

Transcendence of simple geodesics on finite modular covers

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Abstract

The real projective line \mathbb{RP}^1 is the boundary of $\mathbf{HP} = \{z \in \mathbb{C} : \Im(z) > 0\}$, a model of the hyperbolic plane whose space of geodesics identifies with $\mathcal{G}(\mathbf{HP}) = \mathbb{RP}^1 \times \mathbb{RP}^1 \setminus \text{diagonal}$. The modular group $\Gamma = \text{PSL}_2(\mathbb{Z})$ acts on \mathbf{HP} with quotient the modular orbifold $\mathbf{M} = \Gamma \backslash \mathbf{HP}$. Consider a finite-index subgroup of the modular group $\Gamma' \subset \Gamma = \text{PSL}_2(\mathbb{Z})$ corresponding to a finite cover $\mathbf{M} \rightarrow \mathbf{M}'$. A geodesic $(\xi^-, \xi^+) \in \mathcal{G}(\mathbf{HP})$ projects mod Γ' to a geodesic $\xi' \subset \mathbf{M}'$. We conjecture that if ξ' is simple, then ξ^+ is either rational or quadratic or transcendental. We prove this conjecture for leaves of minimal geodesic laminations. We explain why the conjecture is known for all simple geodesics in the modular torus cover associated to the derived subgroup $\Gamma' = [\Gamma, \Gamma]$.

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0 Introduction

In Section 1 we recall the necessary background about continued fractions and the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, the modular orbifold \mathbf{M} and its geodesics, the Markov-Lagrange spectra and Mahler's exponents, to recall the main background results on transcendence of continued fractions expansions. In Section 2 we describe the topology of finite modular covers $\mathbf{M}' \rightarrow \mathbf{M}$ and their simple geodesics to state our main Conjecture 2.3, then describe the symbolic dynamics of minimal laminations to deduce our main Theorem 2.7. In Section 1, we explain why our main Conjecture 2.3 is known in the case of the modular torus cover $\mathbf{M}' \rightarrow \mathbf{M}$ associated to the derived subgroup $\Gamma' = [\Gamma, \Gamma]$, and follows from a much more precise description of simple geodesics in the corresponding modular torus \mathbf{M}' , involving precisely the numbers whose Lagrange constant > 3 .

The sections of this introduction are roughly organised as the sections of the article, except that we take this opportunity to provide some motivation and position this work at the crossroad of low dimensional topology (of simple geodesics in modular covers), symbolic dynamics (of continued fractions) and Diophantine approximation (of algebraic numbers).

0.1 Motivation: Approximating reals by rational and quadratic

Continued fractions as coding geodesics in the modular surface

Classical diophantine approximation concerns the approximation of numbers in the real projective line $\mathbb{R}\mathbf{P}^1$ by numbers in the rational projective line $\mathbb{Q}\mathbf{P}^1$ whose complexity is measured by height. It began with the Euclidean algorithm: every $\xi^+ \in \mathbb{R}_{>1}$ has a unique continued fraction expansion

$$\xi^+ = [x_0; x_1, \dots] = x_0 + \frac{1}{x_1 + \dots} \quad \text{with } x_j \in \mathbb{N}_{\geq 1}$$

that is finite if and only if ξ^+ is rational in which case it is required to have even length $2k \in 2\mathbb{N}$, and that is pre-periodic if and only if ξ^+ is quadratic.

One may rewrite this in terms of the Euclidean monoid $\mathrm{PSL}_2(\mathbb{N})$ freely generated by the parabolic the linear-fractional transformations $R: z \mapsto z + 1$ and $L: z \mapsto z/(1 + z)$, which acts on $[0, \infty]$ so that $\xi^+ \in (1, \infty)$ is the image of ∞ by the finite or infinite product $X = R^{x_0}L^{x_1} \dots$:

$$\mathrm{PSL}_2(\mathbb{N}) = \{R, L\}^* \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \xi^+ = X \cdot \infty \quad X = R^{x_0}L^{x_1} \dots \infty$$

This yields the correspondence with the symbolic encoding of geodesics on the modular orbifold.

The real projective line $\mathbb{R}\mathbf{P}^1 \subset \mathbb{C}\mathbf{P}^1$ is the boundary of $\mathbf{HP} = \{z \in \mathbb{C}: \Im(z) > 0\}$, a model of the hyperbolic plane whose space of geodesics identifies with $\mathcal{G}(\mathbf{HP}) = \mathbb{R}\mathbf{P}^1 \times \mathbb{R}\mathbf{P}^1 \setminus \text{diagonal}$.

The *modular group* $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ generated by $\{R, L\}$ acts on \mathbf{HP} by linear-fractional transformations with quotient the *modular orbifold* $\mathbf{M} = \Gamma \backslash \mathbf{HP}$, which has a cusp $\Gamma \backslash \mathbb{Q}\mathbf{P}^1$. The geodesic $\xi = (0, \xi^+) \in \mathcal{G}(\mathbf{HP})$ projects mod Γ to a geodesic $\xi' \in \mathcal{G}(\mathbf{M})$, which escapes to the cusp if and only if ξ^+ is rational and accumulates on a periodic geodesic if and only if ξ^+ is quadratic.

The real ξ^+ has rational approximants $\frac{p_n}{q_n} = [x_0, \dots, x_n]$ for coprime $p_n, q_n \in \mathbb{N}$ satisfying $|\xi^+ - \frac{p_n}{q_n}| = \frac{1}{q_n^2 x_{n+1}}$ and with denominators $\prod_0^n x_i \leq q_n \leq \prod_0^n (1 + x_i)$. The geodesic $\xi' \subset \mathbf{M}$ has successive cusp-excursions with peaks of height $\log(x_{n+1})$ between times $\log(q_n)$ and $\log(q_{n+1})$.

More generally, we have a dictionary between the symbolic dynamics of the sequence of convergents $x \in (\mathbb{N}_{\geq 1})^{\mathbb{Z}}$ under the shift map, the homogeneous dynamics of the geodesic $\xi' \subset \mathbf{M}$ as time flows and how it approaches the cusp or shadows periodic geodesics, and the quality of successive best rational or quadratic approximations to ξ^+ measured by their arithmetic complexity.

Continued fractions x	Flow of geodesic ξ' in \mathbf{M}	Diophantine approximation to ξ^+
$x_{2k} = \infty$ $\log(x_{n+1}) > \epsilon \log(q_n)$	escapes to the cusp high excursion very soon	ξ^+ is rational Roth $\implies \xi^+ \notin \bar{\mathbb{Q}}$
x pre-periodic repetitions of length $> \epsilon$	accumulates on a period over-shadowing periods	ξ^+ quadratic irrational Schmidt, Bugeaud $\implies \xi^+ \notin \bar{\mathbb{Q}}$

Approximation of algebraic numbers compared to normal numbers

Remark 0.1 (Gauss-Kusmin statistics for Lebesgues-normal numbers). The fact that the Haar measure on \mathbf{M} is ergodic for the geodesic flow yields the Gauss-Kuzmin statistics for the continued fraction expansions of Lebesgue-normal numbers, hence Khintchine applied Birkhoff's ergodic theorem to discover that normal numbers satisfy $\lim_n \frac{1}{n} \log q_n = \frac{\pi^2}{12 \log(2)}$.

Moreover the mixing property of the geodesic flow yields enough independence to apply Borel-Cantelli lemmas, hence Khintchine could find more precise $0 \vee 1$ laws in the metric theory of diophantine approximation. For example, for almost all ξ^+ , there are infinitely many $n \in \mathbb{N}$ such that $|q_n \xi^+ - p_n| < 1/(q_n \log \log q_n)$, which combined with the previous limit amounts to $\limsup_n \frac{\log x_n}{\log n} = 1$, and this yields Sullivan's law of logarithms for the cusp-excursion of ξ' ([[Sul82](#)]).

One may also use the mixing (rates) of the geodesic flow to measure how well a random geodesic shadows periodic geodesics, and deduce quantitative information about the quality of quadratic approximations to normal numbers from their repetitions in their continued fraction expansions.

Algebraic numbers of degree > 2 are expected to behave like normal numbers with respect to the Gauss-Kuzmin statistics, so that they should be neither too well nor too badly approximable by rationals or quadratics. However at this time of writing, very little is known about this.

Indeed, there is not a single example (explicit or not) of an algebraic number ξ^+ of degree > 2 for which we can decide whether its continued fraction expansion x is bounded or unbounded. A longstanding conjecture going back to Khintchine ([[BS23](#), §4]) asserts that if x is bounded then ξ^+ is rational or quadratic or transcendental. This trichotomy is the leitmotif of this work.

The best result we have concerning good rational approximations to ξ^+ (upper bounds on x), remains Roth's theorem implying that if an irrational ξ^+ is algebraic then $\log(x_{n+1}) = o(\log q_n)$, thus algebraic numbers cannot have exponentially better rational approximations than normal numbers (for which $\limsup \log(x_{n+1}) \sim \log(n) \sim \log \log q_n$). This dramatic improvement on Liouville's theorem enables to extend the construction of transcendental numbers, but we need more.

The main improvement of Roth's theorem is Schmidt's subspace theorem, which implies in particular that if an irrational ξ^+ is algebraic then it cannot have too good quadratic approximations. This means that the geodesic $\xi' \subset \mathbf{M}$ cannot get infinitely often too close to a periodic geodesic. The sequence $x \in (\mathbb{N}_{\geq 1})^{\mathbb{N}}$ has *long repetitions* when there is $\epsilon > 0$ such that there are infinitely many times $n \in \mathbb{N}$ at which the prefix (x_0, \dots, x_n) contains two disjoint occurrences of a word over $\mathbb{N}_{\geq 1}$ of length $> \epsilon n$. This enabled [[Bug13b](#)] to show that if x is bounded aperiodic but with long repetitions then ξ^+ is transcendent.

Let us mention that we may unify the last two transcendence results for irrational ξ^+ as follows. Say that $X \in \{R, L\}^{\mathbb{N}}$ has long repetitions when there is $\epsilon > 0$ such that there are infinitely many times $N \in \mathbb{N}$ at which the prefix (X_0, \dots, X_N) contains two disjoint occurrences of a factor $U \in \{R, L\}^*$ with $\text{len}(U) > \epsilon N$. If X has long repetitions then ξ^+ is quadratic or transcendent.

