{D}_j)$ defines its *horizontal lift* in the total space $\mathbb{U} \setminus (\mathbb{U}_i \cup \mathbb{U}_j)$.

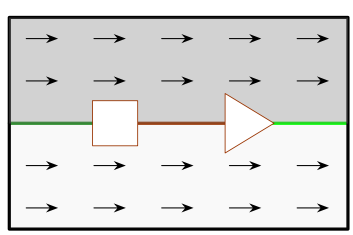


Figure 4.1: Horizontal vector field on $\mathbb{M} \setminus \{i, j\}$: section of its unit tangent bundle.

We thus have a split short exact sequence between their fundamental groups $1 \to \mathbb{Z} \to (\mathbb{Z} * \mathbb{Z}) \times \mathbb{Z} \to \mathbb{Z} * \mathbb{Z} \to 1$ whose section is defined by the horizontal lifts. In particular, we may unambiguously use the same letters $\mathfrak{s}, \mathfrak{t}$ in a presentation for the fundamental group of $\mathbb{U} \setminus (\mathbb{U}_i \cup \mathbb{U}_j)$, and denoting $\mathfrak{u}$ a based simple loop winding once around the fiber we have:

$$\pi_1(\mathbb{U} \setminus (\mathbb{U}_i \cup \mathbb{U}_j)) = \langle \mathfrak{s}, \mathfrak{t}, \mathfrak{u} \mid [\mathfrak{u}, \mathfrak{s}] = \mathfrak{1} = [\mathfrak{u}, \mathfrak{t}] \rangle.$$

The trivialisation of $\mathbb{U} \setminus (\mathbb{U}_i \cup \mathbb{U}_j) \to \mathbb{M} \setminus (\mathbb{U}_i \cup \mathbb{U}_j)$ provides a presentation for the homology of the toric boundaries $\partial\mathbb{U}_i, \partial\mathbb{U}_j$ given by restricting the section to the oriented boundaries $\partial\mathbb{U}_i, \partial\mathbb{U}_j$ followed by any fiber contained in those boundaries. Since $\mathfrak{s}, \mathfrak{t}$ are homotopic to $-\partial\mathbb{D}_i, -\partial\mathbb{D}_j$, and all fibers are homotopic to $\mathfrak{u}$ we have:

$$H_1(\partial\mathbb{U}_i; \mathbb{Z}) = \mathbb{Z} \cdot [\mathfrak{s}^{-1}] \oplus \mathbb{Z}[\mathfrak{u}] \qquad H_1(\partial\mathbb{U}_j; \mathbb{Z}) = \mathbb{Z} \cdot [\mathfrak{t}^{-1}] \oplus \mathbb{Z}[\mathfrak{u}]$$

For any smoothly immersed loop in $\mathbb{M} \setminus (\mathbb{D}_i \cup \mathbb{D}_j)$, its tangent lift in $\mathbb{U} \setminus (\mathbb{U}_i \cup \mathbb{U}_j)$ is homotopic to its horizontal lift multiplied by $\mathfrak{u}^k$, where where k is the index of its unit tangent vector with respect to the horizontal section. For instance the loops corresponding to $\mathfrak{s}$ and $\mathfrak{t}$ have tangent lifts $\mathfrak{s}\mathfrak{u}^{-1}$ and $\mathfrak{t}\mathfrak{u}^{-1}$.

Presenting $\mathbb{U}$ as a Dehn filling $\mathbb{U} \setminus (\mathbb{U}_i \cup \mathbb{U}_j)$

In $\mathbb{U}$ we denote a, b the unit tangent vector fields of the loops $\mathfrak{s}^{-1}, \mathfrak{t}^{-1}$, and c the fiber of the base point. The inclusion $\mathbb{U} \setminus (\mathbb{S}_i \cup \mathbb{S}_j) \to \mathbb{U}$ induces a surjective map:

$$\pi_1(\mathbb{U} \setminus (\mathbb{U}_i \cup \mathbb{U}_j)) \to \pi_1(\mathbb{U}) \qquad \text{defined by} \qquad \mathfrak{s}^{-1}\mathfrak{u} \mapsto a \quad \mathfrak{t}^{-1}\mathfrak{u} \mapsto b \quad \mathfrak{u} \mapsto c$$

Notice that a^2, b^3 are homotopic to a regular fiber c in $\mathbb{U}$ since they lift in the universal cover (which is the unit tangent bundle of $\mathbb{HP}$) to closed loops homotopic to a fiber. The relations $a^2 = c$ and $b^3 = c$ imply that $\mathfrak{s}^{-2}\mathfrak{u} \mapsto 1$ and $\mathfrak{t}^{-3}\mathfrak{u}^2 \mapsto 1$.

The loops $\mathfrak{s}^{-2}\mathfrak{u}$ and $\mathfrak{t}^{-3}\mathfrak{u}^2$ represent the primitive vectors $(2,1) \in H_1(\partial\mathbb{U}_i)$ and $(3,2) \in H_1(\partial\mathbb{U}_j)$, so they are meridians of the solid tori $\mathbb{U}_i$ and $\mathbb{U}_j$. This describes $\mathbb{U}$ as a Dehn filling of $\mathbb{U} \setminus (\mathbb{U}_i \cup \mathbb{U}_j)$. In particular the vanishing loops $\mathfrak{s}^{-2}\mathfrak{u}$ and $\mathfrak{t}^{-3}\mathfrak{u}^2$ generate the kernel of $\pi_1(\mathbb{U} \setminus (\mathbb{U}_i \cup \mathbb{U}_j)) \to \pi_1(\mathbb{U})$, so we have the presentation:

$$\pi_1(\mathbb{U}) = \langle a, b \mid a^2 = b^3 \rangle$$

Remark 4.9. *Consider a 3-manifold U whose boundary contains a component homeomorphic to a torus $\mathbb{S}^1 \times \mathbb{S}^1$. One may perform a Dehn filling of this component by attaching a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$. Such Dehn fillings, well defined up to homeomorphism, are indexed by their slope in $\mathbb{QP}^1$, defined as the image of an unoriented meridian $\partial\mathbb{D}^2 \times \{x\}$ in $\mathbb{P}(H_1(\mathbb{S}^1 \times \mathbb{S}^1; \mathbb{Z})) = \mathbb{QP}^1$.*

Suppose that U is Seifert-fibered. The fibers in the boundary torus are given by a class in $\mathbb{P}(H_1(\mathbb{S}^1 \times \mathbb{S}^1; \mathbb{Z}))$ which one may suppose to be that of $\pm 1/0$. The Dehn fillings yielding a 3-manifold to which the Seifert fibration can be extended are those with finite slope $\pm p/q \in \mathbb{Q}/(\pm 1)$. Such an extension is unique, and it has no additional singular fibers only when $q = \pm 1$, that is for a finite slope $\pm p/q \in \mathbb{Z}/(\pm 1)$.

Let $\mathbb{D}^*_\infty \subset \mathbb{M}$ be a closed punctured disc neighbourhood of the cusp ∞ which is disjoint from $\mathbb{D}_i \cup \mathbb{D}_j$, and denote by $\mathbb{U}_\infty \subset \mathbb{U}$ the union of fibers above it. Then $\mathfrak{s}^{-1}\mathfrak{t}^{-1}$ is homotopic to $-\partial\mathbb{D}^*_\infty$ so the trivialisation presents its homology as:

$$H_1(\mathbb{U}_\infty) = \mathbb{Z} \cdot [\mathfrak{s}^{-1}\mathfrak{t}^{-1}] + \mathbb{Z} \cdot [\mathfrak{u}]$$

The unique Dehn filling of $\mathbb{U} \setminus \mathbb{U}_\infty$ which yields $\mathbb{S}^3$ is obtained by adding the meridian of $\partial\mathbb{U}_\infty$ which, according to [Pin14, Corollary 3.4] or [BP21, Section 3], is the class of $[\mathfrak{s}\mathfrak{t}\mathfrak{u}] \in H_1(\partial\mathbb{U}_\infty)$ (it may be useful to notice that $\mathfrak{s}\mathfrak{t}\mathfrak{u} \mapsto a^{-1}b^{-1}c = ab^{-1}$).

In $H_1(\partial\mathbb{U}_\infty)$, this meridian class intersects once the class $[\mathfrak{u}]$ of the Seifert fibers, so the fibration extends smoothly to a Seifert fibration $\overline{\mathbb{U}} \to \overline{\mathbb{M}}$.

Recovering $\mathbb{U}$ as the complement of a trefoil knot in $\mathbb{S}^3$.

The Seifert fibered manifold $\mathbb{U}$ can be recovered by attaching the Seifert fibered solid tori $\mathbb{U}_i$ and $\mathbb{U}_j$ along their boundary in such a way that the classes of the boundary fibers get identified, and removing one of those common boundary fibers.

In $H_1(\partial\mathbb{U}_i)$ and $H_1(\partial\mathbb{U}_j)$ the meridians $m_i = (2, 1)$ and $m_j = (3, 2)$ have oriented intersection number 1 with $[a] = (1, 1)$ and $[b] = (1, 1)$, so $([m_i], [a])$ and $(m_j, [b])$ form oriented bases of these homology groups. In those coordinates the inclusion maps $H_1(\partial\mathbb{U}_i) \to H_1(\mathbb{U}_i)$ and $H_1(\partial\mathbb{U}_j) \to H_1(\mathbb{U}_j)$ correspond to projecting onto the second factor $[m_i] \mapsto 0$, $[a] \mapsto 1$ and $[m_j] \mapsto 0$, $[b] \mapsto 1$.

The fiber $\mathfrak{u} = (0, 1)$ has intersection numbers -2 and -3 with the meridians $(2, 1)$ and $(3, 2)$, and intersection number -1 with $[a] = (1, 1)$ and $[b] = (1, 1)$ so it decomposes as $[\mathfrak{u}] = -[m_i] + 2[a]$ and $[\mathfrak{u}] = -[m_j] + 3[b]$. In particular it has image $2[a] \in H_1(\mathbb{U}_i)$ and $3[b] \in H_1(\mathbb{U}_j)$.

Now let us attach the fibered solid tori $\mathbb{U}_i$ and $\mathbb{U}_j$ along their boundary. Since $[a]$ and $[b]$ generate $H_1(\mathbb{U}_i)$ and $H_1(\mathbb{U}_j)$ we may use them as generators for the common torus boundary. The common fibers $[\mathfrak{u}]$ represent $2[a] + 3[b]$, that is a $(2, 3)$ torus knot of $\mathbb{U}_i \cap \mathbb{U}_j$.

Remark 4.10. *For the canonical choice of orientation on $\mathbb{U}$, we may ask about the chirality of the trefoil knot or equivalenty of the Hopf link formed by the singular fibers. If the singular fibers have linking number $+1$ then the trefoil is right handed.*

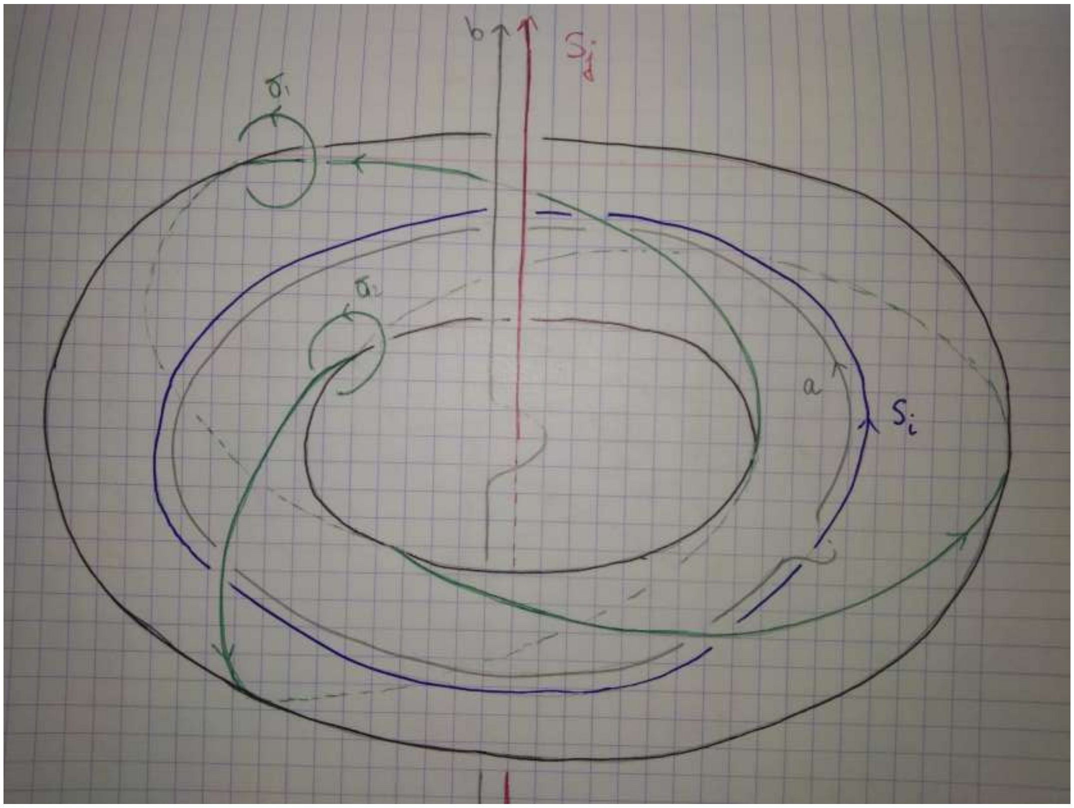


Figure 4.2: In $\mathbb{U} \simeq \mathbb{S}^3$: Hopf link $\mathbb{S}_i \cup \mathbb{S}_j$, torus knot $\mathbb{S}_\infty$, and the loops a, b, c.

Hence the quotient of $\mathrm{PSL}_2(\mathbb{R})$ *by* $\mathrm{PSL}_2(\mathbb{Z})$ *acting on the right or left is homeomorphic to the complement of a left or right handed trefoil knot in the sphere.*

We may sum up part of the previous discussion in the following proposition.

Proposition 4.11. *The space* $\overline{\mathbb{U}}$ *is homeomorphic to the sphere* $\mathbb{S}^3$ *and contains* $\mathbb{S}_i \cup \mathbb{S}_j$ *as a Hopf link, that is a pair of trivial knots such that* $\mathrm{lk}(\mathbb{S}_i, \mathbb{S}_j) = 1$.

The complement $\overline{\mathbb{U}} \setminus (\mathbb{S}_i \cup \mathbb{S}_j)$ *retracts by deformation on a torus containing* $\mathbb{S}_\infty$ *as a* $(2,3)$*-torus knot in the base* $(\mathbb{S}_i, \mathbb{S}_j)$*, so that* $\mathrm{lk}(\mathbb{S}_\infty, \mathbb{S}_i) = 3$ *and* $\mathrm{lk}(\mathbb{S}_\infty, \mathbb{S}_j) = 2$.

In $\overline{\mathbb{U}} \setminus (\mathbb{S}_i \cup \mathbb{S}_j)$ *the loops* a, b *are* $(1,1)$*-torus knots, so* $\mathrm{lk}(a, \mathbb{S}_i) = 1 = \mathrm{lk}(b, \mathbb{S}_j)$.

Lifting loops in the unit tangent bundle

Definition 4.12. *In a* 3*-manifold, a* link *with* $k \in \mathbb{N}$ *components is an embedding of the disjoint union of* k *oriented circles whose components are labelled, considered up to individual reparametrizations. A* knot *is a link with one connected component.*

A knot in $\mathbb{U}$ can be smoothly perturbed to an isotopic knot which is transverse to the fibers of the projection $\mathbb{U} \to \mathbb{M}$, or to a unit vector field carried by an immersed loop in $\mathbb{M}$. It can also be isotoped to a *Legendrian knot*, that is the unit tangent vector field along an immersed loop in $\mathbb{M}$. This can be achieved by smoothly combing the field along the loop; if a vortex appears, then introducing a twiddle in the loop will accommodate it. We refer to [Gei08, Chapter 3] for an introduction to Legendrian knots and their properties.

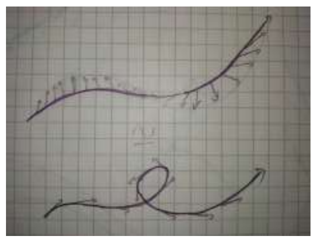


Figure 4.3: Combing vector fields and introducing twildes to accomodate vortices.

Consider Legendrian knots k_1, k_2 in $\mathbb{U}$ and let γ_1, γ_2 be their projections in $\mathbb{M}$. Then k_1, k_2 are homotopic in $\mathbb{U}$ if and only if γ_1, γ_2 are *regular homotopic* in $\mathbb{M}$: that is connected by a sequence of isotopies, Reidemeister moves RII & RIII, and special moves in the neighbourhood of singularities depicted in Figure 4.4. The Legendrian knots k_1, k_2 are isotopic if and only if the immersions γ_1, γ_2 are connected by regular-homotopies such that none of the RII-moves involve strands crossing with the same directions.

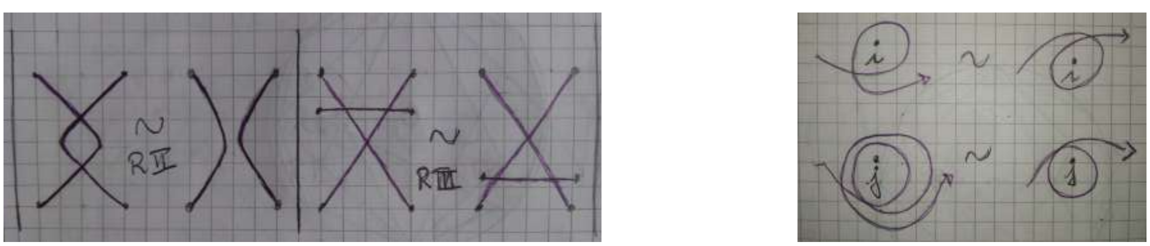


Figure 4.4: Regular homotopies in $\mathbb{M}$: Reidemeister moves II & III, singular moves.

Recall from Section 3.1 that a hyperbolic conjugacy class in $\pi_1(\mathbb{M})$ encodes the homotopy class of two loops in the orbifold $\mathbb{M}$, namely its hyperbolic representative and its linear representative. These representatives are closed geodesics for the hyperbolic and flat metrics on $\mathbb{M}$. They lift in $\mathbb{U}$ to the periodic orbits for the corresponding geodesic flows, which (when they are primitive) are called *modular knots* and *Lorenz knots*.

Any finite collection of modular knots is called a *modular link*, and the collection of all modular knots is called the *master modular link*. Any finite collection of Lorenz knots is called a *Lorenz link*, and the collection of all modular knots is called the *master Lorenz link*. By collection we mean here that the components are labelled by the corresponding primitive hyperbolic conjugacy classes.

Theorem 4.13. *The modular link and Lorenz link associated to a finite number of primitive hyperbolic conjugacy classes in the modular group are isotopic in* $\mathbb{U}$.

Incomplete Proof. Consider the hyperbolic and linear representatives associated to this finite collection of primitive hyperbolic conjugacy classes in the modular group.

By Proposition 3.36, the multiloops obtained as their preimages in the finite Galois cover $\mathbb{T}^* \to \mathbb{M}$ are connected by isotopies and Reidemeister moves RIII. Therefore these preimages have isotopic lifts in the unit tangent bundle of $\mathbb{T}^*$.

Unable to finish the proof, we refer to Theorem 4.24 which cites [Ghy07, ğ3.4]. □

In particular, the homotopy classes of modular knots and Lorenz knots correspond to the same conjugacy classes in the fundamental group $\pi_1(\mathbb{U})$, and we will now describe which ones.