Lagrange and Markov constants measure approximation by rationals

For $\xi^+ = [x] \in \mathbb{R}$, its *Lagrange constant* measures the quality of its best rational approximations: $L(\xi^+)^{-1} = \liminf_n (q_n |q_n \xi^+ - p_n|)$ and $\bar{L}(\xi^+) = \limsup_n ([0, x_{n-1}, \dots, x_0] + [x_n, x_{n+1}, \dots])$. In particular $L(\xi^+) = 0 \iff \xi \in \mathbb{Q}$ and more importantly $L(\xi^+) < \infty \iff \limsup_n (x_n) < \infty$, so it is only of interest for numbers that are badly approximable by rationals. Since $L(\xi^+)$ only depends on the tail of the continued fraction expansion (x_n) , it is invariant by $\mathrm{PSL}_2(\mathbb{N})$.

Mahler's exponents measure approximation by algebraic numbers.

Let us briefly recall Mahler's exponents and hierarchy of real numbers (see [Bug04, BL05]).

The *height* of a polynomial $P(x) = \sum_0^n a_k x^k \in \mathbb{Z}[x]$ is $H(P) = \max\{|a_k| : 0 \leq k \leq n\} \in \mathbb{Z}$, and the *height* of an algebraic number $\alpha \in \mathbb{Q}$ is that of its minimal polynomial in $\mathbb{Z}[x]$.

For a real number $\xi^+ \in \mathbb{R}$ and $n \in \mathbb{N}$, its *Mahler exponent* $w_n(\xi)$ and $\hat{w}_n(\xi)$ for best and uniform approximations are defined as the supremum of $w \in \mathbb{R}$ such that the equations

$$P \in \mathbb{Z}^{1+n} \quad H(P) \leq h \quad 0 < |P(\xi^+)| \leq h^{-w}$$

have solutions for infinitely many $h \in \mathbb{N}$ and for all sufficiently large $h \in \mathbb{N}$, respectively.

By definition and the Schubfachprinzip, for $n \in \mathbb{N}_{\geq 1}$, if $\xi^+ \in \mathbb{R}$ is not algebraic of degree $\leq n$, then $n \leq \hat{w}_n(\xi^+) \leq w_n(\xi^+)$. For Lebesgues normal numbers we have $n = \hat{w}_n(\xi^+) = w_n(\xi^+)$.

By definition, \hat{w}_n and w_n are increasing. Letting $w_\infty(\xi^+) = \limsup_n \frac{1}{n} w_n(\xi^+) \in [0, \infty]$, we may now give Mahler's hierarchy of numbers: the number ξ^+ is of *Mahler class*

- A when $w_\infty(\xi^+) = 0$
- S when $0 < w_\infty(\xi^+) < \infty$
- T when $w_\infty(\xi^+) = \infty$ but for all $n \in \mathbb{N}_{\geq 1}$ it has $w_n(\xi^+) < \infty$
- U when there exists $n \in \mathbb{N}_{\geq 1}$ such that $w_n(\xi^+) = \infty$

The class A coincides with the set of algebraic numbers, and the Schmidt subspace theorem implies that if $\xi^+ \in \mathbb{R}$ is algebraic of $\deg(\xi) = d \in \mathbb{N}_{\geq 2}$ then $w_n(\xi^+) = \hat{w}_n(\xi^+) = \min\{n, d-1\}$. Moreover, if numbers are algebraically dependent then they belong to the same Mahler class.

On the other hand, if $\xi^+ \in \mathbb{R}$ is neither rational nor quadratic, then denoting $\frac{1+\sqrt{5}}{2}$ we have $2 \leq \hat{w}_2(\xi) \leq 1 + \phi$, and there are such ξ^+ that achieve equalities (see [BL05, Theorem 2.7]).

Since our geometric setting naturally leads to approximations by (pairs of) quadratic numbers, we will mostly be interested in w_2 .

0.2 Transcendence of simple geodesics in modular covers

Main conjecture: transcendence of simple geodesics in modular covers

Consider now a finite-index subgroup $\Gamma' \subset \Gamma$, corresponding to a finite cover $\mathbf{M}' \rightarrow \mathbf{M}$. The cusps of \mathbf{M}' correspond to the orbits of rational points $\mathbb{Q}\mathbf{P}^1 \bmod \Gamma'$, hence to the conjugacy classes of parabolic elements in Γ' . A cusp $o \in \mathbb{Q}\mathbf{P}^1 \bmod \Gamma'$ has a *width* defined as the index of stabilisers $w = [\mathrm{Stab}(o, \Gamma) : \mathrm{Stab}(o, \Gamma')]$.

The space of geodesics of \mathbf{HP} identifies with $\mathcal{G}(\mathbf{HP}) = \mathbb{R}\mathbf{P}^1 \times \mathbb{R}\mathbf{P}^1 \setminus \text{diagonal}$. The space of geodesics on \mathbf{M}' identifies with the quotient $\mathcal{G}(\mathbf{M}) = \Gamma' \backslash \mathcal{G}(\mathbf{HP})$ by the diagonal Γ' -action. A geodesic $\xi' \subset \mathbf{M}'$ has a future whose geometry and dynamics are governed by the continued fraction expansion of ξ^+ ; in particular: ξ' escapes into a cusp if and only if ξ' is rational, whereas ξ' accumulates on a periodic geodesic if and only if ξ^+ is quadratic.

A geodesic $\xi' \in \mathcal{G}(\mathbf{M}')$ is *simple* when there are no $\gamma \in \Gamma'$ such the pairs $\{\xi^-, \xi^+\}$ and $\{\gamma\xi^-, \gamma\xi^+\}$ are disjoint and linked with respect to the cyclic order on $\partial\mathbf{HP}$, namely it has no transverse intersection points in \mathbf{M}' . The space of simple geodesics on \mathbf{M}' is a closed subset $\mathcal{S}(\mathbf{M}') \subset \mathcal{G}(\mathbf{M}')$. The following well known result was extensively used by [?]

Lemma 0.2 (simple geodesics have bounded cusp excursions). *Consider irrationals $(\xi^-, \xi^+) \in (-1, 0) \times (1, \infty)$ with continued fraction expansions $-1/\xi^- = [x_{-1}; x_{-2}, \dots]$ and $\xi^+ = [x_0; x_1, x_2, \dots]$. The geodesic $\xi = (\xi^-, \xi^+) \subset \mathbf{HP}$ projects mod Γ' to a geodesic $\xi' \subset \mathbf{M}'$. If ξ' is simple then for all $n \in \mathbb{N}$ we have $x_n < M$ where M is the maximal cusp-width of Γ' .*

In the light of Khintchine's conjecture, it is natural to propose the following.

Conjecture 0.3 (transcendence of simple geodesics in finite modular covers). Consider a finite index subgroup $\Gamma' \subset \Gamma$ corresponding to a finite cover $\mathbf{M}' \rightarrow \mathbf{M}$.

A geodesic $\xi = (\xi^-, \xi^+) \in \mathcal{G}(\mathbf{HP})$ projects mod Γ' to a geodesic $\xi' \in \mathcal{G}(\mathbf{M}')$.

If ξ' is simple then $\xi^+ \in \mathbb{R}\mathbf{P}^1$ is either rational or quadratic or transcendental.

Main result: Mahler measures of minimal geodesics in modular covers

We will prove Conjecture 0.3 under an additional topological assumption on the simple geodesic. A *geodesic lamination* of \mathbf{M}' is a compact subset $\Xi' \subset \mathbf{M}'$ which is a disjoint union of complete simple geodesics called its leaves (it identifies with a certain compact of $\mathcal{S}(\mathbf{M}')$). It is *minimal* when every one of its leaves ξ' has a past and future that are dense in Ξ' .

The closure of a simple geodesic $\xi' \in \mathcal{G}(\mathbf{M}')$ is a geodesic lamination $\Xi' \subset \mathbf{M}'$.

Theorem 0.4 (transcendence of minimal lamination leaves). *If $\Xi' \subset \mathbf{M}'$ is minimal, then either ξ^\pm are quadratic conjugates or else ξ^\pm are both transcendental. More precisely in the transcendental case, either $w_2(\xi^\pm) = \infty$ (of type U_2) or else there exists $c \in \mathbb{R}_{>0}$ such that for all $n \in \mathbb{N}_{\geq 1}$ we have $w_n(\xi^+) \leq \exp(c(\log 3d)^3(\log \log 3d)^2)$ (of class S or T).*

Outline and moral of the proof. We may act by $\mathrm{PSL}_2(\mathbb{Z})$ to assume that $(\xi^-, \xi^+) \in (-1, 0) \times (1, \infty)$ so that the continued fraction expansions of $-1/\xi^-, \xi^+$ encode the trajectory of ξ' on \mathbf{M}' as a bounded sequence $x = (x^-, x^+) \in (\mathbb{N}_{\geq 1})^{\mathbb{Z}}$. If the geodesic ξ' is a leaf of a minimal lamination, then it is (uniformly) recurrent in space, and each time its position returns to a small neighbourhood in space of the base point, its momentum must also align closely to that of the base point, so much so that the geodesic will travel almost parallel to itself for some time, which is proportional to the return time. Thus minimality and topological simplicity implies the bounded continued fraction expansion $x^\pm \in (\mathbb{N}_{\geq 1})^{\mathbb{N}}$ has sub-affine factor complexity, hence) long repetitions. This enables to apply [Bug13a, Theorem 1.1] on the transcendence of bounded continued fractions with long repetitions and [Bug12, Theorem 2.2] to control the Mahler exponents of ξ^\pm from the subaffine complexity of x .

In conclusion, we are able to prove transcendence not only because x is bounded (which only says that ξ^\pm is badly approximable by rationals), but especially because x has long repetitions and subaffine factor complexity (which is saying that ξ has too good quadratic approximations). \square

Remark 0.5 (Arithmetic Fuchsian groups). We will generalise all this work from $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ (the fundamental group of the modular orbifold) to an arithmetic cocompact Fuchsian group $F \subset \mathrm{PSL}_2(\mathcal{O}_{\mathbb{K}})$ (the fundamental group of a compact Shimura curve). This will first require to generalize the (hard) work in [Bug13b, Bug12], from bounded continued fractions to certain

products of hyperbolic matrices in F . For this, we will need to apply the (quantitative) Schmidt subspace theorem over the invariant trace field of F , and use the property that the trace of hyperbolic matrices in F dominates its conjugates in absolute value (see [BHS13, Theorem 2.4]).

Definition and questions on profinitely simple geodesics

Let us give a name to the (pairs of) numbers covered by Conjecture 0.3 (and Theorem 0.4).

Definition 0.6 (profinite simple). Define the subset $\tilde{\mathcal{S}}(\Gamma) \subset \mathcal{G}(\mathbf{HP})$ of *profinite simple geodesics* (for Γ) as consisting of those $\xi \in \mathcal{G}(\mathbf{HP})$ such that there exists a finite index subgroup $\Gamma' \subset \Gamma$ such that $\xi' = \text{mod}\Gamma' \in \mathcal{S}(\mathbf{M}')$. Define the subset $\tilde{\mathcal{S}}^+(\Gamma) \subset \mathbb{R}\mathbf{P}^1$ of *profinite simple numbers* (for Γ) as its image by the projection on (any) one of the two factors $\mathcal{G}(\mathbf{HP}) \rightarrow \mathbb{R}\mathbf{P}^1$.

We may define the subsets of *profinite congruence simple* geodesics and numbers $\mathcal{CS}^+(\Gamma)$, by restricting to congruence covers $\mathbf{M}(N) \rightarrow \mathbf{M}$ associated to the congruence subgroups $\Gamma(N) \subset \Gamma$ defined for $N \in \mathbb{N}_{>2}$ as the kernel of the reduction mod N morphism $\text{PSL}_2(\mathbb{Z}) \rightarrow \text{PSL}_2(\mathbb{Z}/N)$.

Remark 0.7 (Hausdorff dimension). It follows from [BS85] that for every finite cover $\mathbf{M}' \rightarrow \mathbf{M}$ we have $\dim_H \mathcal{G}(\mathbf{M}') = 0$. Thus $\tilde{\mathcal{S}}(\Gamma) \subset (\mathbb{R}\mathbf{P}^1)^2$ hence $\tilde{\mathcal{S}}^+(\Gamma) \subset \mathbb{R}\mathbf{P}^1$ have Hausdorff dimension 0.

We ask a couple of questions regarding profinitely (congruence) simple congruence numbers, namely 2.13 on the symbolics of their continued fraction expansions, as well as 2.14 and 2.15 about the topology of their Lagrange spectra $L(\mathcal{CS}^+(\Gamma)) \subset L(\mathcal{S}^+(\Gamma)) \subset L(\mathbb{R})$, and Mahler spectra $w_2(\mathcal{CS}(\Gamma)) \subset w_2(\tilde{\mathcal{S}}(\Gamma))$ as well as $\hat{w}_2(\mathcal{CS}(\Gamma)) \subset \hat{w}_2(\tilde{\mathcal{S}}(\Gamma)) \subset [2, 1 + \phi]$.

0.3 Simple geodesics in the modular torus: Christoffel, Markov, Sturm

In Section 1 we explain why for the derived subgroup $\Gamma' = [\Gamma, \Gamma]$, the full Conjecture 2.3 is known. Indeed, the corresponding modular cover $\mathbf{M}' \rightarrow \mathbf{M}$ has total space a hyperbolic once-punctured torus, in which all simple geodesics either escape to a cusp or belong to a minimal lamination.

In fact, there is a much more precise description of simple geodesics in the corresponding modular torus \mathbf{M}' . We recall this well known story in the spirit of [Ser85], while emphasizing some details either missing from or dispersed in the literature.

The trichotomy distinguishes Markov rational numbers, Markov (pairs of) quadratic numbers and Sturmian (pairs of) transcendent numbers, which are precisely those whose (Markoff and) Lagrange constant lie before and on the first accumulation point 3 of the spectra.

The Lagrange spectrum $L(\mathbb{R}_{>1})$ begins with a discrete countable set increasing to its first accumulation point 3, and the corresponding ξ^+ are precisely the $\text{PSL}_2(\mathbb{Z})$ orbits of Markov quadratic irrationals corresponding to simple periodic geodesics in \mathbf{M}' . There are uncountably many numbers $\xi^+ \in \mathbb{R}_{>1}$ with $L(\xi^+) = 3$, which we call Sturmian since up to the action of $\text{PGL}_2(\mathbb{Z})$ they are precisely the numbers whose continued fraction expansions are the aperiodic Sturmian sequences on $\{1, 2\}$, and they correspond to simple aperiodic geodesics in \mathbf{M}' . The transcendence of Sturmian numbers was shown in [ADQZ01, Theorem 7].

We will also take this opportunity to describe the structure of the metabelian quotient Γ/Γ'' (in Theorem 3.1, first observed in [Sim22, Section 3.2]), since we believe that it deserves more attention. We are working on its generalisation to modular groups of higher genus Shimura curves and its relation to certain modular forms for symplectic groups.

Remark 0.8. Our survey of diophantine properties about Sturmian numbers will not be exhaustive. We refer to [BL05], as well as the very recent works [Roy11, RZ11] about the fine diophantine properties of characteristic Sturmian numbers whose slope is in the $\text{PGL}_2(\mathbb{Z})$ orbit of ϕ .

1 The modular orbifold and continued fractions

This background section recalls some material about the modular orbifold and continued fractions from [Sim22, Chapter 2] and diophantine approximation from [Haa87] and results on transcendence of bounded continued fractions iwth low complexity from [AB10, Bug12].

1.1 Modular group and its conjugacy classes

The modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ acts on the ideal triangulation Δ of \mathbf{HP} with vertex set $\mathbb{Q}\mathbf{P}^1$ and edges $\left(\frac{a}{c}, \frac{b}{d}\right)$ such that $ad - bc = 1$, freely transitively on its flags, corresponding to the half-edges of its dual trivalent tree \mathcal{T} . The connected components of $\mathbf{HP} \setminus \mathcal{T}$ correspond to the vertices of Δ , parametrized by $\Gamma/\langle R \rangle$. Denote

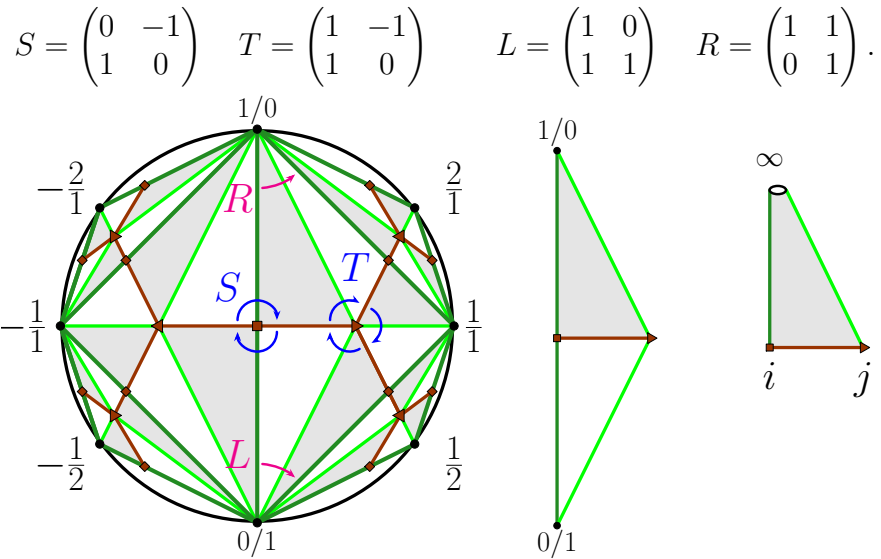


Figure 1: Projective model of \mathbf{HP} , with its ideal triangulation Δ and dual tree \mathcal{T} . Fundamental domain under the action of $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. The orbifold $\mathbf{M} = \Gamma \backslash \mathbf{HP}$.

The modular orbifold $\mathbf{M} = \Gamma \backslash \mathbf{HP}$ has genus zero, a cusp associated to the fixed point $\infty \in \mathbb{Q}\mathbf{P}^1 \subset \partial\mathbf{HP}$ of R , as well as two conical singularities of order two and three associated to the fixed points $i, j \in \mathbf{HP}$ of S and T .

Thus $\Gamma = \pi_1(\mathbf{M})$ is the free amalgam of its subgroups $\mathbb{Z}/2$ and $\mathbb{Z}/3$ generated by S and T . (More precisely $\mathrm{SL}_2(\mathbb{Z})$ is the amalgam of its subgroups $\mathbb{Z}/4$ and $\mathbb{Z}/6$ generated by S and T over their intersection $\mathbb{Z}/2$ generated by $S^2 = -\mathbf{1} = T^3$.)

It follows that in Γ , a finite order element is conjugate to a power of S or T ; whereas an infinite order A is conjugate to $\prod_1^l T^{\epsilon_i} S^{-\epsilon_i}$ for a unique $l \in \mathbb{N}^*$ and a unique sequence of $\epsilon_i = \pm 1$ up to cyclic permutation: this corresponds to a cyclic word in $R = TS^{-1}$ and $L = T^{-1}S$, and we may thus define its $\{R, L\}$ -length $\mathrm{len}(A) = l = \#R + \#L$ and its Rademacher number $\mathrm{Rad}(A) = \sum_1^l \epsilon_i = \#R - \#L$.

Conjugacy classes in Γ correspond to orbifold homotopy classes of loops on \mathbf{M} . The elliptic and parabolic classes correspond to loops circling around the singularities and cusp, whereas every hyperbolic class is represented by a unique closed geodesic.

An element of Γ is primitive when it generates a maximal cyclic subgroup. This notion is invariant by conjugacy. Correspondingly, a cyclic $\{R, L\}$ -word is primitive when it is not a proper power, and a closed geodesic is primitive when it does not wind several times around itself.

A complete geodesic of \mathbf{HP} is uniquely determined by the sequence of triangles in Δ that it intersects: it corresponds to a geodesic of the dual tree \mathcal{T} which is not horocyclic (bounding a region of the complement $\mathbf{HP} \setminus \mathcal{T}$). Such geometric and combinatorial geodesics are equally determined by their common endpoints in $\mathbb{R}\mathbf{P}^1$.

A closed geodesic of \mathbf{M} lifts to the geodesics in \mathbf{HP} corresponding to the periodic geodesics of \mathcal{T} whose period is given by $\{R, L\}$ -cycle of the associated conjugacy class. Their endpoints form the Γ -orbit of a pair of Galois-conjugate quadratic irrationals. Such geodesics determine a unique hyperbolic conjugacy class in Γ which is primitive.

1.2 Euclidean monoid and continued fraction expansions

The monoid $\mathrm{SL}_2(\mathbb{N}) \subset \mathrm{SL}_2(\mathbb{Z})$ is freely generated by the transvections L and R , and it identifies with its image $\mathrm{PSL}_2(\mathbb{N}) \subset \mathrm{PSL}_2(\mathbb{Z})$ which we call the Euclidean monoid.