Conjugacy classes of modular knots

The Seifert fibration $\mathbb{U} \to \mathbb{M}$ induces an extension of the fundamental group of the orbifold base by the fundamental group of a generic fiber:

$$1 \to \pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{U}) \to \pi_1(\mathbb{M}) \to 1$$

It is central (the kernel is contained in the center) because composing by a loop going around the fiber in the total space corresponds after projection in the base to composing with a twiddle, which can be sled along the projected loop without changing its regular homotopy class.

Consider preimages $a, b \in \pi_1(\mathbb{U})$ of $S^{-1}, T^{-1} \in \pi_1(\mathbb{M})$ obtained by lifting in $\mathbb{U}$ simple embedded based loops in $\mathbb{M}$ which circle once around the conical points. This yields the usual presentation for the fundamental group of a $(2, 3)$ torus knot:

$$\pi_1(\mathbb{U}) = \langle a, b \mid a^2 = b^3 \rangle$$

Observe that $c = a^2 = b^3$ generates the center of $\pi_1(\mathbb{U})$, and since it represents a generic fiber of the projection, the center equals the kernel $\mathbb{Z}$ of the projection.

If in the presentation of $\pi_1(\mathbb{U})$ we change the variables $a \mapsto \sigma_1\sigma_2\sigma_1$ and $b \mapsto \sigma_1\sigma_2$, then we find Artin's presentation for the braid group on three strands:

$$\mathcal{B}_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

The inverse isomorphism is given by $\sigma_1 \mapsto b^{-1}a$ and $\sigma_2 \mapsto ab^{-1}$.

Proposition 4.14. *The braid group has abelianisation* $\mathrm{lk}\colon \mathcal{B}_3 \to \mathbb{Z}$ *defined by* $\sigma_j \mapsto 1$ *on the generators of the Artin presentation. The abelianisation injects the kernel* $\mathbb{Z}$ *of the central extension* $\pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{U}) \to \pi_1(\mathbb{M})$ *to the subgroup of index* 6 *in* $\mathbb{Z}$*.*

Proof. To obtain the abelianisation from a group presentation, first abelianise the free group on the generators and then quotient by the image of the relations.

Using Artin's presentation, this amounts to letting the σ_j commute so we are left with words in $\sigma_1^{n_1}\sigma_2^{n_2}$, and the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ yields $\sigma_1^2\sigma_2 = \sigma_1\sigma_2^2$ which further identifies σ_1 and σ_2, so we are left with the sum $n_1 + n_2$ of all exponents. □

We obtain a central extension $\mathbb{Z} \to \mathcal{B}_3 \to \mathrm{SL}_2(\mathbb{Z})$ by imposing the relation $c^2 = 1$. It is defined (up to an automorphism of the target) by $a \mapsto S^{-1}$, $b \mapsto T^{-1}$, $c \mapsto -1$ so it sends $\sigma_1 = b^{-1}a$ to $R = TS^{-1}$ and $\sigma_2^{-1} = ba^{-1}$ to $L = T^{-1}S$. Composing with the central extension $\{\pm 1\} \to \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z})$ recovers $\mathbb{Z} \to \mathcal{B}_3 \to \mathrm{PSL}_2(\mathbb{Z})$.

Corollary 4.15. *Two elements in* $\mathcal{B}_3$ *are conjugate if and only if they have the same abelianisation, and their projections in* $\mathrm{PSL}_2(\mathbb{Z})$ *are conjugate.*

Proof. The implication is obvious. Suppose $x, y \in \mathcal{B}_3$ have the same abelianisation, and project to $X, Y \in \mathrm{PSL}_2(\mathbb{Z})$ which are conjugate by Z. Choose a lift $z \in \mathcal{B}_3$ of Z and consider the element $zxz^{-1}y^{-1}$. It projects to the identity in $\mathrm{PSL}_2(\mathbb{Z})$ so it must belong to the center. It also abelianises to the identity, but Proposition 4.14 says that the center injects into the abelianisation, so it must be trivial. □

A primitive modular geodesic in $\mathbb{M}$ lifts to a modular knot in $\mathbb{U}$, and we shall now describe how to perform this geodesic lift at the level of conjugacy classes.

Extend the lifts $L \mapsto \sigma_1^{-1}$ and $R \mapsto \sigma_2$ to a morphism $\mathrm{PSL}_2(\mathbb{N}) \to \mathcal{B}_3$ from the free monoid on $L\&R$ to the free monoid on $\sigma_1^{-1}\&\sigma_2$. This yields a map σ from the set of infinite order conjugacy classes in $\pi_1(\mathbb{M})$ to some conjugacy classes in $\pi_1(\mathbb{U})$.

Theorem 4.16. *The map* σ *sends the conjugacy class associated to a closed geodesic in* $\mathbb{M}$ *to the conjugacy class associated the the corresponding modular knot in* $\mathbb{U}$*.*

Proof. Consider a hyperbolic conjugacy class in $\mathrm{PSL}_2(\mathbb{Z})$, encoded by an $L\&R$-cycle.

Theorem 4.13 implies that the corresponding modular knot (defined as the lift of its hyperbolic representative), and the corresponding Lorenz knot (defined as the lift of its linear representative), are homotopic.

By Definition 3.17, its linear representative is a loop in $\mathbb{M}$ whose homotopy class in $\mathbb{M} \setminus \{i, j\}$ is encoded by the $\mathfrak{s}^{\pm 1}\&\mathfrak{t}^{\pm 1}$-cycle obtained by the replacement rules $L \rightsquigarrow \mathfrak{s}^{-1}\mathfrak{t}^{-1}$ and $R \rightsquigarrow \mathfrak{s}^{+1}\mathfrak{t}^{+1}$. It lifts in the unit tangent bundle $\mathbb{U}\setminus(\mathbb{S}_i \cup \mathbb{S}_j)$ of $\mathbb{M}\setminus\{i,j\}$ to the loop given by the replacement rules $L \rightsquigarrow \mathfrak{s}^{-1}\mathfrak{t}^{-1}\mathfrak{u}^{-1}$ and $R \rightsquigarrow \mathfrak{s}^{+1}\mathfrak{t}^{+1}\mathfrak{u}^{+1}$.

Recall that the inclusion map $\mathbb{U} \setminus (\mathbb{S}_i \cup \mathbb{S}_j) \to \mathbb{U}$ induces a map between the fundamental groups defined on the generators by $\mathfrak{s}, \mathfrak{t}, \mathfrak{u} \mapsto a^{-1}c, b^{-1}c, c$. Hence, lifting a linear representative from $\mathbb{M}$ to $\mathbb{U}$ results in translating the $L\&R$-cycle according to the replacement rules $L \rightsquigarrow abc^{-1} = \sigma_1^{-1}$ and $R \rightsquigarrow a^{-1}b^{-1}c = \sigma_2$. ☐

We say that a conjugacy class in $\mathcal{B}_3 = \widetilde{\mathrm{PSL}}_2(\mathbb{Z}) = \pi_1(\mathbb{U})$ is modular if it corresponds to a modular knot.

Corollary 4.17. *A conjugacy class $[\beta]$ in $\pi_1(\mathbb{U}) = \widetilde{\mathrm{PSL}}_2(\mathbb{Z})$ is modular if and only if it projects to a primitive hyperbolic conjugacy class $[\gamma]$ in $\pi_1(\mathbb{M}) = \mathrm{PSL}_2(\mathbb{Z})$, and abelianises to its Rademacher invariant:*

$$\mathrm{lk}([\beta]) = \mathrm{Rad}([\gamma]).$$

Scholium 4.18. *In terms of loops in $\mathbb{U}$, the abelianisation $\mathrm{lk}\colon \mathcal{B}_3 \to \mathbb{Z}$ of the braid group corresponds to the linking number with the trefoil given by $\mathrm{lk}\, \pi_1(\mathbb{U}) \to H_1(\mathbb{U})$. Hence, once the proof of Theorem 4.13 is completed independently from [Ghy07], its Corollary 4.17 provides an alternative explanation for the equality between the Rademacher invariant of a hyperbolic element of the modular group and the linking number of the associated modular knot with the trefoil.*

Remark 4.19. *Note that the conjugacy class of a braid $\beta \in \mathcal{B}_3$ defines, by taking its cyclic closure, a link $\bar{\beta}$ which has at most three components.*

Hence the conjugacy class of a primitive hyperbolic $A \in \mathrm{PSL}_2(\mathbb{N})$ yields a modular geodesic in $\mathbb{M}$, which lifts to a modular knot in $\mathbb{U}$ corresponding the conjugacy class of the braid $\sigma(A) \in \widetilde{\mathrm{PSL}}_2(\mathbb{Z})$ described in Theorem 4.16. Its closure yields a link $\overline{\sigma}(A)$ in the solid tori $\mathbb{D} \times \mathbb{S}^1$, and the branched double cover of this solid tori over the link yields the punctured torus bundle $(\mathbb{T}^* \times [0,1])/A$ with monodromy $A \in \mathrm{SL}_2(\mathbb{N})$.

Let us mention here that [Fun13] is dedicated to the investigation of such torus bundles having the same quantum invariants. This should be compared with the various equivalence relations on modular conjugacy classes mentioned in the introduction (especially at the end of Section 0.1 and 0.2).

Abelianisation of a central sequence

In this paragraph we work in the category of groups, but we will name objects and morphisms so as to arouse the topological pictures and hint to the applications we have in mind.

Lemma 4.20. *Let* $1 \to \mathit{Fibre} \to \mathit{Total} \to \mathit{Base} \to 1$ *be a central extension of groups. The pull-back* $p' \colon \mathit{Total}' \to \mathit{Base}'$ *of the projection* $p \colon \mathit{Total} \to \mathit{Base}$ *along the inclusion* $\mathit{Base}' \to \mathit{Base}$ *is an isomorphism.*

Proof. Choose a set theoretic section $s \colon \mathit{Base} \to \mathit{Total}$, so that every $t \in \mathit{Total}$ writes uniquely as $t = f.s(b)$ for $f \in \mathit{Fibre}$ and $b \in \mathit{Base}$. The equality of commutators $[t_1, t_2] = [f_1.s(b_1), f_2.s(b_2)] = [s(b_1), s(b_2)]$ shows that s is a section to p' which is thus surjective, and that $\mathit{Fiber} \cap \mathit{Total} = 1$ so p' is injective. □

In other terms, the abelianisation functor yields a commutative diagram of short exact sequences in the category of groups, the columns being central extensions:

$$\begin{array}{ccccc} \mathit{Fibre}' & \longrightarrow & \mathit{Fibre} & \longrightarrow & \mathit{Fibre}^{ab} \\ \downarrow & & \downarrow & & \downarrow \\ \mathit{Total}' & \longrightarrow & \mathit{Total} & \longrightarrow & \mathit{Total}^{ab} \\ \downarrow & & \downarrow & & \downarrow \\ \mathit{Base}' & \longrightarrow & \mathit{Base} & \longrightarrow & \mathit{Base}^{ab} \end{array} \qquad \begin{array}{ccccc} 0 & \longrightarrow & \mathit{Fibre} & \longrightarrow & \mathit{Fibre} \\ \downarrow & & \downarrow & & \downarrow \\ \mathit{Page} & \longrightarrow & \mathit{Total} & \longrightarrow & \mathit{Dromy} \\ \downarrow & & \downarrow & & \downarrow \\ \mathit{Cover} & \longrightarrow & \mathit{Base} & \longrightarrow & \mathit{Galois} \end{array}$$

Let us explain why we rename the groups in left diagram as those in the right. Since Fibre is contained in its own center it is abelian, equal to Fibre^{ab}, so $\mathit{Fibre}' = 0$. We write $\mathit{Page} = \mathit{Total}'$ and $\mathit{Total}^{ab} = \mathit{Dromy}$ in reference to an open book decomposition (the fundamental group of the binding's complement in the total space acts by monodromy on that of the page). We write $\mathit{Cover} = \mathit{Base}'$ and $\mathit{Base}^{ab} = \mathit{Galois}$ in reference to a Galois cover with abelian symmetry group.

We just proved that $\mathit{Page} = \mathit{Cover}$, and we also deduce that the monodromy group of the open book decomposition is a central extension of the Galois group of the cover by the fundamental group of the fibre.

Corollary 4.21. *The euler class in* $H^2(\mathit{Base}; \mathit{Fibre})$ *classifying the central extension*

$$1 \to \mathit{Fibre} \to \mathit{Total} \to \mathit{Base} \to 1$$

is the pull-back by $\mathit{Abel} \colon \mathit{Base} \to \mathit{Galois}$ *of the euler class in* $H^2(\mathit{Galois}; \mathit{Fibre})$ *classifying the central extension*

$$1 \to \mathit{Fibre} \to \mathit{Dromy} \to \mathit{Galois} \to 1$$

Derived subgroup of the braid group $\mathcal{B}_3$

The complement $\mathbb{U}$ of the trefoil knot fibers over the circle $\mathbb{S}^1$ with fiber a punctured torus $\mathbb{T}^*$. This fibration is part of an open book decomposition of $\mathbb{S}^3$: the pages are Seifert surfaces turning around the knotted binding, in this case punctured tori with the trefoil as common boundary, as suggested in figure 4.5.

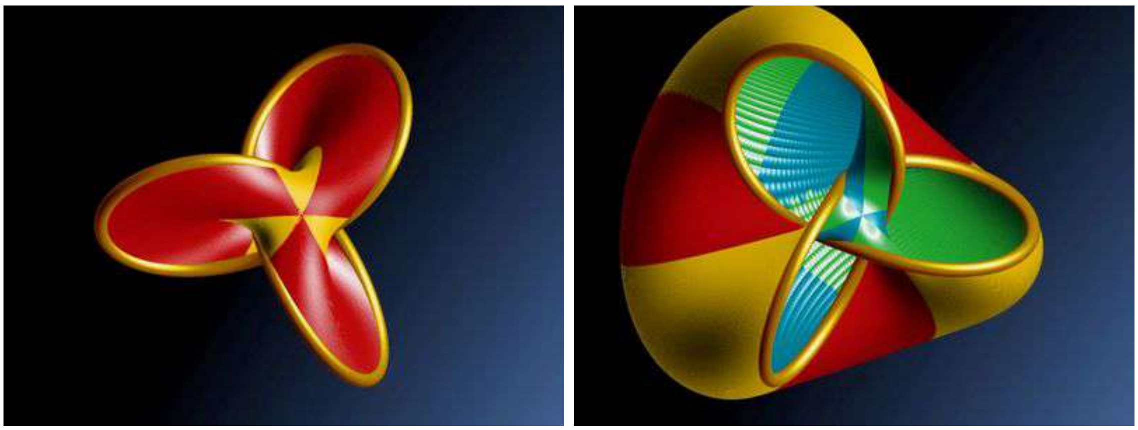


Figure 4.5: Open book decomposition of $\mathbb{S}^3$ with toric pages and trefoil binding. Those images were created by Jos-Leys and Étienne Ghys.

The associated short exact sequence of fundamental groups:

$$1 \to \pi_1(\mathbb{T}^*) \to \pi_1(\mathbb{U}) \to \pi_1(\mathbb{S}^1) \to 1$$

corresponds to the abelianisation of $\pi_1(\mathbb{U})$, and is given by the linking number with the trefoil, or the algebraic intersection number with its Seifert surface $\mathbb{T}^*$. Consequently the kernel $\pi_1(\mathbb{T}^*) = \mathcal{F}_2$ is the derived subgroup of $\pi_1(\mathbb{U}) = \mathcal{B}_3$.

By Lemma 4.20 The central extension $\pi_1(\mathbb{U}) \to \pi_1(\mathbb{M})$ restricts and corestricts to an isomorphism between the derived subgroups.

Besides, Proposition 4.14 says that the abelianisation map $\pi_1(\mathbb{U}) \to H_1(\mathbb{U})$ injects the kernel $\pi_1(\mathbb{S}^1)$ of the central extension to its subgroup of index 6.

Corollary 4.22. *We have a commutative diagram of short exact sequences (implicit trivial groups are omitted), with abeliansations for lines and central extensions for* *columns:*

$$\begin{array}{ccccc} 0 & \longrightarrow & \pi_1(\mathbb{S}^1) & \longrightarrow & H_1(\mathbb{S}^1) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1(\mathbb{T}^*) & \longrightarrow & \pi_1(\mathbb{U}) & \longrightarrow & H_1(\mathbb{U}) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1(\mathbb{T}^*) & \longrightarrow & \pi_1(\mathbb{M}) & \longrightarrow & H_1(\mathbb{M}) \end{array} \qquad \begin{array}{ccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_2 & \longrightarrow & \mathcal{B}_3 & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_2 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z}/6 \end{array}$$

The outer automorphism action $\mathbb{Z} \to \mathrm{Out}(\mathcal{F}_2)$ given by the middle line factors through the outer automorphism action $\mathbb{Z}/6 \to \mathrm{Out}(\mathcal{F}_2)$ given by the bottom line.

The euler class $H^2(\Gamma; \mathbb{Z})$ classifying the central extension of the middle column is the pull back by the abelianisation map $\Gamma \to \mathbb{Z}/6$ of the euler class $H^2(\mathbb{Z}/6; \mathbb{Z})$ classifying the central extension of the right column.

The fibration $\mathbb{T}^* \to \mathbb{U} \to \mathbb{S}^1$ yields a monodromy action of the fundamental group of the circular base on the toral page by homeomorphisms well defined up to isotopy. The homeomorphisms of the punctured torus up to isotopy form its mapping class group, isomorphic to the automorphisms of its fundamental group up to inner-morphisms, that is $\mathrm{Out}(\mathcal{F}_2) = \mathrm{GL}_2(\mathbb{Z})$. The monodromy representation is the geometric counterpart of the outer automorphism representation $\mathbb{Z} \to \mathrm{Out}(\mathcal{F}_2)$ arising from the short exact sequence $\mathcal{F}_2 \to \mathcal{B}_3 \to \mathbb{Z}$. The previous diagram implies that the image of the generator is an orientation preserving mapping class of the punctured torus with order 6.

Consequently, the toral page of the open book decomposition provides a geometric model for the toral cover of the modular orbifold, and the monodromy action factors through the Galois action: the quotient is the modular orbifold.

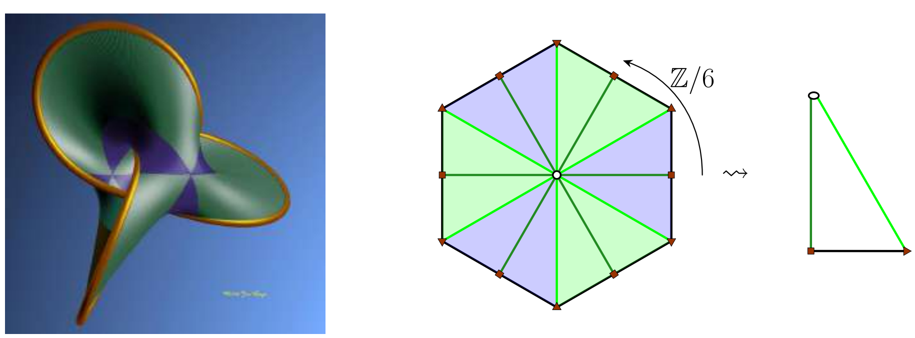


Figure 4.6: The toric pages $\mathbb{T}^*$ of the open book decomposition quotient to $\mathbb{M}$.

4.2 Linking numbers of modular knots

In this section we provide several formulae for computing the linking numbers of modular knots.

We first describe the topology of the Lorenz template and the master modular link consisting of all modular knots. This yields an algorithmic formula for linking numbers counting the number of crossings in the corresponding link diagrams.

As an excursion, we show a variation on this formula which involves an infinite sum: it opens a door onto the Hilbertian analysis of linking quadratic forms, although we shall not pursue this direction here.

Then we rewrite the algorithmic formula in terms of the intrinsic algebra of $\mathrm{PSL}_2(\mathbb{Z})$, involving a summation over double cosets (but with finite support). Thus it will recast the linking form in terms of the general formalism we developed for invariants of pairs of conjugacy classes.

The Lorenz template in the unit tangent bundle

We now use the projective model $\nabla_1 \to \mathbb{M}$ represented in figure 3.2 to construct a multivalued section of the unit tangent bundle $\mathbb{U} \to \mathbb{M}$. Its image will be a branched surface $\mathbb{Y} \subset \mathbb{U}$ called the Lorenz template, and will carry a semi-flow which is conjugate to the geodesic flow of $\mathbb{M}$. In particular the template $\mathbb{Y}$ will handle all periodic orbits for the geodesic flow.

Consider first the branched surface $\mathbb{Y} = \triangle_1 \bmod \mathrm{PSL}_2(\mathbb{N})$ obtained from ∇_1 by identifying its left and right edges to its hypothenuse according to L^{-1} and R^{-1}. The branch locus is the segment formed by the identified edges, it is parametrized by $\alpha \in [0, \infty] \mapsto V_\alpha \cap [v_0, v_\infty]$.

The radial vector field on $\mathbb{R}^2$ restricts to $\triangle_1$ and projects to $\mathbb{Y}$, on which it defines a semi-flow. The branched locus provides a section of the semi-flow, on which the first return map is given by the inverse action of $\mathrm{PSL}_2(\mathbb{N})$ on $\alpha \in]0, \infty[$ deleting the first letter of its continued fraction expansion (also known as a Bernoulli shift). Hence the orbit emerging from α is the projection of the line $V_\alpha \cap \triangle_1$ in $\mathbb{Y}$, and can be represented in the fundamental domain ∇_1 as a union of disjoint segments which piece together according to the action of $L\&R$.

The past of the vertex $v_0 \simeq v_1 \simeq v_\infty$ in $\mathbb{Y}$ is formed by the union of all segments with rational inclination: let $\mathbb{Y}^\circ$ be the complement of this orbit. The quadratic irrationals have ultimately periodic orbits, and the periodic numbers have periodic orbits. Every other number $\alpha \in]0, \infty[$ whose continued fraction tails are all aperiodic, has an aperiodic orbit which is an embedded half line $V_\alpha \cap \triangle_1 \to \mathbb{Y}$.

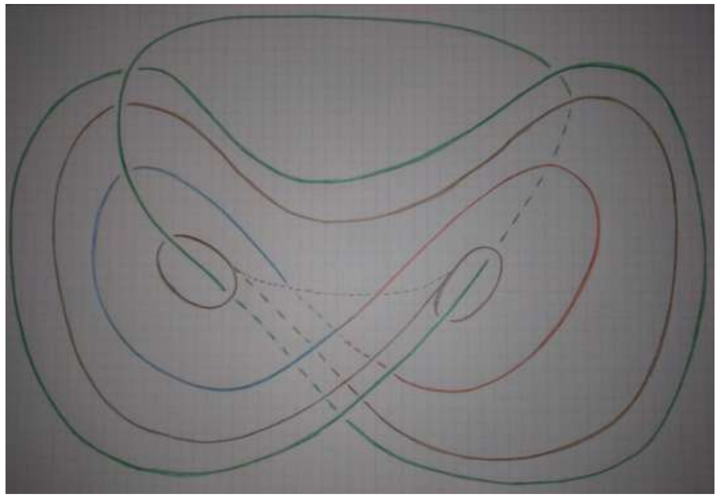


Figure 4.7: The Lorenz template with the trefoil (and the loops $L\&R$).

Now recall the projective model $\nabla_1 \to \mathbb{M}$, represented in figure 3.2. It pushes down the radial field to the modular orbifold defining a 3-valued section of its unit tangent bundle $\mathbb{U} \to \mathbb{M}$ whose image is a branched surface which identifies with $\mathbb{Y} \subset \mathbb{U}$, we call it the *Lorenz template*.

Proposition 4.23. *The embedding of the Lorenz template $\mathbb{Y}$ in $\mathbb{U}$ is such that the trefoil knot follows its boundary and branch locus as depicted in figure 4.7.*

Moreover, its left ear (corresponding to R-turns) arrives from above, and its right ear (corresponding to L-turns) arrives from beneath.

Proof. Of course, one may push-forward the radial vector field by $\mathbb{P}\bar{\psi} \colon \nabla_1 \to \nabla_2$ to work with the hyperbolic model $\nabla_2 \subset \mathbb{HP}$.

Then up to orientation matters, the embedding of the template and its position with respect to the trefoil are shown in [BP21], following the ideas of [Ghy07, ğ3.4]. One may also consult [Pin14] for more concerning templates carrying the periodic orbits for the geodesic flow on the Hecke (p, q, r)-orbifolds. □

Theorem 4.24. *The master modular link is isotopic to the master Lorenz link.*

Proof. By Proposition 4.23, this statement is equivalent to that of Theorem 4.13, for which we only provided an incomplete proof. This time we refer to [Ghy07, ğ3.4].

The idea is to deform the hyperbolic metric on $\mathbb{M}$ in such a way to open the cusp. This yields a one-parameter family of hyperbolic orbifolds whose unit tangent bundles retract onto a Lorenz template. We do not delve upon these ideas since they will be the subject of Chapter 5 as we deform the representation $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R})$. □

Linking numbers between all pairs of hyperbolic matrices

Since $\mathbb{U}$ is homeomorphic to the complement of a trefoil knot in the sphere, any two components of a link have a well defined linking number.

Let us define the linking number between any two hyperbolic matrices of $\mathrm{PSL}_2(\mathbb{Z})$. Such hyperbolic matrices correspond to periodic orbits for the modular flow.

This orbit is a knot when the matrix is primitive, otherwise the orbit travels several times along a knot. When two matrices are coprime (meaning they are not positive powers of a same element as we saw in Proposition 2.48), their corresponding periodic orbits are disjoint. Thus coprime hyperbolic matrices correspond to disjoint periodic orbits whose linking number is well defined. Since $\mathrm{lk}(A^m, B^n) = mn\,\mathrm{lk}(A, B)$ it is enough to understand the linking number between primitive elements, but most of our formulae will be written to hold without this assumption. We finally remove the coprimality hypothesis by defining the self-linking number of a modular knot.

A band in a 3-manifold M is an embedding of the annulus $S^1 \times [-1, 1]$ in M. Up to isotopy, a band is equivalent to a knot together with a framing, that is section of its normal bundle defined up to multiplication by a scalar function.

Definition 4.25 (Lorenz framing & self-linking number)**.** *The Lorenz framing of a modular knot $k_A \subset \mathbb{U}$ is the band obtained from a tubular neighbourhood of k_A in $\mathbb{Y}$. Its boundary consists in two parallel copies of k_A and their linking number defines the self-linking number of the modular knot k_A for the Lorenz framing.*

Remark 4.26. *We know by Proposition 3.4 that in $\mathbb{M}$, the modular geodesics avoid the point j, and we also defined a canonical perturbation of those which contain i. Besides, we also know that the linear representatives avoid both i and j.*

Thus one may speak of the linking numbers between special fibers of the Seifert fibration (which form a Hopf link in the complement of the trefoil), with modular knots and Lorenz knots. However, since we showed in 3.1 that the hyperbolic and linear representatives lift to different homotopy classes in $\mathbb{M} \setminus \{i, j\}$, these linking numbers may not coincide !

This implies in particular that during the deformation of the hyperbolic metric on $\mathbb{M}$ alluded to in the proof of Theorem 4.24, the periodic orbits for the geodesic flow will cross the singular fibres, so the projected geodesics will perform special moves to pass over the conical singularities.

Still, for every hyperbolic matrix in $\mathrm{PSL}_2(\mathbb{Z})$, one may define two conjugacy-invariants given by the linking numbers with the special fibers: those will be combinations of the exponents appearing in the $\mathfrak{s}\&\mathfrak{t}$-cycles encoding their hyperbolic or linear representative lifted in $\mathbb{M} \setminus \{i, j\}$.

Algorithmic formula

Let us derive from Theorems 4.23 & 4.24 the algorithmic formula of Proposition 4.27 used by Pierre Dehornoy in [Deh11] to compute linking numbers of modular knots. For this, we must introduce some notations relying on Section 2.2.

We endow the submonoid $\mathrm{PSL}_2(\mathbb{N})$ of $\mathrm{PSL}_2(\mathbb{Z})$, which is freely generated by $L\&R$, with the lexicographic order extending $L < R$.

In the group $\mathrm{PSL}_2(\mathbb{Z})$ the conjugacy class of an infinite order element intersects the monoid $\mathrm{PSL}_2(\mathbb{N})$ along its Lyndon representatives, which consist in all cyclic permutations of a non-empty $L\&R$-word. The primitivity of the conjugacy class is equivalent to the primitivity of the cyclic words, and the conjugacy class is hyperbolic when both letters L and R appear.

The set $\{L, R\}^{\mathbb{N}}$ of infinite binary sequences on the letters $L\&R$ is given the lexicographic order extending $L < R$. The monoid $\mathrm{PSL}_2(\mathbb{N})$ maps to $\{L, R\}^{\mathbb{N}}$ by sending a finite word A to its periodisation A^∞. This map is increasing, and injective in restriction to primitive elements.

We use σ to denote the Bernoulli shift on $\{L, R\}^{\mathbb{N}}$ which removes the first letter, as well as the cyclic shift on $\mathrm{PSL}_2(\mathbb{N})$ which moves the first letter at the end. These shifts are intertwined by the periodisation map $A \mapsto A^\infty$, namely for all $A \in \mathrm{PSL}_2(\mathbb{N})$ we have $(\sigma^j A)^\infty = \sigma^j(A^\infty)$.

In particular, the Lyndon representatives for the conjugacy class of $A \in \mathrm{PSL}_2(\mathbb{N})$ are the cyclic permutations $\sigma^i A$ for $1 \leq i \leq \mathrm{len}(A)$, and we shall consider them with multiplicity when A is not primitive.

Denote by $W[-1] \in \{L, R\}$ the last letter of a non-empty word $W \in \mathrm{PSL}_2(\mathbb{N})$. Thus for instance, $(\sigma^1 W)[-1]$ is the first letter of W.

Following Iverson [Knu92], denote $[\![P]\!] \in \{0, 1\}$ the truth value of a property P, which satisfies the usual rules of boolean algebra.

Proposition 4.27. *For all hyperbolic matrices $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ we have:*

$$\mathrm{lk}(A, B) = \frac{1}{2} \sum_{i=1}^{\mathrm{len}(A)} \sum_{j=1}^{\mathrm{len}(B)} \begin{pmatrix} [\![(\sigma^i A)[-1] > (\sigma^j B)[-1]]\!]\, [\![\sigma^i A^\infty < \sigma^j B^\infty]\!] \\ + \\ [\![(\sigma^i A)[-1] < (\sigma^j B)[-1]]\!]\, [\![\sigma^i A^\infty > \sigma^j B^\infty]\!] \end{pmatrix} \tag{Algo-Sum}$$

This Algo-Sum counts the pairs of Lyndon representatives whose periodisations are ordered in the opposite way to their last letters.

Proof. First we isotope the Lorenz template to look like in Figure 4.8. The projection on the plane of the paper yields link diagrams for the collections of periodic orbits, whose crossings all contribute positively to the intersection.

Thus we must simply count these crossings and divide by two. Clearly, the crossings are in bijection with the terms of the double sum. □

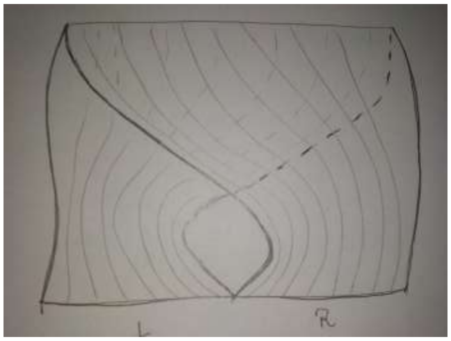


Figure 4.8: Lorenz template positioned for a visual proof of the algorithmic formula.

Remark 4.28 (Primitivity). *The Algo-Sum holds without the primitivity assumption provided we sum over all cyclic permutations of the Lyndon representatives (and in the proof we must count crossing numbers with appropriate multiplicities).*

Remark 4.29 (Self-linking number). *The proof shows that when $A = B$ the Algo-Sum computes the self-linking number of the modular knot for the Lorenz framing.*

Remark 4.30 (Algorithmic). *This formula, although mathematically cumbersome, yields a computation of $\mathrm{lk}(A, B)$ with complexity $O(\operatorname{len} A^2 B^2)$ provided we recall Proposition 2.48 saying that $A^\infty = B^\infty \iff AB = BA$.*

Symbolic dynamics: summing occurrence of linked patterns

In this paragraph we propose a combinatorial formula for the linking number which arises from a different count of the crossings in the Lorenz template. Then we interpret it as a factorisation (or polarisation) of the quadratic linking form, thus opening a door onto its Hilbertian analysis.

Definition 4.31. *For a pattern $P \in \mathrm{PSL}_2(\mathbb{N})$ and a hyperbolic $A \in \mathrm{PSL}_2(\mathbb{N})$, let $\mathrm{pref}(P, A^\infty) = [\![A^\infty \in P \cdot \mathrm{PSL}_2(\mathbb{N})]\!] \in \{0, 1\}$ tell whether P is a prefix of A^∞, and:*

$$\mathrm{occ}(P, A) = \sum_{j=1}^{\operatorname{len} A} \mathrm{pref}\left(P, \sigma^j A^\infty\right)$$

count the number of occurrences of P in A^∞ beginning before the index $\operatorname{len}(A)$, or equivalently the number of cyclic occurrences of P in $A \bmod \sigma$.

Remark 4.32 (Long patterns)**.** *For* $\operatorname{len}(P) \geq \operatorname{len}(A)$ *we have* $\operatorname{occ}(P, A) > 0$ *if and only if* $A^\infty = P^\infty \bmod \sigma$ *which is equivalent to saying that* P *and* A *are not coprime.*

Remark 4.33 (Primitivity)**.** *The definition holds for non primitive* $A \in \mathrm{PSL}_2(\mathbb{N})$ *and we have* $\operatorname{occ}(P, A^n) = n \operatorname{occ}(A)$*.*

Proposition 4.34 (Sum of linked patterns)**.** *For coprime hyperbolic* $A, B \in \mathrm{PSL}_2(\mathbb{N})$ *the corresponding modular knots have linking number:*

$$\operatorname{lk}(A, B) = \frac{1}{2} \sum_w \begin{pmatrix} \operatorname{occ}(RwL, A) \cdot \operatorname{occ}(LwR, B) \\ + \\ \operatorname{occ}(RwL, B) \cdot \operatorname{occ}(LwR, A) \end{pmatrix} \tag{Symb-Dyna-Sum}$$

where the summation extends over all words $w \in \mathrm{PSL}_2(\mathbb{N})$ *including the empty one.*

Proof. The proof consists in rearranging the terms of the Algo-Sum rewritten as:

$$\operatorname{lk}(A, B) = \frac{1}{2} \sum_{i=1}^{\operatorname{len} A} \sum_{j=1}^{\operatorname{len} B} \begin{pmatrix} [\![\sigma^i A[-1] = R]\!]\, [\![\sigma^i B[-1] = L]\!]\, [\![\sigma^i A^\infty < \sigma^j B^\infty]\!] \\ + \\ [\![\sigma^i B[-1] = R]\!]\, [\![\sigma^i A[-1] = L]\!]\, [\![\sigma^i B^\infty < \sigma^j A^\infty]\!] \end{pmatrix}$$

Note that $[\![\sigma^i A[-1] = R]\!] = \operatorname{pref}(R, \sigma^{i-1} A)$ and $[\![\sigma^j B[-1] = L]\!] = \operatorname{pref}(L, \sigma^{j-1} B)$. Moreover $\sigma^i A^\infty < \sigma^j B^\infty$ if and only if the words $\sigma^i A^\infty$ and $\sigma^j B^\infty$ have prefixes of the form wL and wR for some $w \in \mathrm{PSL}_2(\mathbb{N})$ which is uniquely determined, thus

$$\left[\!\left[\sigma^i A^\infty < \sigma^j B^\infty\right]\!\right] = \sum_w \left[\!\left[\operatorname{pref}(wL, \sigma^i A^\infty)\right]\!\right] \cdot \left[\!\left[\operatorname{pref}(wR, \sigma^j B^\infty)\right]\!\right]$$

Of course we have $\operatorname{pref}(R, \sigma^{i-1} A) \operatorname{pref}(wL, \sigma^i A^\infty) = \operatorname{pref}(RwL, \sigma^{i-1} A^\infty)$ and $\operatorname{pref}(L, \sigma^{j-1} B) \operatorname{pref}(wR, \sigma^j B^\infty) = \operatorname{pref}(LwR, \sigma^{j-1} B^\infty)$.

By replacing all terms in the Algo-Sum we find:

$$\operatorname{lk}(A, B) = \frac{1}{2} \sum_{i=1}^{\operatorname{len} A} \sum_{j=1}^{\operatorname{len} B} \sum_w \begin{pmatrix} \operatorname{pref}(RwL, \sigma^{i-1} A^\infty) \cdot \operatorname{pref}(LwR, \sigma^{j-1} B^\infty) \\ + \\ \operatorname{pref}(RwL, \sigma^{i-1} A^\infty) \cdot \operatorname{pref}(LwR, \sigma^{j-1} B^\infty) \end{pmatrix}$$

It is then a matter of reordering terms so that sums over w appears on the outside, and recognising the definition of occ inside the sums over i, j. □

Remark 4.35 (Finite support)**.** *By the Remark 4.32 concerning long patterns, the coprimality assumption on* A *and* B *ensures that the support of the Symb-Dyna-Sum is contained in the set of* w *such that* $\operatorname{len} w < \max\{\operatorname{len} A, \operatorname{len} B\}$*.*

Remark 4.36 (Self-linking)**.** *We may use the same sum to define* $\mathrm{lk}(A, B)$ *even when they are not coprime by imposing the bound* $\operatorname{len} w < \max\{\operatorname{len} A, \operatorname{len} B\}$ *on its indices, as suggested by the previous remark.*

In that case the proof shows that the Symb-Dyna-Sum returns the same number as the Algo-Sum, that is (a multiple) of the self-linking number of their common primitive root.

Remark 4.37 (Algorithmic)**.** *This formula has little algorithmic interest: first we must list for* A *and* B *all words* w *(enriched with two distinct letters) which appear in one of their cyclic permutations and count the number of occurrences, then compute the sum: the cost is exponential in* $\operatorname{len}(AB)$*.*

We have programmed it literally to ensure the coincidence with the Algo-Sum, which is much more efficient as it is polynomial in $\operatorname{len}(AB)$*.*

The previous proof does not do justice to the way we found the formula: a clever surgery of the Lorenz pattern (the seeds of which are to be found in the original article of Birman and Williams [BW83]), and a careful examination of the location of the crossings, reveals its recursive structure.

If we split the template (in its usual representation) by extending the dividing line backwards in time, as suggested in the sequence of thumbnails 4.9, we observe that the crossings occur in regions arranged according to a binary tree. As one cuts farther back in time, the binary tree becomes deeper, and the crossing regions break up into sub-regions: what is left in the limit is a fractal Cantor set.