In $\mathrm{PSL}_2(\mathbb{Z})$, the conjugacy classes of torsion elements are those of $\mathbf{1}, S, T, T^{-1}$, whereas the conjugacy class of an infinite order element intersects the Euclidean monoid $\mathrm{PSL}_2(\mathbb{N})$ along the cyclic permutations of a unique $\{R, L\}$ -word. Hence the hyperbolic geodesics of \mathbf{M} are indexed by cyclic words in $\{R, L\}$ containing both letters.

Observe that for $A \in \mathrm{SL}_2$ we have $A^\dagger = SA^{-1}S^{-1}$, hence transposition of cyclic words in $\mathrm{PSL}_2(\mathbb{N})$ corresponds to inversion of the associated conjugacy classes in $\mathrm{PSL}_2(\mathbb{Z})$.

Every number $\xi \in (0, +\infty]$ admits a unique Euclidean continued fraction expansion:

$$\lfloor x_0, x_1, \dots \rfloor = x_0 + \frac{1}{x_1 + \dots} = R^{x_0} L^{x_1} \dots \infty \quad \text{with } x_j \in \mathbb{N} \quad \text{and } \forall j > 0, n_j > 0$$

which is finite if and only if ξ is rational, in which case it must have an even number of terms (as the notation implies that the $\{R, L\}$ -word begins with a power of R and ends with a power of L). The involution $\xi \mapsto S(\xi) = -1/\xi$ with no fixed points yields a partition $\mathbb{R}\mathbf{P}^1 = (0, \infty] \sqcup S \cdot (0, \infty]$, hence every $\xi \in \mathbb{R}\mathbf{P}^1$ admits a unique representation of the form $\xi = \lfloor x_0, x_1 \dots \rfloor$ or $\xi = -1/\lfloor x_0, x_1 \dots \rfloor$ (including $\infty = \lfloor \rfloor$ and $0 = -1/\lfloor \rfloor$).

Now consider the action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{R}\mathbf{P}^1$ and of its Euclidean submonoid $\mathrm{PSL}_2(\mathbb{N})$ on $[0, \infty]$. If ξ_i denotes the i^{th} remainder of $\xi \in (0, \infty]$ given by the tail $\lfloor x_i, \dots \rfloor$ of its continued fraction expansion, then $\xi_0 = (R^{x_0} L^{x_1}) \xi_2$. Hence the orbits of $\alpha, \beta \in (0, \infty]$ under $\mathrm{PSL}_2(\mathbb{N})$ have non-empty intersection if and only if there exist even starting points i, j at which the tails α_i and β_j coincide. Consequently $\alpha, \beta \in (0, \infty]$ belong to the same $\mathrm{PSL}_2(\mathbb{Z})$ -orbit if and only if there exist even starting points i, j at which the tails α_i and β_j coincide.

Example 1.1 (translation axes of elements in $\mathrm{PSL}_2(\mathbb{Z})$). A hyperbolic $C \in \mathrm{PSL}_2(\mathbb{Z})$ has translation axis $(\gamma^-, \gamma^+) \in \partial\mathbf{HP} \times \partial\mathbf{HP}$ connecting a pair of Galois conjugate real quadratic irrationals. We have $C \in \mathrm{PSL}_2(\mathbb{N}) \iff \gamma^- < 0 < \gamma^+$ and $C \in R\mathrm{PSL}_2(\mathbb{N})L \iff (\gamma^-, \gamma^+) \in (-1, 0) \times (1, \infty)$.

The axis of $C = R^{c_0} \dots L^{c_k} \in R \cdot \mathrm{PSL}_2(\mathbb{N}) \cdot L$ has endpoints the Galois conjugate pair of quadratic numbers $(\gamma^-, \gamma^+) \in (-1, 0) \times (1, \infty) \subset \partial\mathbf{HP} \times \partial\mathbf{HP}$ given by the purely periodic continued fraction expansions:

$$\gamma^+ = \lfloor (c_0, \dots, c_k)^{\mathbb{N}} \rfloor \quad \text{and} \quad -1/\gamma^- = \lfloor (c_k, \dots, c_0)^{\mathbb{N}} \rfloor.$$

Any geodesic $(\xi^-, \xi^+) \subset \mathbf{HP}$ intersects $(\infty, 0)$ positively if and only if $\xi^- < 0 < \xi^+$. Moreover if $-1 < \xi^- < 0$ and $1 < \xi^+ < \infty$, then it intersects Δ along a sequence of triangles whose encoding in $\{R, L\}^{\mathbb{Z}}$ is obtained from the continued fraction expansions of ξ^+ and $-1/\xi^-$ by concatenating the transpose of the latter with the former.

Every geodesic on \mathbf{M} has a lift in \mathbf{HP} (namely every geodesic in \mathbf{HP} has a $\mathrm{PSL}_2(\mathbb{Z})$ -translate) whose endpoints ξ^\pm satisfy $-1 < \xi^- < 0$ and $1 < \xi^+ < \infty$: it intersects Δ along a sequence of triangles whose encoding in $\{R, L\}^{\mathbb{Z}}$ is obtained from the continued fraction expansions of ξ^+ and $-1/\xi^-$ by concatenating the transpose of the latter with the former. The closed geodesics on \mathbf{M} correspond to the periodic sequences, hence to the Γ -orbits of pairs (γ^-, γ^+) of Galois-conjugate quadratic irrationals.

For distinct $\xi^-, \xi^+ \in \mathbb{R}\mathbf{P}^1$, the geodesic $(\xi^-, \xi^+) \bmod \mathrm{PSL}_2(\mathbb{Z}) \subset \mathbf{M}$ escapes (in the past or future) to the cusp when ξ^- or ξ^+ is rational. Otherwise, the geodesic intersects the segment $[i, j] \subset \mathbf{M}$ infinitely many times, with intervals given by the entries in the continued fraction expansions of $-1/\xi^-$ and ξ^+ , as we now explain.

1.3 Rational approximations and cusp-excursions

Let us recall from [Aig13] the Lagrange constant of a real number and the Markov constant of a pair of distinct real numbers (r equivalently of a geodesic).

For $\xi^+ \in \mathbb{R}$, its *Lagrange constant* $L(\xi^+)$ is the supremum of $L \in \mathbb{R}_+$ such that there are infinitely many $p, q \in \mathbb{Z} \times \mathbb{N}^*$ with $|\xi - p/q| < 1/(Lq^2)$, in formula $L(\xi^+)^{-1} = \liminf_q (q^2 |\xi - p/q|)$. It may be expressed in terms of the continued fraction expansion $\xi^+ = [x_0, x_1, \dots]$:

$$L(\xi^+) = \limsup_n ([0, x_{n-1}, \dots, x_0] + [x_n, x_{n+1}, \dots])$$

In particular $L(\xi^+) = 0 \iff \xi^+ \in \mathbb{Q}$ and more importantly $L(\xi^+) = +\infty \iff \limsup_n (x_n) = +\infty$. Moreover, for $\xi \notin \mathbb{Q}$ we have $L(\xi) \geq \limsup_n ([0, 1, \dots, 1] + [1, 1, \dots]) = \sqrt{5}$. The Lagrange constant $L(\xi^+)$ only depends on the tail of the continued fraction expansion (x_n) (by the lim sup), hence it is invariant by the action of $\mathrm{PSL}_2(\mathbb{N})$.

In \mathbf{M} , the horoball neighborhood $\mathbf{B}(h)$ of the cusp ∞ at height $h \geq 1$ is the projection of $\{\Im(z) > h\} \subset \mathbf{HP}$, having area $1/h$. The geodesic $(\infty, \xi) \in \mathbf{HP}$ projects to a geodesic on \mathbf{M} whose penetrations into $\mathbf{B}(h)$ are indexed by $n \in \mathbb{N}$ such that $\frac{1}{2} ([0, x_{n-1}, \dots, x_0] + [x_n, x_{n+1}, \dots]) > h$.

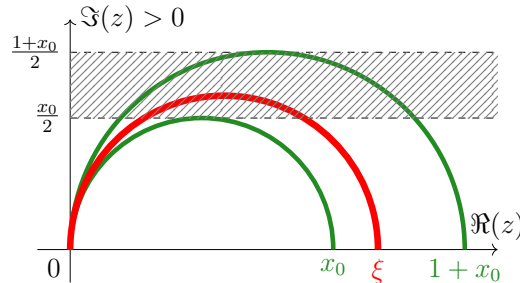


Figure 2: The geodesic $(0, \xi) \bmod \mathrm{PSL}_2(\mathbb{Z})$ penetrates $\mathbf{B}(h) \subset \mathbf{M}$ each time $n \in \mathbb{N}$ satisfies $\frac{1}{2} ([0, x_{n-1}, \dots, x_0] + [x_n, x_{n+1}, \dots]) > h$.

For a geodesic $(\xi^-, \xi^+) \in \mathcal{G}(\mathbf{M})$, its *Markov constant* $C(\xi^-, \xi^+)$ is the supremum of $2h \in \mathbb{R}_+$ such that it is disjoint from $\mathbf{B}(h)$. If $(\xi^-, \xi^+) \in [-1, 0) \times [1, \infty)$ with $-1/\xi^- = [x_{-1}; x_{-2}, \dots]$ and $\xi^+ = [x_0, x_1, x_2, \dots]$, the geodesic $(\xi^-, \xi^+) \subset \mathbf{HP}$ intersects the ideal triangles of Δ starting from the base edge $(0, \infty) \subset \mathbf{HP}$ according to the sequence of (x_n) . It projects to a geodesic on \mathbf{M} which penetrates $\mathbf{B}(h)$ each time $\frac{1}{2} ([0, x_{n-1}, x_{n-2}, \dots] + [x_n, x_{n+1}, \dots]) > h$, thus

$$C(\xi^-, \xi^+) = \sup_n ([0, x_{n-1}, x_{n-2}, \dots] + [x_n, x_{n+1}, \dots]).$$

Remark 1.2 (Gauss-Kuzmin statistics). The continued fraction expansions of random numbers follow the Gauss-Kuzmin statistics as we now recall.

The Gauss-Kuzmin measure μ_{GK} on $(0, 1)$ is defined on intervals $I \subset (0, 1)$ by $\mu_{GK}(I) = \frac{1}{2} \int_I \frac{dt}{1+t}$. For a finite sequence of positive integers $x = (x_1, \dots, x_n) \in (\mathbb{N}^*)^n$ the interval of real numbers in $(0, 1)$ whose continued fraction expansion begins with $(0, x_1, \dots, x_n)$ forms an interval denoted $I(x)$. For a real number sampled according to the Lebesgue measure, the frequency with which the finite sequence x of positive integers appears in its continued fraction expansion is given by $\mu_{GK}(I(x))$. For instance on cylinders $x = (x_1)$ of length 1, we find that the probability to witness the integer x_1 is given by $\log_2 \left(1 + \frac{1}{x_1(2+x_1)}\right)$.