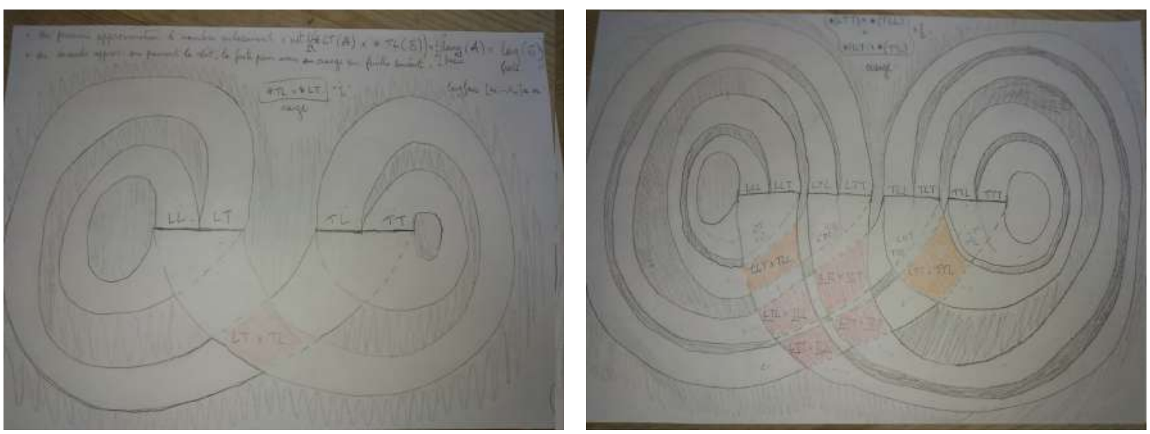


Figure 4.9: Overlapping regions of the split template arranged in a tree-like pattern.

Visual proof. Let us provide a contemplative argument to show Proposition 4.34.

Cut the template in the past for a finite time N counted as the number of intersections with the branching interval (which is $2, 3$ in the thumbnails). The overlapping regions corresponds to the pairs of branches leading to it, and those are indexed (remounting them backwards in time) by the pairs (RwL, LwR) for words $w \in \mathrm{PSL}_2(\mathbb{N})$ with length $\operatorname{len} w < N$. Two coprime modular knots cross inside a certain number of overlaps with depth at most l, and this number is given by the Symb-Dyna-Sum with indices restricted to $\operatorname{len} w < N$. In the limit we recover the infinite sum. □

Remark 4.38 (Taylor expansion)**.** *The previous proofs also shows that the truncated Symb-Dyna-Sum indexed by* $\operatorname{len} w < N$ *yields the exact linking number provided* $\max\{\operatorname{len} A, \operatorname{len} B\} \leq N$. *This partial sum can thus be understood as Taylor approximation to the order N of the linking pairing.*

Scholium 4.39 (Binomial statistics)**.** *The previous analysis also has the advantage of exhibiting the statistical behaviour of the bilinear linking pairing between modular knots. One can make precise statements about the set of all modular knots whose L&R-length is bounded by a constant N, and let it go to infinity.*

On the one hand the areas of the overlapping regions reflect the distribution of the number of crossings between all such modular knots. In particular the distribution of these modular knots according to the abscissa of crossing regions follows a binomial distribution. Note that this abscissa can be parametrized by the Rademacher function.

On the other hand the modular knots that spend a long time in the deepest regions of the tree which are located in the centre (such as $(LR)^n$) will tend to have large linking numbers with most of the other knots, whereas the modular knots which escape into the ears (such as L^nR^n) will have weak intersection with all modular knots except those presenting a very similar behaviour.

Consider the free $\mathbb{Z}$-module generated by the set $\mathrm{PSL}_2(\mathbb{N})/\sigma$ of all cyclic words, endowed with the symmetric bilinear form lk. Let us reformulate Proposition 4.34 as a factorisation of the corresponding symmetric matrix.

Definition 4.40 (Occurrence matrices)**.** *Denote $P(w, A)$ the infinite rectangular matrix with entries* $\operatorname{occ}(RwL, A)$ *indexed by* $w \in \mathrm{PSL}_2(\mathbb{N})$ *and* $A \in \mathrm{PSL}_2(\mathbb{N})/\sigma$.

Denote $P^\#(w, A) = P(w^\#, A^\#)$ where $w^\#$&$A^\#$ are the mirror images of w&A. Its entries are given by $\operatorname{occ}(LwR, A) = \operatorname{occ}(Lw^\# R, A^\#)$.

Finally we define $Z = P + iP^\#$ over the ring $\mathbb{Z}[i]$ of Gaussian integers.

Corollary 4.41 (Factorising the linking matrix)**.** *The matrix of the bilinear form* $\operatorname{lk}(A, B)$ *is the imaginary part of the product* ${}^tZZ^\#$.

Proof. The Symb-Dyna-Sum amounts to the relation $2\operatorname{lk} = {}^tPP^\# + {}^tP^\#P$, which is twice the imaginary part of the product $\left({}^tP + i \cdot {}^tP^\#\right) \cdot \left(P + i \cdot P^\#\right)$. □

Group theory: orbit average of the meyer cocycle

The aim of this paragraph is to recast the Algo-Sum formula for linking numbers in terms of the action of the group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ on the geometree $\Sigma = (\mathcal{T}, \mathrm{cord})$.

Recall that the action of the group $\mathrm{PSL}_2(\mathbb{Z})$ on the trivalent tree $\mathcal{T}$ preserves the cyclic order structure, and is freely transitively on the set of oriented edges.

An infinite order element $A \in \mathrm{PSL}_2(\mathbb{Z})$ acts by translation of $\mathcal{T}$ along an oriented combinatorial axis g_A with endpoints $\alpha', \alpha \in \partial\mathcal{T}$.

Consider two primitive hyperbolic matrices $A, B \in \mathrm{PSL}_2(\mathbb{Z})$. By Proposition 2.48 they are coprime when they have distinct translation axes $g_A, g_B \subset \mathcal{T}$, thus when they have distinct fixed points $\alpha', \alpha, \beta', \beta$.

The algebraic intersection number $\mathrm{cross}(g_A, g_B) \in \{-1, 0, 1\}$ between their translation axes was introduced in the paragraph containing Scholia 1.95. We shall mostly need the geometric intersection number $|\mathrm{cross}|(g_A, g_B) \in \{0, 1\}$ which takes the value 1 or 0 according to whether the fixed points α', α of A and β', β of B are linking on the boundary $\partial\mathcal{T}$ or not.

Corollary 4.42. *For coprime hyperbolic matrices $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ we have:*

$$\mathrm{lk}(A, B) = \frac{1}{2} \sum_{A_i \nparallel B_j} |\mathrm{cross}|(A_i, B_j)$$

where the sum extends over all Lyndon conjugates $A_i, B_j \in \mathrm{PSL}_2(\mathbb{N})$ of A and B such that their last letters differ, which we denote $A_i \nparallel B_j$.

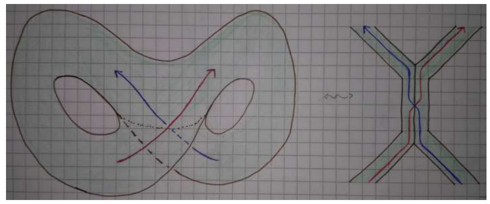
Proof. If hyperbolic coprime matrices $A, B \in \mathrm{PSL}_2(\mathbb{N})$ satisfy $A \in \mathrm{PSL}_2(\mathbb{N}).R$ and $B \in \mathrm{PSL}_2(\mathbb{N}).L$, then $\mathrm{cross}(A, B) = [\![A^\infty < B^\infty]\!]$. More generally if hyperbolic coprime matrices $A, B \in \mathrm{PSL}_2(\mathbb{N})$ have distinct last letters $A[-1] \neq B[-1]$ then:

$$\mathrm{cross}(A, B) = \begin{pmatrix} [\![A[-1] = R]\!]\, [\![B[-1] = L]\!]\, [\![A^\infty < B^\infty]\!] \\ - \\ [\![B[-1] = R]\!]\, [\![A[-1] = L]\!]\, [\![B^\infty < A^\infty]\!] \end{pmatrix}$$

and since at least one of the terms in this difference is 0 we have:

$$|\mathrm{cross}|(A, B) = \begin{pmatrix} [\![A[-1] = R]\!]\, [\![B[-1] = L]\!]\, [\![A^\infty < B^\infty]\!] \\ + \\ [\![B[-1] = R]\!]\, [\![A[-1] = L]\!]\, [\![B^\infty < A^\infty]\!] \end{pmatrix}$$

which completes the proof. □



Template crossings $\leftrightarrow \{(g_A, g_B)\colon\ |\mathrm{cross}|(g_A, g_B) = 1 = \mathrm{cosign}(g_A, g_B)\} \bmod \Gamma \times \Gamma$.

Definition 4.43. *For oriented bi-infinite geodesics $g_A, g_B \subset \mathcal{T}$ with distinct ends we define:*

$$\mathrm{coc}(g_A, g_B) = \left(|\mathrm{cross}| \times \frac{1 + \mathrm{cosign}}{2}\right)(g_A, g_B) = \left(\frac{1 + \ \mathrm{cross}}{2} \times \frac{1 + \mathrm{cosign}}{2}\right)(g_A, g_B)$$

Theorem 4.44. *For coprime hyperbolic matrices $A, B \in \Gamma = \mathrm{PSL}_2(\mathbb{Z})$ we have:*

$$\mathrm{lk}(A, B) = \frac{1}{2} \sum_{U,V} \mathrm{coc}(U \cdot g_A, V \cdot g_B) \qquad \text{(Group-Coset-Sum)}$$

where the sum is over the product of right cosets of $\mathrm{PSL}_2(\mathbb{Z})$ under the stabilisers of the translation axes g_A and g_B modulo the left action of $\mathrm{PSL}_2(\mathbb{Z})$ by translation:

$$(U, V) \in (\Gamma / \operatorname{Stab} g_A) \underset{\Gamma}{\times} (\Gamma / \operatorname{Stab} g_B)$$

This can also be written as the sum over double cosets $W \in \operatorname{Stab} g_A \backslash \Gamma / \operatorname{Stab} g_B$:

$$\mathrm{lk}(A, B) = \frac{1}{2} \sum_{W} \mathrm{coc}(A, WBW^{-1}) \qquad \text{(Double-Coset-Sum)}$$

Proof. Let us show that the sum over (U, V) coincides with that of Proposition 4.42.

Recall that by Lemma 2.43 the infinite order elements $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ can be simultaneously conjugated in $\mathrm{PSL}_2(\mathbb{N})$ if and only if their combinatorial axes in $\mathcal{T}$ share an oriented edge, that is when $\mathrm{cosign}(A, B) = 1$. Hence by Definition 4.43, the term $\mathrm{coc}(U \cdot g_A, V \cdot g_B)$ is 0 unless the axes cross and their orientation coincides along the intersection, in which case it is 1, so one may restrict the sum to such pairs.

Since the action of $\mathrm{PSL}_2(\mathbb{Z})$ is freely transitive on the oriented edges of $\mathcal{T}$, every coset $(U, V) \in \Gamma / \operatorname{Stab} g_A \times_\Gamma \Gamma / \operatorname{Stab} g_B$ contains exactly one representative (U_0, V_∞) such that the axes of $U_0 A U_0^{-1}$ and $V_\infty B V_\infty^{-1}$ intersect along an oriented segment of $\mathcal{T}$ starting at the base edge $\vec{e}_i$. This precisely means that $U_0 A U_0^{-1}$ and $V_\infty B V_\infty^{-1}$ belong to $\mathrm{PSL}_2(\mathbb{N})$ and finish by different letters.

Consequently, we have a bijection between the non-zero terms of the Algo-Sum and Group-Coset-Sum, and this completes the proof. □

Remark 4.45 (Stabilisers)**.** *Recall that for infinite order* $A \in \mathrm{PSL}_2(\mathbb{Z})$ *the stabiliser* $\mathrm{Stab}\, g_A$ *is the cyclic subgroup generated by the primitive root of* A*.*

Remark 4.46 (Intersection from linking)**.** *We recover in particular the intersection number of the modular geodesics as:*

$$\mathrm{lk}(A,B) + \mathrm{lk}(A,B^{-1}) = \frac{1}{2}\sum |\mathrm{cross}|(A_u, B_v) = \tfrac{1}{2}\cdot I(A,B)$$

whereas the sum of the cosign over pairs of intersecting axes yields:

$$\mathrm{lk}(A,B) - \mathrm{lk}(A,B^{-1}) = \frac{1}{2}\sum \left(|\mathrm{cross}| \times \mathrm{cosign}\right)(A_u, B_v).$$

Recall that B^{-1} *and* ${}^tB$ *are conjugate by* S *so* $\mathrm{lk}(A,B) = \mathrm{lk}(A, {}^tB)$*, and given the* $L\&R$ *factorisation of* $B \in \mathrm{PSL}_2(\mathbb{N})$ *it is immediate to deduce that of* ${}^tB \in \mathrm{PSL}_2(\mathbb{N})$*. We deduce an efficient algorithm to compute the intersection number* $I(A,B)$ *from the* $L\&R$*-factorisation of* A, B *by applying Algo-Sum formula to the linking numbers* $\mathrm{lk}(A,B)$ *and* $\mathrm{lk}(A, {}^tB)$*.*

What about the meyer cocyle ? Let us finish this chapter with a conjectural relationship between the quantity $\mathrm{coc}(A,B)$ and the Meyer cocycle.

The Rademacher function $\mathrm{Rad}\colon \Gamma \to \mathbb{Z}$ is a non bounded function. However its formal coboundary $d\,\mathrm{Rad}(A,B) := \mathrm{Rad}(B) - \mathrm{Rad}(AB) + \mathrm{Rad}(A)$ defines a bounded function on $\Gamma \times \Gamma$. We say that Rad is a quasi-morphism.

Its formal coboundary $d\,\mathrm{Rad}$ defines a bounded 2-cocycle, namely an element of $H^2_b(\Gamma;\mathbb{R})$, called the Meyer cocycle.

One may consult to [BG92, Bou16] for much more concerning the Meyer cocycle, bounded cohomology and quasimorphisms.

Conjecture 4.47 (Meyer cocycle)**.** *We believe that* $\mathrm{coc}(A,B)$ *is strongly related to the Meyer cocycle* $d\,\mathrm{Rad}(A,B^{-1})$*. Indeed, for* $A, B \in \mathrm{PSL}_2(\mathbb{N})$ *we observe that:*

$$\mathrm{coc}(A,B) = \begin{cases} \mathrm{Rad}(B) + \mathrm{Rad}(AB^{-1}) - \mathrm{Rad}(A) & \text{if } A < B \\ \mathrm{Rad}(A) + \mathrm{Rad}(A^{-1}B) - \mathrm{Rad}(B) & \text{if } A > B \end{cases}$$

where as usual $\mathrm{PSL}_2(\mathbb{N})$ *is endowed with the lexicographic order extending* $L < R$*.*

Chapter 5

Bilinear forms on the $\mathrm{PSL}_2(\mathbb{R})$-character variety of $\mathrm{PSL}_2(\mathbb{Z})$

Outline of the chapter

In this chapter we denote the modular group by $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, which Theorem 2.11 presented as the free amalgam of its subgroups $\mathbb{Z}/2$ and $\mathbb{Z}/3$ generated by S and T.

Let us recall the following elements of $\mathrm{GL}_2(\mathbb{Z})$, all in $\mathrm{SL}_2(\mathbb{Z})$ except J and K. They satisfy $S^2 = T^3 = -\mathbb{1}$ and $J^2 = K^2 = \mathbb{1}$ as well as $T^{-1}S = L$ and $TS^{-1} = R$.

$$S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \quad T = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) \qquad J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \quad K = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \qquad L = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \quad R = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$$

Although T and T^{-1} are not conjugate in $\mathrm{PSL}_2(\mathbb{Z})$, they are conjugate by J.

The space of representations $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R})$

The first section describes the $\mathrm{PSL}_2(\mathbb{R})$-character variety of Γ, defined as the space of representations $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ considered up to $\mathrm{PSL}_2(\mathbb{R})$-conjugacy at the target. A representation $\rho\colon \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ is uniquely determined by a pair $\rho(S), \rho(T)$ such that $\rho(S)^2 = \mathbb{1} = \rho(T)^3$. We are mostly interested in those which are both faithful and discrete, called *Fuchsian representations*, because they correspond to the holonomy maps of hyperbolic orbifolds with the same two conical singularities as $\mathbb{M}$. They are characterised by the condition that $\rho(S)$ and $\rho(T)$ act on $\mathbb{HP}$ as rotations of order 2 and 3 whose fixed points are separated by a distance $\lambda \geq \lambda_0 = d(i, j)$. This distance λ, together with the orientation of $\rho(T)$, determine a unique Fuchsian representation up to conjugacy.

Geometrically, the positively oriented Fuschsian representations are the holonomy maps of hyperbolic orbifolds with conical singularities of order 2 and 3, so their conjugacy classes they form the Teichmüller space of complete hyperbolic metrics on such an orbifold. As soon as $\lambda > \lambda_0$ the quotient orbifold has a funnel with a unique collar geodesic, and all other geodesics remain beneath it, in the so called convex core. This is the deformation which was used by É. Ghys in [Ghy07, ğ3.6] to isotope the master modular link into the Lorenz template (Theorem 4.24).

We then provide an algebraic parametrization by $q \in \mathbb{R}_+^*$ for the set of positively oriented Fuchsian representations up to conjugacy. It is obtained by fixing $S_q = S$, and considering for T_q the conjugate of T by $\exp\left(\frac{l}{2}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\right)$ where $q = \exp(l)$:

$$T_q = \begin{pmatrix} 1 & -q \\ q^{-1} & 0 \end{pmatrix} \quad \text{so} \quad R_q = T_q S_q^{-1} = \begin{pmatrix} q & 1 \\ 0 & q^{-1} \end{pmatrix} \quad \text{and} \quad L_q = T_q^{-1} S_q = \begin{pmatrix} q & 0 \\ 1 & q^{-1} \end{pmatrix}$$

In particular, the Teichmüller space of the abstract orbifold $\mathbb{M}$ is a closed connected real semi-algebraic set parametrized by $q \in \mathbb{R}_+^*$, and we denote $\mathbb{M}_q = \rho_q(\Gamma)\backslash\mathbb{HP}$.

One may consult [Thu97, FLP12, Bus92] to learn about the Teichmüller space of (mostly compact) Riemann surfaces.

The Universal and the Burau representations

Actually, we have just defined a representation $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}[q, q^{-1}])$ denoted $A \mapsto A_q$ which we call the *universal representation*. The matrix A_q is obtained from any $S\&T$-factorisation of A by replacing $T \mapsto T_q$, or equivalently from any $L\&R$-factorisation of A by replacing $L \mapsto L_q$ and $R \mapsto R_q$. Since every irreducible representation $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R})$ can be lifted to $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{R})$, this universal representation actually contains the information of all irreducible representations $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{C})$ up to conjugacy. In other terms the irreducible component of the $\mathrm{SL}_2(\mathbb{C})$-character variety of $\mathrm{PSL}_2(\mathbb{Z})$ is an affine line parametrized by $q \in \mathbb{C}$.

Every element $A \in \Gamma$ defines a function $\mathrm{disc}(A_q) = \mathrm{Tr}(A_q)^2 - 4 \in \mathbb{Z}[q, q^{-1}]$ on the Teichmüller space of $\mathbb{M}$, which is a reciprocal Laurent polynomial. When A is hyperbolic, this yields the length of the corresponding hyperbolic geodesic in $\mathbb{M}_q$. In that case, we show (in 5.10) that $\mathrm{disc}(A_q)$ is unitary of degree $2\,\mathrm{len}(A)$, that is twice the minimum displacement length for the action of A on the trivalent tree $\mathcal{T}$. This computation, which will serve in the next section, is a manifestation of the fact that as $q \to \infty$, the action of Γ on the hyperbolic plane $\mathbb{HP}$ through the representation $\rho_q \colon \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ converges in a suitable sense to its action on the trivalent tree $\mathcal{T}$.

Let us explain how one can make sense of this convergence, even though we will not need it here. Geometrically, the convex core of the orbifold $\mathbb{M}_q$ lifts in $\mathbb{HP}$ to an

ϵ-tubular neighbourhood of $\mathcal{T}$ with $\epsilon = \Theta(1/q^2)$ as $q \to \infty$. A dynamical viewpoint is to consider the actions of Γ on the boundaries $\partial\mathbb{HP}$ & $\partial\mathcal{T}$: this leads to the study of groups acting on the circle, for which we refer to [Ghy01]. An algebraic viewpoint, which is more in the spirit of this work, is to consider the Riemann-Zariski compactification of the character variety: the point $q = \infty$ corresponds to the valuation $-\deg_q$ centered at infinity. This approach for studying degenerations of hyperbolic structures was developped by Culler, Morgan and Shalen in [CS83, MS84], for which we refer to [Ota15].

We saw in Chapter 4 that the unit tangent bundle $\mathbb{U} \to \mathbb{M}$ of the modular orbifold has total space homeomorphic to the trefoil complement, and this yields a central extension $\mathcal{B}_3 \to \mathrm{PSL}_2(\mathbb{Z})$ of the modular group by the braid group on three strands. As for $\mathrm{PSL}_2(\mathbb{Z})$, the $\mathrm{SL}_2(\mathbb{C})$-character variety of $\mathcal{B}_3$ has its irreducible component (coming from irreducible representations, but which turns out to be irreducible as an algebraic variety) equal to an affine line. Hence it is not a surprise that we may relate the universal representation $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}[q, q^{-1}])$ to the Burau representation $\mathcal{B}_3 \to \mathrm{GL}_2(\mathbb{Z}[t, t^{-1}])$ of the braid group on three strands.

Indeed, we explicit a representation $\mathrm{Sq}\colon \mathcal{B}_3 \to \mathrm{SL}_2(\mathbb{Z}[q, q^{-1}])$ which is conjugate to the Burau representation with $q = \sqrt{-t}$, defined on the Artin generators by $\mathrm{Sq}(\sigma_1^{-1}) = q^{-1} \cdot L_q$ and $\mathrm{Sq}(\sigma_2) = q \cdot R_q$. In particular, we find that for $A \in \mathrm{PSL}_2(\mathbb{N})$ we have $\mathrm{Sq}(A) = q^{\mathrm{Rad}(A)} A_q$ where $\mathrm{Rad}(A) = \#R - \#L$ is its Rademacher invariant. Now recall from Chapter 4 that lifting modular geodesics to modular knots associates to the conjugacy class of a hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$ the conjugacy class of a braid $\sigma_A \in \mathcal{B}_3$, whose closure is a well defined link in the solid torus. We deduce that the Alexander polynomial of this link $[\sigma_A]$ is equal to

$$\Delta([\sigma_A]) = \frac{q^{\mathrm{Rad}(A)} - \mathrm{Tr}(A_q) + q^{-\mathrm{Rad}(A)}}{(q - q^{-1})^2}$$

This formula should be confronted with the observations in [Bir85] and the investigations of [Fun13]. Altogether, they support the conjecture 0.1 advanced at the end of Sections 0.1 & 0.2, namely that the arithmetic $\mathbb{Q}$-equivalence of $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ implies the "quantum equivalence" given by $\mathrm{Tr}(A_q) = \mathrm{Tr}(B_q)$ & $\mathrm{Rad}(A) = \mathrm{Rad}(B)$. More recently, the polynomials $\mathrm{Tr}(A_q)$ have been studied (after a change of variables) in [MGO20], and related to the Jones polynomials of certain knots.

In spite of the fully fledged theories concerning character varieties of Fuchsian groups and their compactifications, it seems that no one had bothered focusing on the character variety of the modular group, to recognise its unique boundary point as the well known action on the trivalent tree, and relate the universal representation of the modular group to the Burau representation of the braid group.

Asymptotic values of functions on the character variety

In this last section we prove Theorem 0.25 expressing the linking numbers between modular knots in terms of functions defined on the character variety of $\mathrm{PSL}_2(\mathbb{Z})$, by taking their limits at the boundary point $q = \infty$.

Fix two hyperbolic matrices $A, B \in \mathrm{PSL}_2(\mathbb{Z})$. Recalling the formalism developed at the end of Section 2.3, we define functions $\mathrm{L}_q([A],[B])$ and $\mathrm{C}_q([A],[B])$ of their conjugacy classes, by averaging conjugacy-invariants of pairs of conjugacy classes obtained from $\mathrm{bir}(A_q, B_q)$ and $\cos(A_q, B_q)$. Geometrically, they can be expressed as the following sums extending over the oriented intersection angles θ between the geodesics of $\mathbb{M}_q$ associated to A and B:

$$\mathrm{L}_q([A],[B]) = \sum \left(\cos\tfrac{\theta}{2}\right)^2 \qquad \mathrm{C}_q([A],[B]) = \sum \left(\cos\theta\right).$$

In Section 2.3, we described the relative positions between the combinatorial axes of hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ acting on $\mathcal{T}$. In particular we introduced the $\mathrm{cosign}(A, B)$ to compare their orientations along their intersection when it is not empty, and Proposition 2.44 showed that $\mathrm{cosign}(A, B) = \mathrm{len}(AB) - \mathrm{len}(AB^{-1})$. Using Proposition 5.10 computing $\deg \mathrm{Tr}(C_q) = \mathrm{len}(C)$ we deduce in Corollary 5.19 that $\cos(A_q, B_q) \to \mathrm{cosign}(A, B)$. Combining this with Theorem 4.44 expressing the linking number in terms of cosign, we find our Theorem 5.24, saying that:

$$\tfrac{1}{2}\,\mathrm{L}_q([A],[B]) \xrightarrow[q\to\infty]{} \mathrm{lk}(A,B) \qquad \tfrac{1}{2}\,\mathrm{C}_q([A],[B]) \xrightarrow[q\to\infty]{} 2\,\mathrm{lk}(A,B) - \tfrac{1}{2}I(A,B).$$

To finish, let us compare the definitions of the functions L_q and C_q and their limiting behaviour at $q = \infty$ with similar considerations which have been made for non-oriented loops in a closed surface S of genus $g \geq 2$. Such loops, corresponding to the conjugacy classes of $\alpha, \beta \in \pi_1(S)$ up to inversion, define trace functions $\mathrm{Tr}(\alpha), \mathrm{Tr}(\beta)$ on the $\mathrm{SL}_2(\mathbb{C})$-character variety of $\pi_1(S)$ (whose real locus contains the Teichmüller space of S as a Zariski dense open set). This character variety carries a natural symplectic structure [Gol84], given by the Weil-Petersson symplectic form.

The sum $\mathrm{C}_q(A, B)$ looks very much like Wolpert's cosine formula [Wol82, Wol81] computing the Poisson bracket $\{\mathrm{Tr}(\alpha), \mathrm{Tr}(\beta)\}$ of the trace functions. The major difference is that Wolpert's formula is a skew-symmetric expression in two non-oriented loops. In fact, we are able to define an analog of Wolpert's formula by summing the product $\mathrm{cross}(A, B) \times \cos(A, B)$. However, the character variety of $\mathrm{PSL}_2(\mathbb{Z})$ being one-dimensional, any Poisson structure in the usual sense would be trivial, so we expect this function to be zero (and this is confirmed by our computer experimentation).

Moreover, the Weil-Petersson symplectic form has been extended to several compactifications of the character variety [PP91, SB01, MS]. The limits of the Poisson bracket $\{\mathrm{Tr}(\alpha), \mathrm{Tr}(\beta)\}$ at the respective boundary points have been interpreted in [Bon92, Proposition 6] and [MS]. Let us also mention [Ota92] which was a source of inspiration in his understanding of cross-ratios.

Still, the definitions of our functions $\mathrm{L}_q \,\&\, \mathrm{C}_q$ may be generalised to oriented geodesics in hyperbolic surfaces, and we may wonder about *their limits at the boundary points of Teichmüller space*. Besides, we believe that the functions L_q would yield some kind of Killing form on Goldman's Lie algebra of oriented loops [Gol86].

5.1 The space of representations $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R})$

In this section, we describe the $\mathrm{PSL}_2(\mathbb{R})$-character variety of Γ, defined as the space of representations $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ considered up to $\mathrm{PSL}_2(\mathbb{R})$-conjugacy at the target.

Theorem 2.11 presented Γ as the free amalgam of its (cyclically ordered) subgroups $\mathbb{Z}/2$ and $\mathbb{Z}/3$ generated by S and T respectively. Consequently, a representation $\rho \colon \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ is uniquely determined by a pair $\rho(S), \rho(T)$ such that $\rho(S)^2 = \mathbb{1} = \rho(T)^3$, so their collection $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ is a real algebraic subvariety of $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$.

Among all representations, some are faithful, some are discrete, and we are mostly interested in those matching both conditions called *Fuchsian representations*.

Abelian representations

Let us first describe those representations ρ for which $\rho(S)$ and $\rho(T)$ have a common fixed point for their action on the hyperbolic plane. The image of ρ is thus contained in an abelian subgroup of $\mathrm{PSL}_2(\mathbb{R})$, thus in a conjugate of $\mathrm{PSO}_2(\mathbb{R})$, and factors through the abelianisation $\Gamma/[\Gamma, \Gamma] = \mathbb{Z}/2 \times \mathbb{Z}/3 = \mathbb{Z}/6$. Such representations belong to six conjugacy classes enumerated as follows.

One may have the trivial representation for which both $\rho(S)$ and $\rho(T)$ trivial, or else only $\rho(S)$ trivial in which case $\rho(\Gamma) \simeq \mathbb{Z}/3$ and there are two possibilities given by the orientation of rotation, or else only $\rho(T)$ trivial in which case $\rho(\Gamma) \simeq \mathbb{Z}/2$, or neither of them trivial in which case $\rho(\Gamma) \simeq \mathbb{Z}/6$ and again there are two possibilities given by the orientation of $\rho(T)$.

Non-abelian representations

From now on, the representation ρ is non-abelian, so $\rho(S)$ and $\rho(T)$ are rotations of order 2 and 3 around distinct fixed points. Then ρ is uniquely described by those fixed points, along with the orientation of the order three rotation. Its conjugacy class is uniquely determined by the distance $\lambda \in]0, \infty[$ between those fixed points, along with the orientation of $\rho(T)$.

The space of representations admits an involution (conjugating by J) exchanging the orientation of $\rho(T)$, whose only fixed points are the two abelian representations with $\rho(T) = \mathbb{1}$. The same goes for the character variety since T and T^{-1} are not conjugate in $\mathrm{PSL}_2(\mathbb{R})$. We speak of (conjugacy classes of) *positive* and *negative* non-abelian representations.

Hence there are two components in the space of non-abelian representations, and the same goes for the character variety which is thus homeomorphic to $\mathbb{R}_+^* \times \{\pm 1\}$. This homeomorphism is analytic but not algebraic.

Proposition 5.1. *Let $\rho \colon \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ be a positive representation, and λ_0 be the hyperbolic distance between the fixed points of $\rho(S)$ and $\rho(T)$, given by $\cosh \lambda_0 = 2$.*

If $\lambda \in [\lambda_0, \infty[$ then the representation is faithful and discrete, that is Fuchsian.

If $\lambda \in]0, \lambda_0[$ then the representation is not Fuchsian.

If $\lambda = 0$ then the representation is abelian, thus discrete but not faithful.

Proof. The image $\rho(\Gamma)$ is generated by $\rho(L)$ and $\rho(R)$, whose trace equal $\cosh(\lambda)$. Hence the result follows from the classification of Fuchsian groups [dSG10, VI]. □

One may replace the analytic parameter given by the hyperbolic distance λ between the fixed points of $\rho(S)$ and $\rho(T)$, by the algebraic parameter given by the discriminant of their product $\delta = \mathrm{disc}\, \rho(R)$. Recall that $\mathrm{disc} = \mathrm{Tr}^2 - 4$, so to change parameters we must relate $|\mathrm{Tr}\, \rho(R)|$ to λ. As we shall see, this relation is given by $\mathrm{Tr}\, \rho(R) = 2 \cosh \lambda' + \sinh \lambda'$ with $\lambda' = \lambda - \lambda_0$.

Corollary 5.2. *The non-abelian representations are determined by their orientation and discriminant $\delta > -4$; they satisfy the following dichotomy.*

If $\delta \in]-4, 0[$ then the representation is not Fuchsian.

If $\delta \in [0, \infty[$ then the representation is Fuchsian.

The value $\delta = 0$ for the algebraic parameter corresponds either to an abelian representation or to the inclusion $\mathrm{PSL}_2(\mathbb{Z}) \subset \mathrm{PSL}_2(\mathbb{R})$ which is Fuchsian.

Remark 5.3 (Non-faithful)**.** *The non-abelian and non-faitfull representations form a countable set of points in the interval parametrized either by $\lambda \in]0, \lambda_0[$ or by $\delta =$*

$\mathrm{disc}\,\rho(R) \in]-4, 0[$ *indexed by the integers* $n > 6$. *More precesely, the image of such a representation* ρ *is a hyperbolic triangle group* $(2, 3, n)$ *for some* $n > 6$ *which is minimal with the property that* $\rho(R)^n = \mathbb{1}$.

Analytic parametrization by the distance

The subset of non-abelian positive representations projects onto the character variety: let us parametrize a section, first analytically using the distance $\lambda' = \lambda - \lambda_0$.

Proposition 5.4. *For* $\lambda' \in \mathbb{R}$, *define the representation* $\rho_{\lambda'} \colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R})$ *by*

$$T_{\lambda'} = T \qquad S_{\lambda'} = \begin{pmatrix} \sinh\lambda' & -\cosh\lambda' \\ \cosh\lambda' & -\sinh\lambda' \end{pmatrix}.$$

The family of representations $\rho_{\lambda'}$ *analytically parametrized by* $\lambda' \in]-\lambda_0, \infty[$ *projects to all positive non-abelian representations up to conjuacy, and* $\lambda' \geq 0$ *if and only if the representation is Fuchsian.*

Proof. Consider a positive non-abelian representation ρ. Up to conjugacy, we may fix $\rho(T) = T_{\lambda'} = T$ and let the center of $\rho(S)$ vary along the half geodesic extending from the fixed point of T towards the fixed point of S. Let us show that when it reaches a distance λ' from S we have $\rho(S) = S_{\lambda'}$.

For this we work in the linear and projective models of the hyperbolic plane described in Section 1.6. The fixed points of S and T in $\mathbb{P}(\mathbb{H})$ lift to $\mathrm{pr}\,S = S$ and $\mathrm{pr}\,T$ in the hyperboloid $\mathbb{H}$. The line passing through them is $\{\mathrm{pr}\,S, \mathrm{pr}\,T\}^{\perp}$.

Since $T = S + \frac{1}{2}(\mathbb{1} - K)$, we have $\mathrm{pr}\,T = S - \frac{1}{2}K$, so $\{\mathrm{pr}\,S, \mathrm{pr}\,T\} = +\frac{1}{2}J$. Consequently, by Proposition 1.57, the hyperbolic translation of distance λ' along the oriented axis from the center of T to the center of S is:

$$\exp\left(\tfrac{\lambda'}{2}J\right) = \cosh(\lambda'/2)\mathbb{1} + \sinh(\lambda'/2)J = \begin{pmatrix} \cosh\lambda'/2 & \sinh\lambda'/2 \\ \sinh\lambda'/2 & \cosh\lambda'/2 \end{pmatrix}$$

and this matrix conjugates S to the matrix $S_{\lambda'}$.

The rest of the statement now follows from Proposition 5.1. □

Remark 5.5. *Notice that, denoting* $c = \cosh\lambda'$ *and* $s = \sinh\lambda'$, *the values of*

$$L_{\lambda'} = T_{\lambda'}^{-1} S_{\lambda'} = \begin{pmatrix} c & -s \\ c-s & c-s \end{pmatrix} \quad \text{and} \quad R_{\lambda'} = T_{\lambda'} S_{\lambda'}^{-1} = \begin{pmatrix} c-s & c-s \\ -s & c \end{pmatrix}$$

are still conjugate by J, *but are transpose to one another only when* $\sinh\lambda' = 0$.

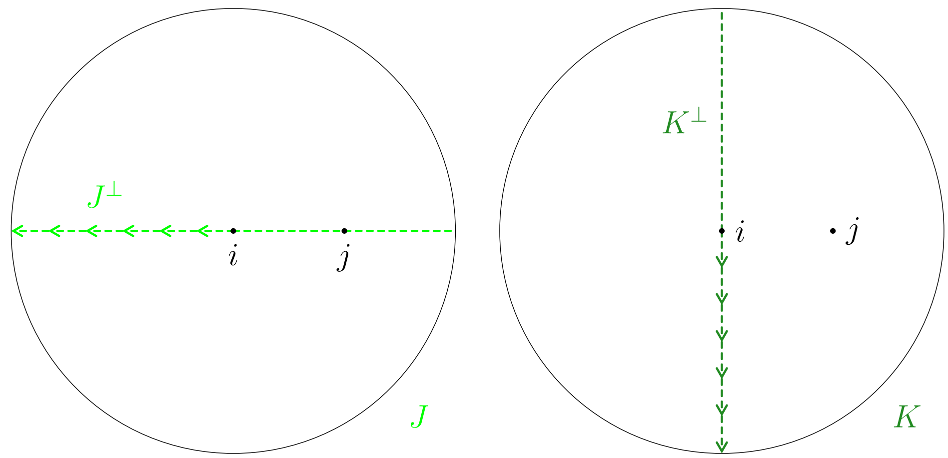


Figure 5.1: Parametrizing families of representations by moving S along $J^\perp$ or $K^\perp$.

Algebraic parametrization by the discriminant

The set of Fuchsian representations projects to (an open in) the character variety: let us parametrize a section by $q \in \mathbb{R}^*$ algebraically related to $\delta = (q - q^{-1})^2$.

This time we let S vary along the geodesic $\operatorname{pr} K^\perp$ joining the centers of S and T (that is the orthogonal to $\operatorname{pr} J^\perp$ passing through $\operatorname{pr} S$). This amounts to conjugating S by $\exp\left(\frac{l}{2}K\right)$ for some $l \in \mathbb{R}$ given by the distance to S.

For more symmetric formulae, we conjugate this whole family of representations to fix $S_q = S$ and let T_q be the conjugate of T by $\exp\left(-\frac{l}{2}K\right)$ where $q = \exp(l)$:

$$T_q = \begin{pmatrix} 1 & -q \\ q^{-1} & 0 \end{pmatrix} \quad \text{so} \quad R_q = T_q S_q^{-1} = \begin{pmatrix} q & 1 \\ 0 & q^{-1} \end{pmatrix} \quad \text{and} \quad L_q = T_q^{-1} S_q = \begin{pmatrix} q & 0 \\ 1 & q^{-1} \end{pmatrix}$$

which implies in particular that $\delta = \operatorname{disc} R_q = (q - q^{-1})^2$. This defines for every $q \in \mathbb{R}_+^*$ a representation

$$\bar{\rho}_q \colon \operatorname{PSL}_2(\mathbb{Z}) \to \operatorname{PSL}_2(\mathbb{R})$$

which is Fuchsian since $\delta > 0$. Note that $\bar{\rho}_q$ still makes sense for $q \in \mathbb{R}_-^*$.

Proposition 5.6. *The conjugacy classes of Fuchsian representations* $\operatorname{PSL}_2(\mathbb{Z}) \to \operatorname{PSL}_2(\mathbb{R})$ *form a closed real semi-algebraic set parametrized by* $(\operatorname{sign}(q), (q - q^{-1})^2)$ *for* $q \in \mathbb{R}^*$ *whose sign provides the orientation of the action on* $\mathbb{HP}$*.*

Proof. By Corollary 5.2, a value of $\delta = (q - q^{-1}) > 0$ determines a unique Fuchsian representation up to change of orientation, and it determines a unique value of q up to change of sign and inversion.