We refer to [SZH25] for an elegant characterisation of the Gauss-Kuzmin measure.

Remark 1.3 (transcendence badly approximable numbers?). A positive real number whose continued fraction expansion has bounded entries is called *badly approximable*. A longstanding conjecture going back to Khintchine (see [BS23, §4]) asserts a badly approximable number is rational or quadratic or transcendental. This trichotomy is the leitmotiv of this work. Let us warn that to this date, there is not a single example (explicit or non-explicit) of algebraic number with degree > 2 for which we can decide whether its continued fraction expansion is bounded or unbounded.

In fact, it is more generally expected that algebraic numbers of degree > 2 are normal with respect to the Gauss-Kuzmin statistics, which would imply in particular that their corresponding $\{R, L\}$ -expansion has exponential factor complexity (a notion that we will introduce in Section 1.4).

1.4 Transcendence of continued fractions with low complexity

Let us recall two transcendence criteria from [Bug13a, Bug13b, Bug12, AB10].

Recall that $\text{PSL}_2(\mathbb{N})$ is the free monoid of words over the letters $\{R, L\}$, and that the transposition $A \mapsto A^\dagger$ is the anti-involution that reverses the word while exchanging the letters R, L .

We define a *set of syllables* as a finite subset $\Sigma \subset R\text{PSL}_2(\mathbb{N})L$ that is invariant by transposition, and let $\Sigma^* \subset \text{PSL}_2(\mathbb{N})$ be the free monoid of *words* over those syllables $\Sigma^* = \bigcup_{n=1}^{\infty} \Sigma^n$. Note that transposition restricts to an anti-involution of $\Sigma^* \subset \text{PSL}_2(\mathbb{N})$.

Example 1.4 ($\Sigma \subset \Sigma_M^n$). For $M \in \mathbb{N}_{\geq 1}$, the set $\Sigma_M = \{R^{c_0}L^{c_1} : 1 \leq c_0, c_1 \leq M\}$ is a set of syllables. A syllable set Σ can equivalently be defined as a \dagger -invariant subset of Σ_M^n for some $n, M \in \mathbb{N}_{\geq 1}$.

Definition 1.5 (diophantine exponent). Consider a sequence $X \in \Sigma_M^{\mathbb{N}}$.

For $\rho \in [1, +\infty)$, say that X has *exponent* $\geq \rho$ when for all $N \in \mathbb{N}$ there are words $U, W \in \Sigma^*$ and $r \in \mathbb{R}_{\geq 0}$ such that WU^r is a prefix of X with $\text{len}(WU^r) \geq N$ and $\text{len}(WU^r)/\text{len}(WU) \geq \rho$. Its *diophantine exponent* $\text{Dio}(X) \in [1, \infty]$ is the supremum of ρ such that X has exponent $\geq \rho$.

Definition 1.6 (factor complexity). For a sequence $X \in \Sigma_M^{\mathbb{N}}$, its *language of subwords* is the subset $\mathcal{L}_X \subset \Sigma_M^*$ defined by $\mathcal{L}_X = \{X_{[k, k+n]} \in \Sigma_M^n : n, k \in \mathbb{N}\}$ and its *factor complexity* is the function $\text{fac}(X) : \mathbb{N} \rightarrow \mathbb{N}$ whose value at $n \in \mathbb{N}$ is $\text{fac}_n(X) = \text{Card}(\mathcal{L}_X \cap \Sigma_M^n)$.

A sequence $X \in \Sigma_M^{\mathbb{N}}$ has an attractive fixed point $\xi^+ = X \cdot \infty \in \mathbb{R}_{>1}$, whose continued fraction expansion $\xi^+ = [x]$ has continuants the exponents of the $\{R, L\}$ -conversion of X ; in particular the sequence $x \in (\mathbb{N}_{\geq 1})^{\mathbb{N}}$ is bounded by M .

Remark 1.7 (aperiodic complexity gap). Let $X \in \Sigma_M^{\mathbb{N}}$ and $\xi^+ = X \cdot \infty \in \mathbb{R}_{>1} \setminus \mathbb{Q}$.

If $\liminf_n \frac{1}{n} \text{fac}_n(X) > 0$ or $\text{Dio}(X) < \infty$ then X is aperiodic hence ξ^+ is not quadratic. Conversely if X is aperiodic then we have $\liminf_n \frac{1}{n} \text{fac}_n(X) \geq 1$ as we explain now. A classical result from [MH38] (see [CN10, §4.3]) states that if there exists $n \in \mathbb{N}$ such that $\text{fac}_n(X) \leq n$

then X must be periodic. Hence the least possible complexity function for an aperiodic sequence is $n \mapsto n + 1$ (this characterises Sturmian sequences, which will appear in Section ??).

It follows from the boundedness of x that $w_2(\xi^+) \geq \text{Dio}(X)$, in particular if $\text{Dio}(X) > 2$ then ξ^+ is quadratic or transcendent. In fact, it suffices to have $\text{Dio}(X) > 1$ to deduce transcendence, as we now recall from [Bug13a, Theorem 1.1].

Theorem 1.8 (linear lim inf complexity). *Let $X \in \Sigma_M^{\mathbb{N}}$ and $\xi^+ = X \cdot \infty \in \mathbb{R}_{>1} \setminus \mathbb{Q}$.*

If $\liminf_n \frac{1}{n} \text{fac}_n(X) \in [0, +\infty)$, then $\text{Dio}(X) \in (1, +\infty]$, hence ξ^+ is quadratic or transcendent.

Outline of the proof. If $\liminf_n \frac{1}{n} \text{fac}_n(X) < \infty$ then [Bug13a, §4] deduces from the Schubfachprinzip that $\text{Dio}(X) > 1$, which is equivalent to saying that for some $\epsilon > 0$ there are infinitely many triples of words $U, V, W \in \Sigma^*$ such that $WUVU$ is a prefix of X and $\text{len}(U) > \epsilon \text{len}(WUVU)$. A fantastic argument involving (repeated applications of) the Schmidt subspace theorem shows that ξ^+ is quadratic or transcendent. \square

We will mostly be interested in sequences X with subaffine complexity, and for those [Bug12, Theorem 2.2] employs the quantitative Schmidt subspace theorem to bound the Mahler measures of ξ^+ in terms of the symbolics of X .

Theorem 1.9 (linear lim sup complexity). *Let $X \in \Sigma^{\mathbb{N}}$ and $\xi^+ = X \cdot \infty \in \mathbb{R}_{>1}$.*

Assume that $\limsup \frac{1}{n} \text{fac}_n(X) < \infty$.

If $\text{Dio}(X) = \infty$ then ξ^+ is quadratic or transcendent of Mahler class U_2 (namely $w_2(\xi^+) = \infty$).

If $\text{Dio}(X) < \infty$ then ξ^+ is transcendent of Mahler class S or T , and more precisely there exists $c \in \mathbb{R}_{>0}$ such that for all $d \in \mathbb{N}_{\geq 1}$ we have $w_d(\xi^+) \leq \exp(c(\log 3d)^5(\log \log 3d)^4)$.

A special class of sequences with subaffine complexity are the morphic sequences, arising from fixed points of an endomorphism of the monoid Σ^* . A sequence $X \in \Sigma^{\mathbb{N}}$ is purely morphic when it is a fixed point of an endomorphism of monoid $\varphi: \Sigma^* \rightarrow \Sigma^*$, namely there exists a syllable $A \in \Sigma$ such that $X = \lim_n \varphi^n(A)$. A sequence is morphic when it is the image by a morphism $\psi: \tilde{\Sigma}^* \rightarrow \Sigma^*$ of a fixed point of an endomorphism of $\tilde{\Sigma}^*$. Let us warn of certain subtle differences between purely morphic and morphic sequences [CN10, DLR13].

A sequence $X \in \Sigma^{\mathbb{N}}$ is recurrent when every subword of X occurs infinitely many times (if X is purely morphic, then it is enough to check that the first syllable occurs at least twice).

Remark 1.10 (from recurrence to uniform recurrence and linear recurrence). Let us note that for a morphic sequence X , its recurrence is equivalent to its uniform recurrence (every subword w appears infinitely often with bounded gaps, namely for every $l \in \mathbb{N}$ there is $c(l) \in \mathbb{N}$ such that every subword of length $c(l)$ contains all subwords of length l) and even to linear recurrence (the gaps are linear in the size of the word, namely there exists $c \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, every subword of length cl contains all subwords of length l). We refer to [Dur13] and references therein.

We may now recall [AB10, Theorem 2.2.2]

Theorem 1.11 (morphic continued fractions). *Let $X \in \Sigma^{\mathbb{N}}$ and $\xi^+ = X \cdot \infty \in \mathbb{R}_{>1} \setminus \mathbb{Q}$.*

If $X \in \Sigma^{\mathbb{N}}$ is recurrent and fixed point of a morphism, then either ξ^+ is quadratic or else it is transcendent of Mahler class S or T , and more precisely there exists $c \in \mathbb{R}_{>0}$ such that for all $n \in \mathbb{N}_{\geq 1}$ we have $w_n(\xi^+) \leq \exp(c(\log 3d)^3(\log \log 3d)^2)$.

2 Transcendence of simple geodesics in modular covers

Subsections 2.1 and 2.2 introduce the background to prove the main Theorems 2.7 and 2.7 in Subsection 2.3. Subsection 2.4 defines profinite simple numbers and raises several questions.

2.1 Topology and combinatorics of simple geodesics in modular covers

Topology and combinatorics of finite modular covers

Consider a finite-index subgroup $\Gamma' \subset \Gamma$, corresponding to a finite regular cover $\Gamma' \subset \Gamma$. Assume that Γ' is torsion free, so that \mathbf{M}' is a surface with no conical singularities: it has genus $g \in \mathbb{N}$ and $p \in \mathbb{N}_{>0}$ cusps, hence Euler characteristic $\chi = 2 - 2g - p \in \mathbb{Z}_{<0}$. The surface \mathbf{M}' has an ideal triangulation $\Delta' = \Delta \bmod \Gamma'$ which is dual to the embedded trivalent graph $\mathcal{T}' = \mathcal{T} \bmod \Gamma'$. The surface \mathbf{M}' retracts by deformation onto \mathcal{T}' , hence $\Gamma' = \pi_1(\mathcal{T}', i')$ is a free group of rank $2g + p$.