If $q > 0$ then T_q acts like a positive rotation since it is conjugate to T in $\mathrm{SL}_2(\mathbb{R})$. If $q < 0$ then T_q acts like a negative rotation since $T_{-q} = KT_qK^{-1}$ and $\det(K) = -1$.

The representations $\bar{\rho}_q$ and $\bar{\rho}_{q^{-1}}$ are conjugate in $\mathrm{PSL}_2(\mathbb{R})$ since $T_{q^{-1}} = JT_q^{-1}J^{-1}$ and T_q acts with the same orientation as $\det(J) = -1$. Explicitly $T_{q^{-1}}$ and T_q are conjugate by $\exp(\theta S)$ for a unique $-\pi/4 < \theta < \pi/4$ solving $\tan(\theta) = q - q^{-1}$. □

The matrices $S, T \in \mathrm{PSL}_2(\mathbb{Z})$ are the unique lifts of $S, T \in \mathrm{SL}_2(\mathbb{Z})$ which act as positive orientations on the plane $\mathbb{Z}^2$: this fixes the projection $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z})$. The group $\mathrm{SL}_2(\mathbb{Z})$ is the amalgam of its (oriented) cyclic subgroups $\mathbb{Z}/4$ and $\mathbb{Z}/6$ generated by S and T over their intersection $\mathbb{Z}/2$ generated by $S^2 = T^3 = -\mathbf{1}$.

We actually defined for every $q \in \mathbb{R}^*$ a representation

$$\rho_q \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{R})$$

Corollary 5.7. *The conjugacy classes of faithful discrete representations* $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{R})$ *form a closed semi-algebraic set parametrized by $q + q^{-1}$ for $q \in \mathbb{R}^*$ whose sign determines the orientation of the action on $\mathbb{R}^2$.*

Proof. A faithful discrete representation $\rho \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{R})$ projects to a Fuchsian representation $\bar{\rho} \colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R})$ which is conjugate to $\bar{\rho}_q$ for some $q \in \mathbb{R}^*$ which is unique up to inversion.

Conversely for all $q \in \mathbb{R}^*$, the lift of $\bar{\rho}_q$ to a representation $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R})$ is uniquely determined by the choices we made for the lifts of $S\&T$ This further lifts the double cover $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ in at most two ways ρ_q and ρ_{-q}, each one being determined by the orientation of the action of $\rho_q(S)$ on $\mathbb{R}^2$.

Finally ρ_q and $\rho_{q^{-1}}$ are conjugate by $\exp(\theta S) \in \mathrm{SL}_2(\mathbb{R})$ for $\tan(\theta) = q - q^{-1}$. □

5.2 The Universal and the Burau representations

The universal representation $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}[q, q^{-1}])$

We just introduced a one parameter family of representations $\rho_q \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{R})$ depending algebraically on the parameter $q \in \mathbb{R}^*$ and with integral coefficients. This leads to the definition of the *universal representation*.

Definition 5.8. *The* universal representation $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}[q, q^{-1}])$ *denoted $A \mapsto A_q$ is defined by $S \mapsto S_q$ and $T \mapsto T_q$, or equivalently by $L \mapsto L_q$ and $R \mapsto R_q$ where:*

$$S_q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T_q = \begin{pmatrix} 1 & -q \\ q^{-1} & 0 \end{pmatrix} \qquad R_q = \begin{pmatrix} q & 1 \\ 0 & q^{-1} \end{pmatrix} \quad L_q = \begin{pmatrix} q & 0 \\ 1 & q^{-1} \end{pmatrix}$$

Hence A_q is obtained from any S&T-factorisation of A by replacing $T \mapsto T_q$, or equivalently from any L&R-factorisation of A by replacing $L \mapsto L_q$ and $R \mapsto R_q$.

Definition 5.9. *The* Fricke polynomial *of $A \in \mathrm{SL}_2(\mathbb{Z})$ is $F_A(q) = \mathrm{Tr}(A_q) \in \mathbb{Z}[q, q^{-1}]$. The* discriminant polynomial *of $A \in \mathrm{PSL}_2(\mathbb{Z})$ is $\mathrm{disc}(A_q) = (\mathrm{Tr}\, A_q)^2 - 4 \in \mathbb{Z}[q, q^{-1}]$.*

Of course F_A is invariant by conjugacy and inversion of A, hence by transposition. Since ρ_q and $\rho_{q^{-1}}$ are conjugate, the polynomial F_A is reciprocal $F_A(q) = F_A(q^{-1})$, so its degree $\deg(F_A)$ is unambiguously defined by its highest monomial in q or q^{-1}. For instance $F_{L^n} = q^n + q^{-n} = F_{R^n}$ has degree n.

If $A \in \mathrm{SL}_2(\mathbb{Z})$ has finite order then it is conjugate to a power of S or T, so F_A is a constant determined by its order according to $F_{\pm 1} = \pm 2$, $F_{S^{\pm 1}} = 0$, $F_{T^{\pm 1}} = 1$.

Recall that $\mathrm{len}(A)$ is the minimum displacement length of A acting on $\mathcal{T}$.

Proposition 5.10. *If $A \in \mathrm{SL}_2(\mathbb{N})$ then $\mathrm{Tr}(A_q)$ is unitary of degree $\mathrm{len}(A)$, with non-negative coefficients and constant term $\mathrm{Tr}(A) \geq 2$.*

Hence, if $A \in \mathrm{PSL}_2(\mathbb{Z})$ has infinite order, then $\mathrm{disc}(A_q)$ is unitary of degree $2\,\mathrm{len}(A)$, with positive coefficients and constant term $\mathrm{disc}(A) \geq 0$.

Proof. Let us prove the first assertion, the second follows immediately. Recall that the monoid $\mathrm{SL}_2(\mathbb{N})$ is freely generated by L&R. The non-negativity of the coefficients is obvious and so is $\mathrm{Tr}(A) \geq 2$, so we are left to compute the degree and leading coefficient of $\mathrm{Tr}(A_q)$.

We reason by induction on $\mathrm{len}(A)$ using the trace relation $F_{UV} = F_U F_V - F_{UV^{-1}}$ which was equation 1.5 in Chapter 1. It is true for $\mathrm{len}(A) \leq 1$.

An astute way of performing the inductive step is to successively reduce the number of L's appearing in the factorisation of A to show that:

$$\mathrm{Tr}(A_q) = \mathrm{Tr}\left(R_q^{\mathrm{len}(A)}\right) + o\left(q^{\mathrm{len}(A)} + q^{-\mathrm{len}(A)}\right).$$

If $A = R^{\mathrm{len}(A)}$ then we are done. Otherwize we may assume (after conjugating) that $A = LA' = T^{-1}SA'$ with $\mathrm{len}(A') = \mathrm{len}(A) - 1$, and apply the trace relation:

$$\mathrm{Tr}\left(T_q S_q^{-1} A'_q\right) = \mathrm{Tr}(T_q)\,\mathrm{Tr}(S_q^{-1} A'_q) - \mathrm{Tr}(T_q S_q^{-1} A'_q) = \mathrm{Tr}(T_q S A'_q) - \mathrm{Tr}(A'_q S)$$

and the first term is $\mathrm{Tr}(R_q A'_q)$ and $\mathrm{len}(A'S) < \mathrm{len}(A)$ as reveals a simplification of the last two S's and a combination of the remaining extremal T's. □

Corollary 5.11. *If $A \in \mathrm{PSL}_2(\mathbb{Z})$ has infinite order, then for all $q \in \mathbb{R}^* \setminus \{1\}$ the element $A_q \in \mathrm{PSL}_2(\mathbb{R})$ is hyperbolic.*

The character ring: algebraic presentation & linear basis

This paragraph is not needed in what follows, but is meant show some properties of the character variety of $\mathrm{PSL}_2(\mathbb{Z})$ which can be generalised to other of other Fuchsian groups, while shedding light on its specificity.

Proposition 5.12. *The F_A generate the subring of reciprocal polynomials in $\mathbb{Z}[q, q^{-1}]$. The ideal of relations satisfied between them is generated by the trace relations:*

$$\forall A, B \in \mathrm{SL}_2(\mathbb{Z}) : \; F_{AB} + F_{AB^{-1}} = F_A F_B \quad \text{and} \qquad F_S = 0, \; F_T = 1.$$

Proof. The first sentence is now obvious: in fact the F_{R^n} provide a linear basis for the sub-algebra of reciprocal Laurent polynomials, which is graded by the degree.

Note that the $(F_R)^n$ provides another graded basis, in which the expression for the F_{R^n} is given by a family of Chebychev polynomials since they satisfy the recurrence relations $F_{R^n} = F_R F_{R^{n-1}} - F_{R^{n-2}}$.

To prove that the ideal of relations satisfied by the F_A is generated by the trace and unit relations, it is enough to show how they enable to decompose F_A in the linear basis spanned by the F_{R^n}. This follows by induction as in the previous proof. □

Let us simply enunciate the following description for the algebra of functions on the $\mathrm{SL}_2(\mathbb{R})$-character variety of a finitely generated group Γ in terms of the trace functions $F_C : [\rho] \mapsto \mathrm{Tr}\,\rho(C)$ for $C \in \Gamma$.

Theorem 5.13. *The algebra of functions on the $\mathrm{SL}_2(\mathbb{R})$-character variety of Γ is generated by the F_C for $C \in \Gamma$ and the ideal of relations is generated by the trace relations $F_{AB} + F_{AB^{-1}} = F_A F_B$ for $A, B \in \Gamma$ along with the unit relation $F_{\mathbb{1}} = 2$.*

Remark 5.14 (Exercise)**.** *It is an amusing exercise to recover the invariance of F_C by inversion and conjugacy of C from the trace relations and the unit relation.*

The Burau representation

Recall from [Squ84] that the reduced Burau representation $\mathrm{Br} \colon \mathcal{B}_3 \to \mathrm{GL}_2(\mathbb{Z}[t, t^{-1}])$ is defined on the Artin generators by:

$$\sigma_1 \mapsto \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \qquad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}$$

Conjugating by S, then setting $q = \sqrt{-t}$ and conjugating by the diagonal matrix with entries $1, q$ we obtain a similar representation $\mathrm{Sq} \colon \mathcal{B}_3 \to \mathrm{GL}_2(\mathbb{Z}[q, q^{-1}])$ defined on the Artin generators by:

$$\mathrm{Sq} \colon \sigma_1^{-1} \mapsto \tfrac{1}{q} \times L_q \qquad \mathrm{Sq} \colon \sigma_2 \mapsto q \times R_q$$

Thus for all $\beta \in \mathcal{B}_3$ we have $q^{-\operatorname{lk}(\beta)} \operatorname{Sq}(\beta) \in \mathrm{SL}_2(\mathbb{Z}[q, q^{-1}])$.

Remark 5.15. *The name* Sq *refers to* $\mathrm{SL}_2(\mathbb{Z}[q, q^{-1}])$ *and is meant to recall the similarity with the (slightly different) symplectic representation of Squier [Squ84].*

Recall the discussion in Section 4.1 concerning the abelianisation map $\operatorname{lk}\colon \mathcal{B}_3 \to \mathbb{Z}$, where we defined a morphism of monoids $\sigma\colon \mathrm{PSL}_2(\mathbb{N}) \to \mathcal{B}_3$ by $\sigma(L) = \sigma_1^{-1}$ and $\sigma(R) = \sigma_2$ which satisfies $\operatorname{lk} \sigma(A) = \operatorname{Rad}(A)$.

Proposition 5.16. *For all* $A \in \mathrm{PSL}_2(\mathbb{N})$ *we have* $\operatorname{Sq}(\sigma(A)) = q^{\operatorname{Rad}(A)} A_q$.

The link $\bar{\sigma}(A)$ *obtained from the cyclic closure of the braid* σ_A *has Alexander polynomial* $\Delta(\bar{\sigma}_A)(t) \in \mathbb{Z}[t, t^{-1}]$ *given, up to a multiple of the unit* $q = \sqrt{-t}$*, by:*

$$\Delta(\bar{\sigma}_A)(q) = \frac{q^{\operatorname{Rad}(A)} - \operatorname{Tr}(A_q) + q^{-Rad(A)}}{(q - q^{-1})^2}$$

Proof. The first assertion is immediate.

Following [BB05, 4.2], the Alexander polynomial of a braid $\beta \in \mathcal{B}_3$ is given in terms of the reduced Burau representation by

$$\Delta(\beta)(t) = \frac{\det(\operatorname{Br}(\beta) - \mathbb{1})}{1 + t + t^2}$$

By the Cayley-Hamilton identity, for $M \in \mathfrak{gl}_2$: $\det(M - \mathbb{1}) = \det(M) - \operatorname{Tr}(M) + 1$. Besides $1 + t + t^2 = q^2(q - q^{-1})^2$. Thus:

$$\Delta(\beta)(q) = \frac{\det(q^{\operatorname{Rad}(A)} A_q) - \operatorname{Tr}(q^{\operatorname{Rad}(A)} A_q) + 1}{q(q - q^{-1})^2} = \frac{q^{2\operatorname{Rad}(A)} - q^{\operatorname{Rad}(A)} \operatorname{Tr}(A_q) + 1}{q(q - q^{-1})^2}$$

The Alexander polynomial is only defind up to multiplication by a unit of the ring, and dividing the last expression by $q^{\operatorname{Rad}(A)-1}$ yields the desired result. □

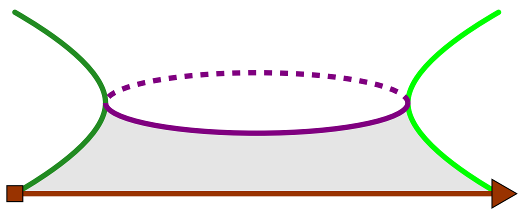
5.3 Asymptotic values of functions in $q \in \mathbb{R}^*$

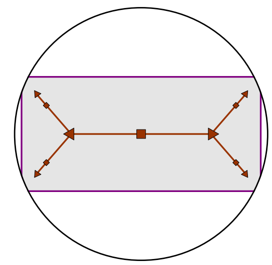
Before embarking on the algebraic discussion, let us think geometrically about the deformation of the hyperbolic metric of the orbifold $\mathbb{M}_q$ as $q \to \infty$.

The hyperbolic orbifold $\mathbb{M}_q = \rho_q(\Gamma) \backslash \mathbb{HP}$ has a convex core which retracts onto a thin neighbourhood of the long geodesic arc (i, j_q) connecting the conical singularities. It lifts in $\mathbb{HP}$ to an ϵ-neighbourhood of $\mathcal{T}_q$ with $\epsilon = \Theta(1/q^2)$, and whose collar-neck has width λ equal to the distance between the translation axes of L_q and R_q, that is such that:

$$\cosh\lambda = \left(\frac{q+q^{-1}}{q-q^{-1}}\right)^2$$

The hyperbolic geodesics γ_{A_q} of $\mathbb{M}_q$ becomes parallel to the edge (i, j_q) so the geometric axis γ_{A_q} comes closer and closer to the combinatorial axis g_A. In particular, the intersection angles converge to $\pm\pi$ and it appears that $\cos(A_q, B_b) \to \mathrm{cosign}(g_A, g_B)$.





The convex core of $\mathbb{M}_q$ lifts in $\mathbb{HP}$ to an ϵ-neighbourhood of $\mathcal{T}_q$ with $\epsilon = \Theta\left(1/q^2\right)$.

This should not surprise someone acquainted with compactifications of Teichmüller space by actions on trees or by valuations [Ota15, MS21]. Here the unique boundary point $q = \infty$ corresponds to the action on $\mathcal{T}$ or to the valuation $-\deg_q$.

The $\mathrm{cosign}(A, B)$ as limit of $\cos(A_q, B_q)$

Let us recall the following computations from chapter 1, in which we used to denote $\mathrm{tr}(M) = \frac{1}{2}\mathrm{Tr}(M)$ the half-trace.

Proposition 5.17. *Consider $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ and let $q \in \mathbb{R}^*$. Denoting a_q and b_q the orthogonal projections of A_q and B_q in $\mathfrak{sl}_2(\mathbb{R})$, we have:*

$$-\langle a_q \mid b_q\rangle = \frac{1}{4}\left(\mathrm{Tr}\, A_qB_q - \mathrm{Tr}\, A_qB_q^{-1}\right)$$

and therefore:

$$\cos(A_q, B_q) = \frac{\mathrm{Tr}(A_qB_q) - \mathrm{Tr}(A_qB_q^{-1})}{\sqrt{\mathrm{disc}(A_q)\,\mathrm{disc}(B_q)}}$$

Proof. Denote $x_q = \mathrm{tr}(A_q)$ and $y_q = \mathrm{tr}(B_q)$ so that $A_q = x_q + a_q$ and $B_q = y_q + b_q$. Then compute:

$$\begin{aligned}-\langle a_q \mid b_q\rangle &= \mathrm{tr}(a_qb_q) = \mathrm{tr}\left((A_q - x_q)(B_q - y_q)\right) = \mathrm{tr}(A_qB_q) - 2x_qy_q + x_qy_q \\ &= \mathrm{tr}(A_qB_q) - x_qy_q = \frac{1}{4}\left(2\,\mathrm{Tr}\, A_qB_q - \mathrm{Tr}\, A_q\,\mathrm{Tr}\, B_q\right)\end{aligned}$$

and the result follows by applying the trace identity.

The formula for $\cos(A_q, B_q)$ then follows from Proposition 1.89 expressing the cosine in terms of the scalar product, or more directly from Remark 1.51. □

For hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{R})$, we defined 1.93 the algebraic intersection number $\mathrm{cross}(A, B) \in \{-1, 0, 1\}$ and geometric intersection number $|\mathrm{cross}|(A, B) \in \{0, 1\}$ between their oriented axes $\gamma_A, \gamma_B \subset \mathbb{HP}$.

For infinite order $A, B \in \mathrm{PSL}_2(\mathbb{Z})$, their $\mathrm{cosign}(A, B) \in \{-1, 0, 1\}$ compares the orientation of their combinatorial axes $g_A, g_B \subset \mathcal{T}$ in the trivalent tree where they intersect, according to Definition 2.42.

The following statements are immediate corollaries to Proposition 5.10 computing the degrees of $\mathrm{Tr}(C_q)$ and $\mathrm{disc}(C_q)$.

Corollary 5.18. *For infinite order elements $A, B \in \mathrm{PSL}_2(\mathbb{Z})$, if $\mathrm{cosign}(A, B) \neq 0$ then $F_{AB} - F_{AB^{-1}}$ has degree $\mathrm{len}(A) + \mathrm{len}(B)$ and leading coefficient $\mathrm{cosign}(A, B)$.*

Corollary 5.19. *For hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ we have $\mathrm{cross}(A_q, B_q) = \mathrm{cross}(A, B)$, whence $|\mathrm{cross}|(A_q, B_q) = |\mathrm{cross}|(A, B)$.*

If $|\mathrm{cross}|(A, B) = 1$ then the hyperbolic elements $A_q, B_q \in \mathrm{PSL}_2(\mathbb{R})$ have oriented geometric axes γ_{A_q} and γ_{B_q} which intersect at an angle with cosine $\cos(A_q, B_q)$. In that case, this algebraic function on the $\mathrm{PSL}_2(\mathbb{R})$-character variety of $\mathrm{PSL}_2(\mathbb{Z})$ has a limit at the boundary point: $\cos(A_q, B_q) \xrightarrow[q\to\infty]{} \mathrm{cosign}(A, B)$.

Linking number as limit of the cross-ratio orbital sum

Now we put into practice the general procedure explained at the end of Section 2.3 to construct invariants of pairs of conjugacy classes: it involves a summation over pairs of representatives in each conjugacy class modulo centralisers and up to the diagonal action of the group.