The cusps of \mathbf{M}' correspond to the orbits of rational points $\mathbb{Q}\mathbf{P}^1 \bmod \Gamma'$, hence to the conjugacy classes of parabolic elements in Γ' . A cusp $o \in \mathbb{Q}\mathbf{P}^1 \bmod \Gamma'$ has a *width* defined as the index of stabilisers $w = [\text{Stab}(o, \Gamma) : \text{Stab}(o, \Gamma')]$, which can be characterised as follows: geometrically, the cusp neighbourhood $\mathbf{B}(o, h)$ of o at height $h > 0$ has area w/h ; combinatorially, the smallest cycle of \mathcal{T}' surrounding o has w edges (when Γ' is normal in Γ all cusps have the same width).

Denote by $V(\mathcal{T}')$ and $E(\mathcal{T}')$ the vertices and edges of \mathcal{T}' . The rank of $\Gamma' = \pi_1(\mathcal{T}')$ equals that of the first homology groups $H_1(\mathcal{T}'; \mathbb{Z}) = H_1(\mathbf{M}'; \mathbb{Z})$, that is $\text{Card } E(\mathcal{T}') - \text{Card } V(\mathcal{T}') = 2g + p$. Denote by $\vec{E}(\mathcal{T}')$ the set of *arcs* of \mathcal{T}' , which correspond to pairs of incident edges and vertices. Since Γ acts freely transitively on arcs of \mathcal{T} , we have a bijection $C \in \Gamma/\Gamma' \mapsto \vec{e}_C \in \vec{E}(\mathcal{T}')$; the *base arc* is \vec{e}_1 (that is the edge i' directed towards the vertex j'). When Γ' is normal in Γ , the quotient group Γ/Γ' acts freely transitively on the arcs of \mathcal{T}' .

An arc $\vec{e} \in \vec{E}(\mathcal{T}')$ has *starting* and *terminating* vertices $(\partial^- \vec{e}, \partial^+ \vec{e}) \in V(\mathcal{T}') \times V(\mathcal{T}')$. (Note that the map $\partial^- \times \partial^+ : \vec{E}(\mathcal{T}') \rightarrow V(\mathcal{T}') \times V(\mathcal{T}')$ may be non-injective: this would be true only if the trivalent graph \mathcal{T}' had no multiple edges, that is when $2 \text{Card } E(\mathcal{T}') = 3 \text{Card } V(\mathcal{T}')$.)

Define the *tree-path semi-groupoid* of \mathcal{T}' as the set of non-empty words $p = p_0 \dots p_l$ over the alphabet $\vec{E}(\mathcal{T}')$ in which every two consecutive letters $p_k p_{k+1}$ satisfy $\partial^+ p_k = \partial^- p_{k+1}$ but $p_k^{-1} \neq p_{k+1}$. The tree-path semi-groupoid corresponds (by recording the $\{R, L\}$ -turns from the starting arc), to some kind of semi-direct product construction $(\Gamma/\Gamma') \rtimes \text{PSL}_2(\mathbb{N})$ (details are left as an exercise).

Define the *tree-loop semigroup* of \mathcal{T}' as the subset of words $p = p_0 \dots p_l$ in the tree-path groupoid starting and terminating with $p_0 = \vec{e}_1 = p_l$. It is freely generated by the subset \mathcal{B} of tree-paths having exactly two occurrences of \vec{e}_1 (which is infinite as soon as $[\Gamma : \Gamma'] > 6$). The tree-path semigroup is in correspondence with the *positive monoid* $\Gamma'_{\geq 0} = \Gamma' \cap \text{PSL}_2(\mathbb{N})$.

Define the *even-positive semigroup* as $\Gamma' \cap R \text{PSL}_2(\mathbb{N}) L$, that is the subsemigroup of $\Gamma'_{\geq 0}$ freely generated by the subset $\mathcal{B}_{RL} \subset \mathcal{B}$ of edge-loops starting with $\vec{e}_1 \vec{e}_R$ and terminating with $\vec{e}_{L-1} \vec{e}_1$. Observe that $S \in \text{PSL}_2(\mathbb{Z})$ normalises the group Γ' if and only if \dagger preserves the even-positive semigroup $\Gamma' \cap R \text{PSL}_2(\mathbb{N}) L$ hence its basis \mathcal{B}_{RL} ; and note that in terms of tree-paths, the transposition reverses the word while inverting the arcs.

Example 2.1 (normal congruence subgroups). For $N \in \mathbb{N}_{\geq 2}$, the morphism $\text{PSL}_2(\mathbb{Z}) \rightarrow \text{PSL}_2(\mathbb{Z}/N)$ has kernel $\Gamma(N) = \{A \in \Gamma \mid A \equiv \mathbf{1} \bmod N\}$, defining the *level- N normal congruence subgroup* $\Gamma(N) \subset \Gamma$ corresponding to the $\text{PSL}_2(\mathbb{Z}/N)$ -Galois cover $\mathbf{M}(N) \rightarrow \mathbf{M}$. It is torsion-free and has cusp-width N . The positive monoid $\Gamma(N)_{\geq 0} = \{A \in \text{PSL}_2(\mathbb{N}) : A \equiv \mathbf{1} \bmod N\}$ is finitely generated if and only if $N = 2$ in which case it is freely generated by L^2, R^2 . Figure 3 depicts the trivalent map $\mathcal{T}(N) \subset \mathbf{M}(N)$ for the values $N \in \{2, 3, 4, 5\}$ such that $\mathbf{M}(N)$ has genus 0.

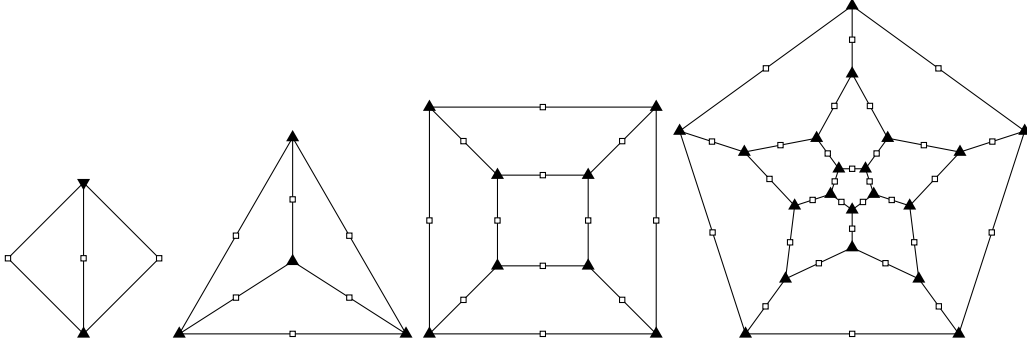


Figure 3: The embedded graphs $\mathcal{T}' \subset \mathbf{M}'$ when $\Gamma' = \Gamma(N)$ for $N \in \{2, 3, 4, 5\}$.

Simple geodesics yield bounded continued fraction expansions

The (*complete oriented*) *geodesics* of \mathbf{HP} identify with the space $\mathcal{G}(\mathbf{HP}) = \partial\mathbf{HP} \times \partial\mathbf{HP} \setminus \text{diagonal}$, and those of \mathbf{M}' with its quotient $\mathcal{G}(\mathbf{M}') = \Gamma' \backslash \mathcal{G}(\mathbf{HP})$ by the diagonal Γ' -action. The geodesics $\xi \in \mathcal{G}(\mathbf{HP})$ with a given projection $\xi' \in \mathcal{G}(\mathbf{M}')$ form the Γ' orbit of $\xi \in \mathcal{G}(\mathbf{HP})$ (note that its Γ -orbit always contains a representative with ends $\in (-1, 0) \times (1, +\infty)$).

A geodesic $\xi \in \mathcal{G}(\mathbf{M}')$ is called *simple* when there are no $\gamma \in \Gamma'$ such the pairs $\{\xi^-, \xi^+\}$ and $\{\gamma\xi^-, \gamma\xi^+\}$ are disjoint and linked with respect to the cyclic order on $\partial\mathbf{HP}$. The simple geodesics form a closed subset $\mathcal{S}(\mathbf{M}') \subset \mathcal{G}(\mathbf{M}')$.

Define a *combinatorial geodesics* of \mathcal{T}' as a sequence $p \in \vec{E}(\mathcal{T}')^{\mathbb{Z}}$ whose consecutive letters $p_k p_{k+1}$ satisfy $\partial^+ p_k = \partial^- p_{k+1}$ but $p_k^{-1} \neq p_{k+1}$. Since \mathbf{M}' retracts by deformation onto \mathcal{T}' , a complete geodesic $\xi' \subset \mathbf{M}'$ is homotopic relative to its ends to a combinatorial geodesic in \mathcal{T}' .

A geodesic $\xi \subset \mathbf{HP}$ follows a combinatorial geodesic in \mathcal{T} which, up to the action of $\text{PSL}_2(\mathbb{Z})$, may be assumed to pass through the directed base edge so that $(\xi^-, \xi^+) \in (-1, 0) \times (1, \infty)$. The continued fraction expansions $-1/\xi^- = [x_{-1}; x_{-2}, \dots]$ and $\xi^+ = [x_0; x_1, x_2, \dots]$ define the sequence $x \in (\mathbb{N}_{\geq 1})^{\mathbb{Z}}$. Consider the sequence $X \in \{R, L\}^{\mathbb{Z}}$ whose exponents are given by $x \in (\mathbb{N}_{\geq 1})^{\mathbb{Z}}$, namely $X = \dots L^{-x_1} R^{x_0} L^{x_1} \dots$. Now the projected geodesic $\xi' \subset \mathbf{M}'$ is homotopic relative to its ends to a combinatorial geodesic in \mathcal{T}' starting at the base arc.

Lemma 2.2 (simple geodesics have bounded cusp excursions). *Consider irrationals $(\xi^-, \xi^+) \in (-1, 0) \times (1, \infty)$ with continued fraction expansions $-\xi^- = [x_{-1}; x_{-2}, \dots]$ and $\xi^+ = [x_0; x_1, x_2, \dots]$.*

Consider a finite index subgroup $\Gamma' \subset \text{PSL}_2(\mathbb{Z})$ corresponding to a finite cover $\mathbf{M}' \rightarrow \mathbf{M}$.

The geodesic $\xi = (\xi^-, \xi^+) \subset \mathbf{HP}$ projects mod Γ' to a geodesic $\xi' \subset \mathbf{M}'$.

If ξ' is simple then for all $n \in \mathbb{N}$ we have $x_n < M$ where M is the maximal cusp-width of Γ' .

Proof. Suppose that there is an index $m \in \mathbb{N}$ such that $x_m \geq M$, and let $C = R^{x_0} \dots L^{x_m}$ or $C = R^{x_0} \dots R^{x_m}$ depending on whether m is odd or even, so that $C^{-1}\xi = \xi_k = [x_k, \dots]$. The geodesic $(C^{-1}\xi^-, C^{-1}\xi^+)$ must also have a simple projection on \mathbf{M}' . However the cusp-width of ∞ is some $k \leq M$ for which $R^{-k} \in \Gamma'$, so the geodesics $(C^{-1}\xi^-, C^{-1}\xi^+)$ and its translate by R^{-k} belong to the same Γ' -orbit, but they intersect in \mathbf{HP} , so we have a contradiction. \square

In the light of Remark 1.3, it is natural to propose the following.

Conjecture 2.3 (transcendence of simple geodesics in finite modular covers). *Consider a finite index subgroup $\Gamma' \subset \Gamma$ corresponding to a finite cover $\mathbf{M}' \rightarrow \mathbf{M}$.*

A geodesic $\xi = (\xi^-, \xi^+) \in \mathcal{G}(\mathbf{HP})$ projects mod Γ' to a geodesic $\xi' \in \mathcal{G}(\mathbf{M}')$.

If ξ' is simple then $\xi^+ \in \mathbb{R}\mathbf{P}^1$ is either rational or quadratic or transcendental.

We will prove Conjecture 2.3 for leaves of minimal geodesic laminations...

2.2 Geodesic laminations and their language complexity

Geodesic laminations

Let us recall some notions about geodesic laminations on \mathbf{M}' , referring to [Haa86, Bon01] for details.

A *geodesic lamination* of \mathbf{M}' is a compact subset $\Xi' \subset \mathbf{M}'$ which is a disjoint union of complete simple geodesics called its leaves (it identifies with a certain compact of $\mathcal{S}(\mathbf{M}')$).

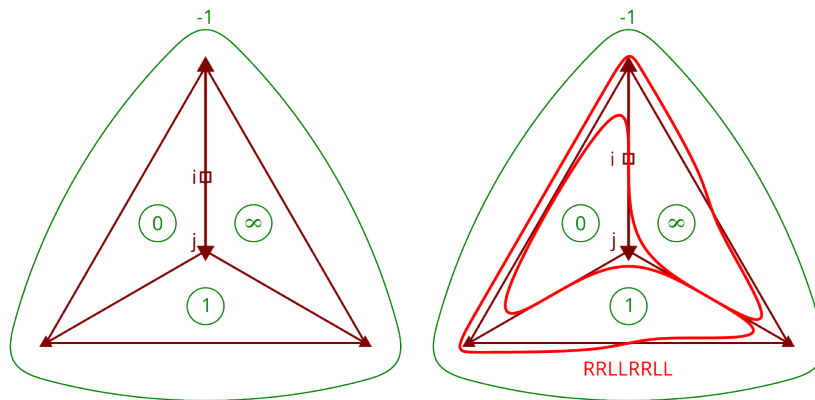


Figure 4: A geodesic (lamination) carried by $\mathcal{T}(3) \subset \Gamma(3)$ yields a train-track structure.

A geodesic lamination $\Xi' \subset \mathbf{M}'$ is *minimal* when every one of its leaves ξ' has both its past and future $\xi^\pm \bmod \Gamma'$ that are dense in Ξ' . By [Bon01, Proposition 3]: every geodesic lamination is a disjoint union of finitely many minimal sublaminations, and finitely many isolated leaves whose ends spiral along the minimal sublamination or escape into a cusp (see [PH92, Corollary 1.7.3]).

Remark 2.4 (the future determines the past). Note that in a minimal geodesic lamination $\Xi' \subset \mathbf{M}'$, every leaf $\xi' \subset \Xi'$ is determined by its future $\xi^+ \bmod \Gamma'$. More generally for $\xi^+ \in \partial\mathbf{HP}$, if there exists $\xi^- \in \partial\mathbf{HP}$ such that the geodesic $\xi' = (\xi^-, \xi^+) \bmod \Gamma'$ on \mathbf{M}' is simple, then there exists a unique such $\xi^- \in \partial\mathbf{HP}$ such that the closure of ξ' is a minimal geodesic lamination.

Since \mathbf{M}' retracts by deformation onto the embedded graph \mathcal{T}' , every geodesic lamination Ξ' is *carried by the embedded graph \mathcal{T}'* : all its leaves are homotopic relative to their ends to a combinatorial geodesic in \mathcal{T}' . The geodesic lamination Ξ' is *orientable* when for every edge in \mathcal{T} , at most one of its directions appears among all codings of its leaves as sequences over $\vec{E}(\mathcal{T}')$.

Moreover, the geodesic lamination Ξ' endows its carrying graph \mathcal{T}' with a *train-track* structure: around each vertex the 3 incident edges are bi-partitionned non-trivially (a singleton and a pair), so that every leaf of always travels from one subset to the other (an arbitrary choice can be made for edges that are not visited).

Subshifts and languages

Let us recall some notions about subshifts and languages (a good general reference is [Fog02]).

Fix a finite set \mathcal{A} called the alphabet of *syllables*, and let $\mathcal{A}^* = \bigcup_{n=1}^{\infty} \mathcal{A}^n$ be the free monoid of *words*. A *language* is a subset $\mathcal{L} \subset \mathcal{A}^*$. It is called *factorial* when it is closed by taking subwords (if $V \in \mathcal{L}$ is a subword of $W \in \mathcal{L}^*$ then $V \in \mathcal{L}$) and is called *bi-extendable* when every word is a bi-proper subword of another one (if $V \in \mathcal{L}$ then there exist non-trivial $U, W \in \mathcal{A}^*$ such that $UVW \in \mathcal{L}$). It is *uniformly recurrent* when for every word $V \in \mathcal{L}$, there is an integer $n \in \mathbb{N}$ such that every word $W \in \mathcal{L} \cap \mathcal{A}^n$ contains V as a subword. Its *factor complexity* is the function $\text{fac}(\mathcal{L}): \mathbb{N} \rightarrow \mathbb{N}$ whose value at $n \in \mathbb{N}$ is $\text{fac}_n(\mathcal{L}) = \text{Card}(\mathcal{L} \cap \mathcal{A}^n)$.

The set of bi-infinite sequences $\mathcal{A}^{\mathbb{Z}}$ with the product topology is a compact space; a basis of neighbourhoods for the topology is given by the clopen sets $\text{Cyl}_k(W) = \{Y \in \mathcal{A}^{\mathbb{Z}} : Y_{[k, k+n)} = W\}$ for $k \in \mathbb{Z}$ and $W \in \mathcal{A}^n$. The function $\text{Shift} : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ defined by $\text{Shift}(Y)_n = Y_{n+1}$ is a homeomorphism. A *subshift* (Ξ', Shift) of $(\mathcal{A}^{\mathbb{Z}}, \text{Shift})$ consists of a compact Shift-invariant subset. A subshift (Ξ', Shift) is *minimal* when every leaf ξ' has an orbit $(\text{Shift}^k \xi')_k$ that is dense in Ξ' as $k \rightarrow +\infty$ and as $k \rightarrow -\infty$.

There is a correspondence between subshifts $(\Xi', \text{Shift}) \subset (\mathcal{A}^*, \text{Shift})$ and languages $\mathcal{L}_{\Xi'} \subset \mathcal{A}^*$ that are factorial and bi-extendable, given by $\mathcal{L}_{\Xi'} = \{\xi'_{[k, k+n)} : \xi' \in \Xi', k \in \mathbb{Z}, n \in \mathbb{N}\}$. For example, a sequence $X \in \mathcal{A}^{\mathbb{Z}}$ has shift-closure $\text{Closure}(\{\text{Shift}^k(X) : k \in \mathbb{Z}\}) \subset \mathcal{A}^{\mathbb{Z}}$ a subshift whose language consists of all subwords of X . A subshift is minimal if and only if its language is uniformly recurrent.

Symbolic complexity of minimal geodesic laminations

Let us assemble the notions introduced in the previous paragraphs to explain Proposition 2.5 (which is well known (from [FZ08], [LN13, LN15], [BDD⁺17])).

From now on, we assume that Γ' is normalized by $S \in \text{PSL}_2(\mathbb{Z})$, so that the even-positive semigroup $\Gamma \cap R\text{PSL}_2(\mathbb{N})L$ is freely generated by the \dagger -invariant basis \mathcal{B}_{RL} . Recall that the transposition involution of $\Gamma_{\geq 0} \cap R\text{PSL}_2(\mathbb{N})L$ yields the involution of the corresponding subsemigroup in $\vec{E}(\mathcal{T}')^*$ that reverses words while inverting its arcs.

Consider a non-empty geodesic lamination $\Xi' \subset \mathbf{M}'$. Since it is carried by \mathcal{T}' , its leaves $\xi' \in \Xi'$ correspond to Shift-orbits of sequences $\vec{E}(\mathcal{T}')^{\mathbb{Z}}$: this defines its *lamination language* $\mathcal{L}_{\Xi'} \subset \vec{E}(\mathcal{T}')^*$, which is factorial and bi-extendable. Up to the action of $\text{PGL}_2(\mathbb{Z})$, we may assume that $\mathcal{L}_{\Xi'} \cap \vec{E}(\mathcal{T}')^3$ contains the word $\vec{e}_{L-1}\vec{e}_1\vec{e}_R$.

Now assume that $\Xi' \subset \mathbf{M}'$ is minimal. By uniform recurrence to $\vec{e}_{L-1}\vec{e}_1\vec{e}_R$, there is a finite subset $\mathcal{A} \subset \mathcal{B}_{RL}$ freely generating a submonoid of $\Gamma' \cap R\text{PSL}_2(\mathbb{N})L$ such that every orbit can be uniquely recoded as a sequence in $\mathcal{A}^{\mathbb{Z}}$. We may (and will) further assume that for all $A \in \mathcal{A}$ either A or A^\dagger occurs, hence \mathcal{A} is uniquely determined. This defines its *involution language* which we also denote $\mathcal{L}_{\Xi'} \subset \mathcal{A}^*$; it is still factorial and bi-extendable and uniformly recurrent.