Definition 5.20. *For conjugacy classes $[A], [B]$ of hyperbolic elements in $\mathrm{PSL}_2(\mathbb{Z})$, consider the functions of q defined by:*

$$\mathrm{L}_q([A],[B]) = \sum_{(U,V)} \left(\frac{[\![\mathrm{bir} > 1]\!]}{\mathrm{bir}}\right)\left(\tilde{A}_q, \tilde{B}_q\right) \qquad (\mathrm{L}_q)$$

$$\mathrm{C}_q([A],[B]) = \sum_{(U,V)} (|\mathrm{cross}| \times \cos)\left(\tilde{A}_q, \tilde{B}_q\right) \qquad (\mathrm{C}_q)$$

where the sums extend over pairs of representatives $\tilde{A} = UAU^{-1}$ and $\tilde{B} = VBV^{-1}$ for the conjugacy classes with $(U, V) \in \Gamma/\mathrm{Stab}(A) \times_\Gamma \Gamma/\mathrm{Stab}(B)$.

Remark 5.21. *Recall that by Corollary 1.94 for all $A_q, B_q \in \mathrm{PSL}_2(\mathbb{R})$ we have $|\mathrm{cross}|(A_q, B_q) = [\![\mathrm{bir}(A_q, B_q) > 1]\!]$, and by Corollary 5.19 for all $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ we have $|\mathrm{cross}|(A_q, B_q) = |\mathrm{cross}|(A, B)$. Thus we could also have written:*

$$\mathrm{L}_q([A],[B]) = \sum_{(U,V)} \frac{[\![\mathrm{bir}(A,B) > 1]\!]}{\mathrm{bir}(\tilde{A}_q, \tilde{B}_q)}$$

$$\mathrm{C}_q([A],[B]) = \sum_{(U,V)} |\mathrm{cross}|(A,B) \times \cos(\tilde{A}_q, \tilde{B}_q)$$

Remark 5.22. *The relation $\frac{1}{\mathrm{bir}} = \frac{1+\cos}{2}$ and Remark 4.46 imply that these functions are related through the geometric intersection function I by:*

$$\mathrm{L}_q = \frac{I + \mathrm{C}_q}{2}$$

Remark 5.23. *The appearance of the factors $[\![\mathrm{bir} > 1]\!] = |\mathrm{cross}|$ in the terms of L_q and C_q restricts the summations over the pairs of matrices whose axes intersect. Hence the support of the sums corresponds to the intersection points of the modular geodesics $[\gamma_A]$ and $[\gamma_B]$ associated to the conjugacy classes, which must be counted with appropriate multiplicity when A or B is not primitive, and we have:*

$$\mathrm{L}_q([A],[B]) = \sum \left(\cos\tfrac{\theta}{2}\right)^2 \qquad \text{and} \qquad \mathrm{C}_q([A],[B]) = \sum \left(\cos\theta\right).$$

Theorem 5.24. *For conjugacy classes of hyperbolic elements $A, B \in \mathrm{PSL}_2(\mathbb{Z})$, the limits of the functions $\mathrm{L}_q(A,B)$ and $\mathrm{C}_q(A,B)$ at the boundary point of the $\mathrm{PSL}_2(\mathbb{R})$-character variety of $\mathrm{PSL}_2(\mathbb{Z})$, recover the linking and intersection function of the corresponding modular knots and modular geodesics:*

$$\tfrac{1}{2}\,\mathrm{L}_q([A],[B]) \xrightarrow[q\to\infty]{} \mathrm{lk}(A,B)$$
$$\tfrac{1}{2}\,\mathrm{C}_q([A],[B]) \xrightarrow[q\to\infty]{} 2\,\mathrm{lk}(A,B) - \tfrac{1}{2}I(A,B)$$

Proof. The first limit follows from Theorem 4.44 and replace the terms of sums defining L_q and C_q by the expression for $\cos(A_q, B_q)$ obtained in corollary 5.19 and take the limit as $q \to \infty$. We recover the formulae obtained for the linking and intersection numbers in Theorem 4.44 and Remark 4.46. □

Computation and analytic behaviour of L_q and C_q

Recall the various expressions we have for the linking numbers $\mathrm{L}(A, B)$. From the topology of the Lorenz template $\mathbb{Y} \subset \mathbb{U}$ we deduced the algorithmic sum in Proposition 4.27, which we reformulated in Corollary 4.42 by exploiting the combinatorics of the trivalent tree $\mathcal{T}$, to finally arrive at the algebraic formulation of Theorem 4.44 in terms of the group $\mathrm{PSL}_2(\mathbb{Z})$.

Now we are facing the opposite problem: from the algebraic definition 5.20 for the q-deformed linking numbers $\mathrm{L}_q(A, B)$ we wish to extract a combinatorial expression which lends itself to algorithmic computations.

From algebra to combinatorics

Let us provide an algorithmic expression for the function $\mathrm{L}_q = \frac{1}{2}(I + \mathrm{C}_q)$ which may come in handy for practical purposes such as drawing its graph. It relies on the general discussion at the very end of Section 2.3.

First choose Lyndon representatives $A, B \in \mathrm{PSL}_2(\mathbb{N})$, defined in Corollary 2.19.

The intersection number $I([A], [B])$ can be computed rapidly in terms of the Lyndon representatives using Proposition 3.10, which as remarked in 4.46 amounts to expressing it in terms of the linking numbers $\frac{1}{2}I(A, B) = \mathrm{lk}(A, B) + \mathrm{lk}(A, {}^tB)$ and applying the Algo-Sum in Proposition 4.27.

To compute $\mathrm{C}_q([A], [B])$ recall the last two paragraphs of Section 2.3. We introduce the function $f(A, B) = |\mathrm{cross}|(A, B) \times \left(\mathrm{Tr}(A_q B_q) - \mathrm{Tr}(A_q B_q^{-1}\right)$ so that

$$\mathrm{C}_q([A], [B]) = \frac{F([A], [B])}{\sqrt{\mathrm{disc}(A_q)\,\mathrm{disc}(B_q)}} = \frac{F_-([A], [B]) + F_0([A], [B]) + F_+([A], [B])}{\sqrt{\mathrm{disc}(A_q)\,\mathrm{disc}(B_q)}}$$

Since f satisfies $(|\mathrm{cross}|(A, B) = 0 \implies f(A, B) = 0)$ we have $F_0 = 0$ and as $f(A, B^{-1}) = -f(A, B)$ we have $F_-([A], [B]) = F_+([A], [{}^tB])$. We are thus reduced to computing sums of the form:

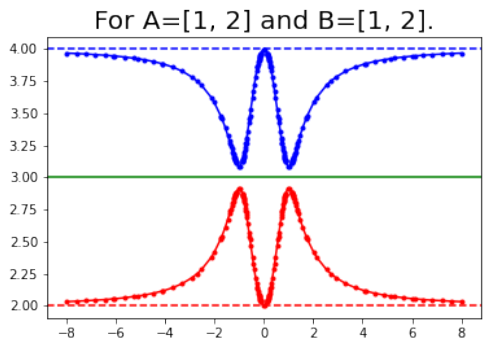
$$F_+([A], [B]) = \sum_{i=1}^{\mathrm{len}(A)} \sum_{j=1}^{\mathrm{len}(B)} f\left(\sigma^i A, \sigma^j B\right)$$

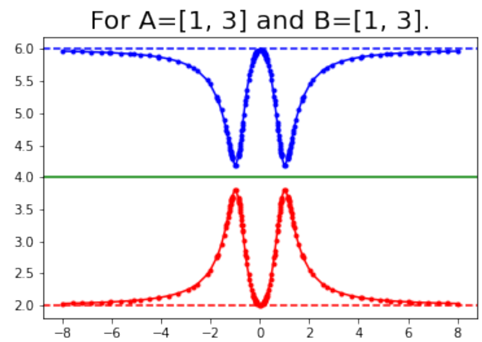
which by definition equals the sum of $\mathrm{Tr}(\sigma^i A_q)(\sigma^j B_q - (\sigma^j B_q)^{-1})$ over pairs of crossing Lyndon representatives. Of course one makes first the cyclic permutations and then replaces $L \mapsto L_q$ and $R \mapsto R_q$, at the last moment changing the word into a matrix with coefficients in $\mathrm{SL}_2(\mathbb{N}[q, q^{-1}])$.

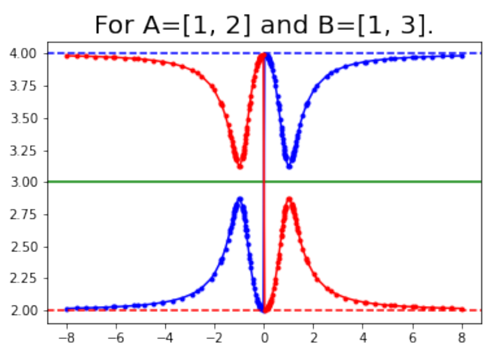
Real graphs of $\mathrm{L}_q(A, B)$

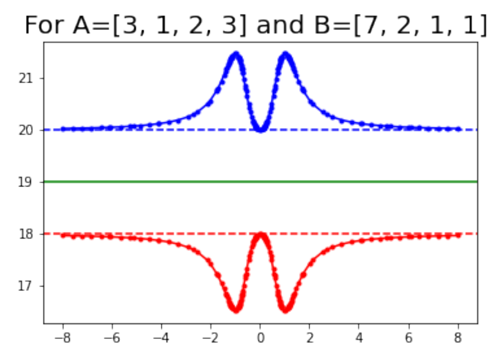
Let us now display some graphs graphs of $\mathrm{L}_q(A, B)$, $\mathrm{L}_q(A, B^{-1})$ and their average $I(A, B) = I(A, B^{-1})$ for various elements $A, B \in \mathrm{PSL}_2(\mathbb{N})$ designed by the sequence of exponents appearing in their $R\&L$ factorisation (starting with an R).

These graphs coroborate Theorem 5.24 and Remark 4.46.









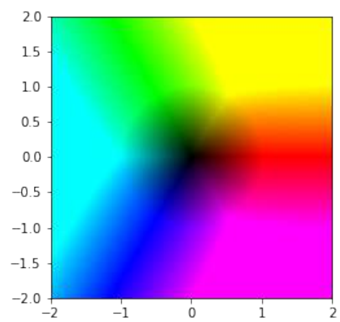
Monotocity at infinity. The functions $\mathrm{L}_q([A], [B])$ seem to be monotonous on the interval $]1, \infty[$. This is neither surprising nor completely obvious.

First note that for all $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ such that $|\mathrm{cross}|(A, B) = 1$, there exists $q_0(A, B) \geq 1$ such that $\cos(A_q, B_q)$ is monotonous on $]q_0(A, B), \infty[$. Indeed $\cos(A_q, B_q)$ is a rational function so it has a finite number of , and it has finite limit $\mathrm{cosign}(A, B)$ at $q = \infty$. If we think of the geodesics $[\gamma_{A_q}]$ and $[\gamma_{B_q}]$ in $\mathbb{M}_q$ while it undergoes the deformation, we expect that $q_0(A, B) = 1$ for all pairs (A, B) associated to an intersecting angle which is far from the singularities. Then we must consider the sum over all pairs of Lyndon representatives (A, B) for $[A], [B]$ such that $|\mathrm{cross}|(A, B) = 1$, and the $\cos(A_q, B_q)$ may not all vary in the same direction inven on a neighbourhood of $+\infty$: some will decrease to -1 and other increase to $+1$.

Complex graphs of $\mathrm{L}_q(A, B)$

Finally, we cannot resist showing the graphs of $\mathrm{L}_q(A, B)$ for $q \in \mathbb{C}$. Since $\mathrm{L}_q = \mathrm{L}_{1/q}$ we bound the module $|q| < 1 + \epsilon$ for a small value of $\epsilon > 0$ chosen according to aesthetic criteria.

For this we assign a colour to each point of the complex plane according to the HSV colour scheme: the hue varies according to the argument, and the brightness varies according to the module. Since a picture is worth a thousand words, we display beneath the graph of the identity map for $|q| < 2$.



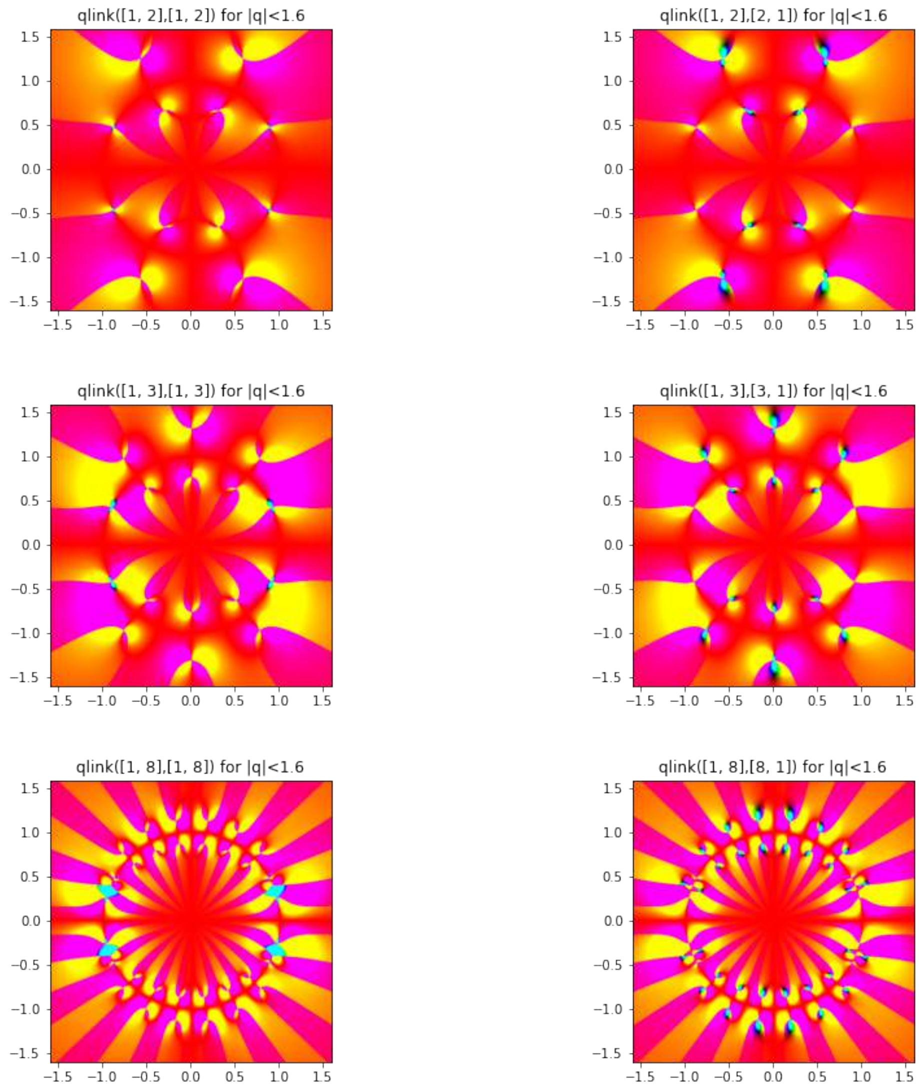
The identity map for $q \in \mathbb{C}$ with $|q| < 2$.

Location of zeros and poles. The main general observation is that the zeros and poles of $\mathrm{L}_q(A, B)$ tend to concentrate on the unit circle.

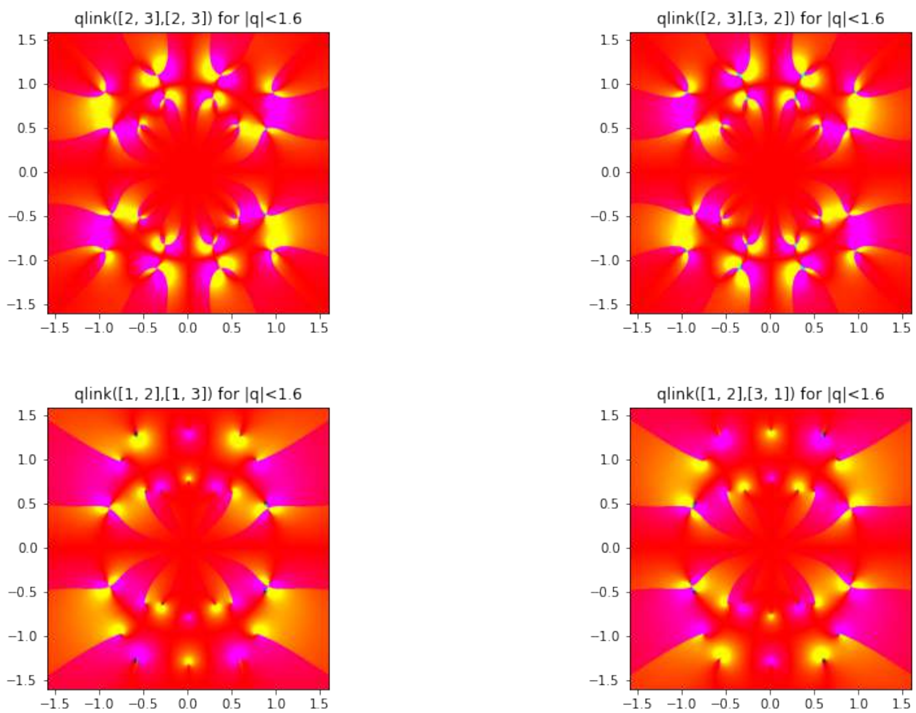
Again this is neither surprising nor obvious. It is not surprising in view of the fact that the polynomials $\mathrm{Tr}(C_q)$ are closely related to Alexander polynomials of braids on three strands as we saw in Proposition 5.16. The zeros of Alexander polynomials of knots and links with braid have been subject to various studies (see for instance [Sto19] for a specific study of those with braid index 3). We should also mention that [Deh15] has shown such a concentration property for the zeroes of the Alexander polynomial of a Lorenz knot: they lie an annulus whose inner and outer radii are bounded in terms the genus and the braid index of the knot. Still, it would remain a challenge to generalise such localisation results to the functions $\mathrm{L}_q([A], [B])$.

Symmetric matrices $\mathrm{L}_q(A, B)$**.** There are of course many other patterns to observe on the shape of $\mathrm{L}_q(A, B)$ when we let A, B vary in special families.

For instance, one may ask about the properties of the symmetric q-linking matrix $\mathrm{L}_q(A, B)$ for elements A, B varying in a given class group $\mathrm{Cl}(\Delta)$.



Graphs of $\mathrm{L}_q(A, B)$ for complex q.



Graphs of $\mathrm{L}_q(A, B)$ for complex q.

Glossary

Generalities for set theory and topological spaces.

Number sets. Inclusion of sets $X \subset Y$ are always understood in the wide sense. In an ordered set we denote by $[x, y[$ the interval closed at x and open at y.

$\mathbb{N}$, $\mathbb{Z}$ Monoid of non-negative integers $\mathbb{N} = \{0, 1, \dots\}$. Ring of integers $\mathbb{Z}$.

$\mathbb{Q}$, $\mathbb{Q}_p$ Field of rational numbers $\mathbb{Q}$. Field of p-adic numbers.

$\mathbb{R}$, $\mathbb{C}$ Field $\mathbb{R}$ of real numbers. Field $\mathbb{C}$ of complex numbers.

$\mathbb{K}$, $\sqrt{\mathbb{K}}$ Field $\mathbb{K}$ of characteristic different from 2. Its quadratic closure $\sqrt{\mathbb{K}}$.

$\mathbb{K}^\times$, $(\mathbb{K}^\times)^2$ The invertible elements of $\mathbb{K}$ form a group $\mathbb{K}^\times$ with subgroup of squares $(\mathbb{K}^\times)^2$.

Topological spaces and geometric constructions. The boundary operator ∂ has various meanings depending on the context: we use it mostly for the boundary of subsets in topological spaces, and the Gromov boundary of hyperbolic spaces.

$\mathbb{P}(V)$ Projectivization of a vector space V.

$\mathbb{KP}^n$ Projective space $\mathbb{P}(\mathbb{K}^{n+1})$ of dimension n over the field $\mathbb{K}$.

$\mathbb{S}^n$ The n-dimensional sphere.

$\mathbb{D}^n$ The n-dimensional disc (sometimes the context refers to its interior).

$\mathbb{HP}$ The hyperbolic plane (without reference to a particular model).

Notations introduced or appearing in Chapter 1

The algebra $\mathfrak{gl}_2(\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 4 with a preferred basis:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

These elements form, together with their opposites, a subgroup of $\mathrm{GL}_2(\mathbb{Z})$ isomorphic to the dihedral group $\mathcal{D}_4$. See Figure 1.1.

$\mathbb{V}$ Vector space of dimension 2 over the field $\mathbb{K}$.

ω Often used to denote a symplectic form on $\mathbb{V}$.

$\mathfrak{gl}(\mathbb{V})$ Algebra of endomorphisms of $\mathbb{V}$, and $\mathfrak{gl}_2(\mathbb{K}) := \mathfrak{gl}_2(\mathbb{K}^2)$.

$\mathrm{GL}(\mathbb{V})$ Group of automorphisms of $\mathbb{V}$ and $\mathrm{GL}_2(\mathbb{K}) := \mathrm{GL}(\mathbb{K}^2)$.

$\mathfrak{sl}(\mathbb{V})$ Endomorphisms of $\mathbb{V}$ with trace 0, and $\mathfrak{sl}_2(\mathbb{K}) := \mathfrak{sl}_2(\mathbb{K}^2)$.

$\mathrm{SL}(\mathbb{V})$ Group of automorphisms of $\mathbb{V}$ with determinant 1 and $\mathrm{SL}_2(\mathbb{K}) := \mathrm{SL}(\mathbb{K}^2)$

$M^\#$ is the adjoint of $M \in \mathfrak{gl}(\mathbb{V})$, or the transpose comatrix of $M \in \mathfrak{gl}_2(\mathbb{K})$.

$\mathrm{Tr}(M)$ Trace of an endomorphism or matrix, given by $M + M^\#$ if $M \in \mathfrak{gl}(\mathbb{V})$.

$\det(M)$ Determinant of an endomorphism or matrix, given by $MM^\#$ of $M \in \mathfrak{gl}(\mathbb{V})$.

$\mathrm{disc}(M)$ $= \mathrm{Tr}(M)^2 - 4\det(M)$ is the discriminant of $M \in \mathfrak{gl}(\mathbb{V})$.

$\langle M, N\rangle$ $= \frac{1}{2}\mathrm{Tr}(MN^\#)$ the bilinear form polarising the quadratic form det on $\mathfrak{gl}(\mathbb{V})$.

$\{M, N\}$ $= \frac{1}{2}(MN - NM)$ the commutator of $M, N \in \mathfrak{gl}(\mathbb{V})$, usually for $M, N \in \mathfrak{sl}(\mathbb{V})$.

$\mathrm{tr}(M)$ Orthogonal projection $\mathrm{tr}\colon \mathfrak{gl}_2(\mathbb{K}) \to \mathbb{K}\mathbb{1}$ with respect to det, which yields the half-trace also denoted $\mathrm{tr}(M) = \frac{1}{2}\mathrm{Tr}(M)$.

$\mathrm{pr}(M)$ Orthogonal projection $\mathrm{pr}\colon \mathfrak{gl}(\mathbb{V}) \to \mathfrak{sl}(\mathbb{V})$ with respect to det. Preserves disc.

$\mathbb{1}, S, J, K$ These matrices defined above, which form an orthogonal-basis of $(\mathfrak{gl}_2(\mathbb{K}), \det)$ respecting the decomposition $\mathfrak{gl}_2(\mathbb{K}) = \mathbb{K}\mathbb{1} \oplus \mathfrak{sl}_2(\mathbb{K})$.

$\mathbb{X}$ Isotropic cone in the quadratic space $(\mathfrak{sl}(\mathbb{V}), \det)$.

$\mathbb{H}$ $= \mathfrak{sl}(\mathbb{V}) \cap \mathrm{SL}(\mathbb{V}) \subset \mathfrak{gl}(\mathbb{V})$ which over $\mathbb{R}$ is a double-sheeted hyperboloid.

$\mathbb{H}'$ $= \{\mathfrak{a} \in \mathfrak{sl}(\mathbb{V}) \mid \det(\mathfrak{a}) = -1\}$ which over $\mathbb{R}$ is a single-sheeted hyperboloid.

$\perp$ The orthogonality relation in the non-degenerate quadratic space $(\mathfrak{sl}_2(\mathbb{K}), \det)$ or the corresponding polarity relation in the projective plane $\mathbb{P}(\mathfrak{sl}_2(\mathbb{K}))$ with respect to the non degenerate conic $\mathbb{P}(\mathbb{X})$.

ψ Parametrization of the isotropic cone $\mathbb{K}^2 \to \mathbb{X} \subset \mathfrak{sl}_2(\mathbb{K})$ or $\mathbb{V} \to \mathbb{X} \subset \mathfrak{sl}(\mathbb{V})$). Defined using coordinates in Lemma 1.33, and intrinsically in Lemma 1.36. The function $\bar{\psi}$ is defined in Proposition 2.5 of Chapter 2.

bir The cross-ratio $\mathrm{bir}(u, v, x, y)$ of $u, v, x, y \in \mathbb{KP}^1$, the cross-ratio $\mathrm{bir}(\mathfrak{a}, \mathfrak{b})$ of $\mathfrak{a}, \mathfrak{b} \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ or the cross-ratio of $A, B \in \mathrm{PGL}(\mathbb{V})$ are defined in Section 1.3.

cos The cosine $\cos(\mathfrak{a}, \mathfrak{b})$ of $\mathfrak{a}, \mathfrak{b} \in \mathfrak{sl}(\mathbb{V}) \setminus \mathbb{X}$ with respect to det. The cosine $\cos(A, B)$ of $A, B \in \mathrm{PGL}(\mathbb{V})$ is defined in Section 1.3.

cord The cyclic order $\mathrm{cord}(x, y, z) \in \{-1, 0, 1\}$ of three points in an oriented circle.

cross $\in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. The algebraic intersection number $\mathrm{cross}(w, z)$ between two chords $w = (u, v)$ and $z = (x, y)$ with endpoints in an oriented circle.

In particular for geodesics in $\mathbb{HP}$ or $\mathcal{T}$, for instance the translations axes of hyperbolic transformations acting on those spaces.

|cross| $\in \{0, \frac{1}{2}, 1\}$. The absolute value of cross.

$\mathcal{Q}(\mathbb{K})$ The vector space of binary quadratic forms.

$\mathcal{Q}(\mathbb{Z})$ The module of integral binary quadratic forms.

$\mathrm{Cl}(\Delta)$ The class group of discriminant Δ.

$\mathrm{Norm}_{\mathbb{K}}$ The norm of an algebraic $\mathbb{K}$-extension. If α is quadratic then $\mathrm{Norm}_{\mathbb{K}}(\alpha) = \alpha\alpha'$.

$\mathcal{P}$ The set of primes (with -1 as prime at infinity), see Section 1.5 for $\mathcal{P}(Q_a, Q_b)$.

$\mathbb{Q}_p$ Field of p-adic numbers, completion of $\mathbb{Q}$ at place p

$(\delta, \chi)_p$ Hilbert symbol of (δ, χ) at prime p defined in Section 1.5.

Notations introduced or appearing in Chapter 2

Throughout the thesis, we stick to the following notations regarding special elements of $\mathrm{SL}_2(\mathbb{Z})$. They satisfy $S^2 = T^3 = -\mathbb{1}$ as well as $L = T^{-1}S$ and $R = TS^{-1}$.

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

We denote by the same letters their classes in $\mathrm{PSL}_2(\mathbb{Z})$. The elements $S\&T$ act on the hyperbolic plane $\mathbb{HP}$ with fixed points denoted $i\&j$.

The submonoid $\mathrm{SL}_2(\mathbb{N}) \subset \mathrm{SL}_2(\mathbb{Z})$ of matrices with non-negative entries is freely generated by $L\&R$ and we identify it with its image $\mathrm{PSL}_2(\mathbb{N}) \subset \mathrm{PSL}_2(\mathbb{Z})$.

We often use $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ to denote hyperbolic elements with attractive fixed points $\alpha, \beta \in \mathbb{RP}^1$ and repulsive fixed points their Galois conjugates $\alpha', \beta' \in \mathbb{RP}^1$.

- $\lfloor x \rfloor$ Euclidean continued fraction expansion of $x \in \mathbb{RP}^1$.
- $\triangle$ The Lagrangian complex $\triangle_2$, the Lotus $\triangle_1$ and its quadruple $\triangle_4$.
- $\triangledown$ The base triangles $\triangledown_2$ of $\triangle_2$ and $\triangle_1$ of $\triangle_1$.
- $\triangle', \triangledown'$ The first barycentric subdivisions of $\triangle$ and $\triangledown$.
- $\bar{\psi}$, $\mathbb{P}\bar{\psi}$ Proposition 2.5 introduces a map $\bar{\psi} \colon \mathbb{R}^2 \to \mathfrak{sl}_2(\mathbb{R})$ rectifying $\psi \colon \mathbb{R}^2 \to \mathbb{X}$ in order to define a simplicial map $\mathbb{P}\bar{\psi} \colon \triangle_4 \to \triangle_2$ between geometric realisations.
- $\mathcal{T}$, $\mathcal{T}'$ The infinite planar trivalent tree $\mathcal{T}$, with first barycentric subdivision $\mathcal{T}'$.
- $\partial\mathcal{T} = \partial\mathcal{T}'$ The boundaries of $\mathcal{T}$ or $\mathcal{T}'$, which are identified with $\partial\mathbb{HP} = \mathbb{RP}^1$.
- $\mathcal{G}$ The subset of bi-infinite geodesics of $\mathcal{T}$ or equivalently of $\mathcal{T}'$.
- g_A The combinatorial axis in $\mathcal{T}$ or $\mathcal{T}'$ of a hyperbolic translation $A \in \mathrm{PSL}_2(\mathbb{Z})$.
- len The minimum displacement length of $A \in \mathrm{PSL}_2(\mathbb{Z})$ acting on $\mathcal{T}$. For torsion A it is zero, and for $A \in \mathrm{PSL}_2(\mathbb{N})$ it is the length of its $L\&R$-factorisation.
- sinc, cosign Definition 2.42: For oriented geodesics $g_a, g_b \subset \mathcal{T}$, let $\mathrm{sinc}(g_a, g_b) \in \mathbb{Z} \cup \{\pm\infty\}$ be the length of their intersection, whose sign $\mathrm{cosign}(g_a, g_b) \in \{-1, 0, +1\}$ compares their orientations along their intersection when it is not empty.

  These functions are defined on $\mathcal{G} \times \mathcal{G}$, whence on pairs of translations axes (g_A, g_B) for hyperbolic transformations $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ of $\mathcal{T}$.
- coc $= \frac{1}{2}(1 + \mathrm{cross}) \times \frac{1}{2}(1 + \mathrm{cosign})$

Notations appearing mostly in Chapters 3, 4, 5

$\mathbb{M}$ $= \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{HP}$ is the modular orbifold, its conical singularities of order 2 & 3 are denoted $[i]$ & $[j]$ or abusively i & j.

$\mathfrak{s}$&$\mathfrak{t}$ Loops encircling the punctures i&j of the thrice punctured sphere $\mathbb{M} \setminus \{i, j\}$. They freely generate its fundamental group. See figures 3.4 and 3.7.

I Geometric intersection number between loops in an orbifold or surface, or the minimal intersection number between homotopy classes thereof.

$\mathbb{T}^*$ The 6-fold abelian Galois cover of $\mathbb{M}$ is a 2-dimensional punctured torus $\mathbb{T}^*$.

$\mathcal{F}_2$ A free group on two generators, often used for $\pi_1(\mathbb{T}^*)$.

$\mathcal{H}$, $\mathcal{H}'$ The planar hexagonal graph $\mathcal{H}$, with first barycentric subdivision $\mathcal{H}'$.

Rad The Rademacher conjugacy invariant of an infinite order $A \in \mathrm{PSL}_2(\mathbb{Z})$. Defined at the end of Section 3.2 as the asymptotic winding number of its combinatorial axis g_A mod $\mathrm{PSL}_2(\mathbb{Z})'' \subset \mathcal{H}$, which is equal to $\#R - \#L$ if $A \in \mathrm{PSL}_2(\mathbb{N})$.

$\mathbb{U}$ $= \mathrm{PSL}_2(\mathbb{Z})\backslash \mathrm{PSL}_2(\mathbb{R})$ the unit tangent bundle to the modular orbifold $\mathbb{M}$, which is homeomorphic to the complement of a trefoil knot in $\mathbb{S}^3$.

$\mathcal{B}_3$, $\sigma(A)$ The braid group on three strands with presentation $\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. It is isomorphic to $\pi_1(\mathbb{U})$ and we also use the presentation $\langle a, b \mid a^2 = b^3 \rangle$ with $a = \sigma_1\sigma_2\sigma_1$ and $b = \sigma_1\sigma_2$. The monomorphism of monoids $\sigma\colon \mathrm{PSL}_2(\mathbb{N}) \to \mathcal{B}_3$ defined by $\sigma(L) = \sigma_1^{-1}$ and $\sigma(R) = \sigma_2$.

$\mathfrak{s}, \mathfrak{t}, \mathfrak{u}$ The unit tangent bundle of the thrice punctured sphere $\mathbb{M} \setminus \{i, j\}$ has fundamental group $(\mathbb{Z} * \mathbb{Z}) \times \mathbb{Z}$ generated by $\mathfrak{s}, \mathfrak{t}, \mathfrak{u}$.

$\mathbb{Y}$ The Lorenz template introduced in Section 4.2.

$[\![P]\!]$ Iverson's convention for the truth value in $[\![P]\!] \in \{0, 1\}$ of a proposition P.

lk The linking number, especially between knots in $\mathbb{U}$.

$\sigma(A)$ Cyclic or Bernoulli shift of a finite or infinite binary sequence like.

Not to be confused with the monoid of morphisms $\sigma\colon \mathrm{PSL}_2(\mathbb{N}) \to \mathcal{B}_3$.

A^∞ Periodisation of a finite binary sequence into an infinite binary sequence.

pref, occ See Definition 4.31. For a pattern $P \in \mathrm{PSL}_2(\mathbb{N})$ and a hyperbolic $A \in \mathrm{PSL}_2(\mathbb{N})$, $\mathrm{pref}(P, A^\infty) = [\![A^\infty \in P \cdot \mathrm{PSL}_2(\mathbb{N})]\!] \in \{0, 1\}$ tells whether P is a prefix of A^∞, and $\mathrm{occ}(P, A)$ counts the number of cyclic occurrences of P in A.

Br, Sq The reduced Burau representation $\mathrm{Br}\colon \mathcal{B}_3 \to \mathrm{PSL}_2(\mathbb{Z}[t, t^{-1}])$ and the representation $\mathrm{Sq}\colon\ \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}[q, q^{-1}])$ are recalled and defined just before 5.15.

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Résumé: Dans cette thèse consacrée au groupe modulaire $\mathrm{PSL}_2(\mathbb{Z})$, on étudie plusieurs structures arithmétiques et topologiques sur l'ensemble de ses classes de conjugaison, comme des relations d'équivalence ou des fonctions bilinéaires.
Le groupe modulaire $\mathrm{PSL}_2(\mathbb{Z})$ agit sur le plan hyperbolique avec pour quotient la surface modulaire $\mathbb{M}$, dont le fibré tangent unitaire $\mathbb{U}$ est une variété de dimension 3 homéomorphe au complémentaire d'un nœud de trèfle dans la sphère. Les nœuds modulaires dans $\mathbb{U}$ sont les orbites périodiques du flot géodésique, relevés des géodésiques fermées orientées de $\mathbb{M}$, qui correspondent aux classes de conjugaison hyperboliques dans $\mathrm{PSL}_2(\mathbb{Z})$. Leur enlacement avec le nœud de trèfle est bien compris. On s'intéresse aux nombres d'enlacement entre ces nœuds modulaires, pour lesquels on détermine plusieurs expressions exploitant la combinatoire, la dynamique ou l'algèbre du groupe modulaire. En particulier, on associe à deux nœuds modulaires une fonction définie sur la variété des caractères de $\mathrm{PSL}_2(\mathbb{Z})$, dont la limite au bord retrouve leur enlacement.
Les matrices hyperboliques de $\mathrm{PSL}_2(\mathbb{Z})$ indexent aussi diverses familles d'objets arithmétiques, telles que les formes quadratiques binaires entières indéfinies. Pour un corps $\mathbb{K}$ contenant $\mathbb{Q}$, on dit que deux matrices de $\mathrm{PSL}_2(\mathbb{Z})$ sont $\mathbb{K}$-équivalentes si elles sont conjuguées par un élément de $\mathrm{PSL}_2(\mathbb{K})$. On décrit comment le groupement des $\mathrm{PSL}_2(\mathbb{Z})$-classes en $\mathbb{K}$-classes varie avec $\mathbb{K}$. Pour $\mathbb{K} = \mathbb{C}$ cela revient à regrouper les formes quadratiques d'un même discriminant, ou les géodésiques modulaires de même longueur. Pour $\mathbb{K} = \mathbb{Q}$ on obtient un raffinement de cette relation d'équivalence, que l'on relie à l'arithmétique des formes quadratiques (symboles de Hilbert) et que l'on décrit en termes des géodésiques modulaires (angles aux points d'intersection et longueurs des ortho-géodésiques).

Abstract: In this work, dedicated to the modular group $\mathrm{PSL}_2(\mathbb{Z})$, we investigate several arithmetical and topological structures underlying its set of conjugacy classes, such as equivalence relations and bilinear pairings.
The modular group $\mathrm{PSL}_2(\mathbb{Z})$ acts on the hyperbolic plane $\mathbb{HP}$ with quotient the modular surface $\mathbb{M}$, whose unit tangent bundle $\mathbb{U}$ is a 3-manifold homeomorphic to the complement of the trefoil knot in the sphere $\mathbb{S}^3$. The modular knots in $\mathbb{U}$ are the periodic orbits for the geodesic flow, the lifts of closed oriented geodesics in $\mathbb{M}$, which correspond to hyperbolic conjugacy classes in $\mathrm{PSL}_2(\mathbb{Z})$. Their linking numbers with the trefoil is well understood. We are concerned with the linking numbers between modular knots and derive several formulae with combinatorial, dynamical or group theoretical flavour. In particular, we associate to a pair of modular knots a function defined on the character variety of $\mathrm{PSL}_2(\mathbb{Z})$, whose limit at the boundary point recovers their linking number.
The hyperbolic matrices in the modular group $\mathrm{PSL}_2(\mathbb{Z})$ also parametrize various families of objects in arithmetics such as indefinite integral binary quadratic forms. For a field extension $\mathbb{K}$ of $\mathbb{Q}$, we consider two matrices in $\mathrm{PSL}_2(\mathbb{Z})$ as $\mathbb{K}$-equivalent when they are conjugate by an element in $\mathrm{PSL}_2(\mathbb{K})$. The set of $\mathrm{PSL}_2(\mathbb{Z})$-classes is thus partitioned into $\mathbb{K}$-classes, and we describe how this varies with $\mathbb{K}$. For $\mathbb{K} = \mathbb{C}$ it amounts to grouping binary quadratic forms with the same discriminant and modular geodesics of the same length. For $\mathbb{K} = \mathbb{Q}$ we obtain a refinement of this equivalence relation, which we relate to the arithmetic of integral binary quadratic forms (Hilbert symbols) and describe in terms of the geometry of modular geodesics (angles at intersection points, and lengths of ortho-geodesics).