Proposition 2.5 (roughly linear complexity). *For a minimal geodesic lamination $\Xi' \subset \mathbf{M}'$ as above, its lamination and involution languages $\mathcal{L}_{\Xi'}$ have roughly linear factor complexity:*

$$\exists \kappa \in \mathbb{N}, \quad \forall n \in \mathbb{N}, \quad |\text{fac}_n(\mathcal{L}_{\Xi'}) - \kappa n| < \infty.$$

Proof. The proposition follows from much more precise results in the literature, as we now explain. All methods of proof boil down to computing and bounding the discrete derivative of the complexity functions $n \mapsto \text{fac}_{n+1}(\mathcal{L}_{\Xi'}) - \text{fac}_n(\mathcal{L}_{\Xi'})$, by enumerating special factors (see [Cas97, Theorem 4.9.3]).

By definition, the involution language $\mathcal{L}_{\Xi'} \subset \mathcal{A}^*$ is the image of the lamination language $\mathcal{L}_{\Xi'} \subset \vec{E}(\mathcal{T}')^*$ by the natural decoding morphism $\mathcal{A}^* \rightarrow \vec{E}(\mathcal{T}')^*$, which is non-erasing and injective. Hence by [CN10, Lemma 4.6.7 and Lemma 4.6.8], their quasi-linear complexities are equivalent.

If Ξ' is orientable then the lamination language $\mathcal{L}_{\Xi'} \subset \vec{E}(\mathcal{T}')^*$ has ultimate affine complexity by [LN13, Theorem 4.1.1]. In fact, the involution language $\mathcal{L}_{\Xi'} \subset \mathcal{A}^*$ is that of an interval exchange transformation for its natural encoding (see [BDD⁺17]), so it satisfies $\text{fac}_n(\mathcal{L}_{\Xi'}) \leq (\text{Card } \mathcal{A} - 1)n + 1$, with equality if and only if the IET satisfies Keane's infinite distinct orbits condition (see [FZ08]).

If Ξ' is non-orientable then by [LN15, §5 Remark 1] there is a ramified double cover of \mathbf{M}' in which Ξ' lifts to an oriented minimal lamination carried by the lift of \mathcal{T}' , whose lamination language has sub-affine complexity: this shows that $\mathcal{L}_{\Xi'}$ is the morphic image of a language with sub-affine complexity, so it also has sub-affine complexity. \square

2.3 Transcendence of minimal laminations in finite covers

We are now ready to prove the main Theorems 2.7 in this work. We make the following assumption.

Context 2.6. Fix distinct $\xi^-, \xi^+ \in \mathbb{R}\mathbf{P}^1$, and behold the geodesic $\xi = (\xi^-, \xi^+) \in \mathcal{G}(\mathbf{HP})$.

Consider a finite index subgroup $\Gamma' \subset \Gamma$ that is torsion free, corresponding to a finite cover $\mathbf{M}' \rightarrow \mathbf{M}$ by a smooth surface. The geodesic $\xi \in \mathcal{G}(\mathbf{HP})$ projects mod Γ' to a geodesic $\xi' \in \mathcal{G}(\mathbf{M}')$. Assume that ξ' is simple, so that its closure $\Xi' \subset \mathbf{M}'$ is a geodesic lamination. Suppose furthermore that $\Xi' \subset \mathbf{M}'$ is minimal (in particular ξ^\pm are irrational).

Theorem 2.7 (transcendence of minimal laminations). *Under assumptions 2.6, $\xi^\pm \in \mathbb{R}$ are either quadratic conjugates or both transcendent, in which case either $w_2(\xi^\pm) = \infty$ (Mahler type U_2) or else $\exists c \in \mathbb{R}_{>0} \forall d \in \mathbb{N}_{\geq 1}: w_d(\xi^\pm) \leq \exp(c(\log 3d)^5(\log \log 3d)^4)$ (Mahler class S or T).*

Proof. Since the conclusion only concerns $\xi^\pm \in \mathbb{R}\mathbf{P}^1$, the statement assumes the existence of a torsion-free finite index $\Gamma' \subset \Gamma$ such that ξ' is simple with closure a geodesic lamination Ξ' that is minimal. Thus, we may pass to any finite index subgroup of Γ' : this will choose a lift of the geodesic ξ' to the corresponding finite cover of \mathbf{M}' , whose closure will remain a minimal geodesic lamination. Hence we may assume Γ' normal in Γ and normalized by S , to fully exploit the discussions from the previous paragraphs leading to Proposition 2.5. In that case, the action of $\mathrm{PSL}_2(\mathbb{Z})$ on \mathbf{HP} descends to the action of the Galois group of Γ/Γ' by deck transformations of the cover $\mathbf{M}' \rightarrow \mathbf{M}$.

The action of $\mathrm{PSL}_2(\mathbb{Z})$ on ξ preserves the conclusion on ξ^\pm and the assumptions on $\Xi' \subset \mathbf{M}'$, so we may use this action to position ξ in a convenient position with respect to \mathcal{T} : the geodesic $\xi \subset \mathbf{HP}$ follows a combinatorial geodesic in \mathcal{T} which may be assumed to pass through the base arc so that $(\xi^-, \xi^+) \in (-1, 0) \times (1, \infty)$. The continued fraction expansions $-1/\xi^- = \lfloor x_{-1}; x_{-2}, \dots \rfloor$ and $\xi^+ = \lfloor x_0; x_1, x_2, \dots \rfloor$ define the sequence $x \in (\mathbb{N}_{\geq 1})^{\mathbb{Z}}$. Consider the sequence $X \in \{R, L\}^{\mathbb{Z}}$ whose exponents are given by $x \in (\mathbb{N}_{\geq 1})^{\mathbb{Z}}$, namely $X = \dots L^{-x_1} R^{x_0} L^{x_1} \dots$.

Now the projected geodesic $\xi' \subset \mathbf{M}'$ is homotopic relative to its ends to a combinatorial geodesic in \mathcal{T}' starting at the base arc then turning left and right according to $X \in \{R, L\}^{\mathbb{Z}}$, which may also be encoded as a sequence $\vec{E}(\xi') \subset \vec{E}(\mathcal{T}')^{\mathbb{Z}}$. The geodesic lamination $\Xi' \subset \mathbf{M}'$ has lamination language in $\vec{E}(\mathcal{T}')$ given by the set of subwords of $\vec{E}(\xi')$.

Note that ξ^\pm are quadratic conjugates if and only if Ξ' consists of a single periodic leaf. Otherwise by minimality X^\pm are not ultimately periodic, and Ξ' has no periodic leaves, its space of leaves forms a Cantor set. By minimality, the languages of X^+ and X^- coincide and $\mathrm{Dio}(X^+) < \infty \iff \mathrm{Dio}(X^-) < \infty$.

Since Ξ' is minimal, we are in the situation leading to Proposition 2.5: there is a syllable set $\mathcal{A} \subset \mathcal{B}_{RL}$ freely generating a submonoid of $\Gamma' \cap R\mathrm{PSL}_2(\mathbb{N})L$ such that X can be uniquely recoded as a sequence in $\mathcal{A}^{\mathbb{Z}}$, and its complexity is bounded by an affine function.

Thus Theorem 1.9 yields the desired result on ξ^\pm . □

Let us recall a few notions on mapping class groups (see [FM12]).

The group of orientation preserving mapping classes $\mathrm{Mod}^o(\mathbf{M}')$ contains the pure mapping class group $\mathrm{Mod}(\mathbf{M}', \partial\mathbf{M}') = \mathrm{Homeo}^o(\mathbf{M}', \partial\mathbf{M}') / \mathrm{Homeo}_0(\mathbf{M}', \partial\mathbf{M}')$ as a finite index normal subgroup: it is the subgroup of $\mathrm{Out}(\Gamma')$ which fixes all parabolic conjugacy classes of Γ' (associated to the so called *peripheral loops* which surround the cusps), or the kernel of the surjective morphism $\mathrm{Mod}^o(\mathbf{M}') \rightarrow \mathfrak{S}(\partial\mathbf{M}')$ to the permutation group of the punctures. The group $\mathrm{Mod}(\mathbf{M}', \partial\mathbf{M}')$ is generated by Dehn-twists along simple closed loops (and to generate $\mathrm{Mod}(\mathbf{M}')$ one must also add half-twists braiding any two punctures, as well as any orientation reversing class). In any case, every element in $\mathrm{Mod}(\mathbf{M}')$ has a finite power in $\mathrm{Mod}(\mathbf{M}', \partial\mathbf{M}')$.

Every element in $\text{Mod}(\mathbf{M}', \partial\mathbf{M}')$ has infinite order, and is represented by an isometry of the hyperbolic surface \mathbf{M}' , which is unique modulo an isometry isotopic to the identity. Hence the group $\text{Mod}(\mathbf{M}', \partial\mathbf{M}')$ acts on the space of simple geodesics, and on the space of geodesic laminations.

Example 2.8 (pseudo-Anosov classes). An element $\varphi' \in \text{Mod}(\mathbf{M}', \partial\mathbf{M}')$ is *pseudo-Anosov* when no simple closed geodesics are invariant under its action. Such a pseudo-Anosov element admits a unique stable fixed geodesic lamination $\Xi' \subset \mathbf{M}'$, it is minimal with no closed leaves, and uniquely ergodic (namely it admits a unique transverse measure [Mas82]). In this stable lamination, there may sometimes also be a simple geodesic that is fixed by φ' .

By Thurston classification of mapping classes (see [?, FM12]), a non-trivial $\varphi' \in \text{Mod}(\mathbf{M}', \partial\mathbf{M}')$ has a maximal closed embedded subsurface with geodesic boundary on which it acts trivially (which may be empty or disconnected), and on each connected component of the complement it acts as a pseudo-Anosov mapping class with a unique stable geodesic lamination, which is minimal aperiodic and filling that subsurface (in the sense that it intersects any non-peripheral simple loop): call those the *φ' -stable minimal aperiodic laminations*. When such a lamination Ξ' contains a geodesic $\xi' \subset \Xi'$ that is preserved by φ' , we will say this geodesic is *preserved and expanded*.

Theorem 2.9 (transcendence of geodesics fixed by isometries). *If $\xi' \in \mathcal{G}(\mathbf{M}')$ is preserved and expanded by a mapping class $\varphi' \in \text{Mod}(\mathbf{M}', \partial\mathbf{M}')$, then ξ^\pm are transcendental numbers of Mahler class S or T, more precisely $\exists c \in \mathbb{R}_{>0}, \forall d \in \mathbb{N}_{\geq 1}: w_d(\xi^\pm) \leq \exp(c(\log 3d)^3(\log \log 3d)^2)$.*

Proof. The mapping class φ' preserves the lamination Ξ' , hence it induces an endomorphism of monoids $\Phi: \Sigma^* \rightarrow \Sigma^*$ which preserves the involution language $\mathcal{L}_{\Xi'}$, and the word X is, up to a shift Shift , a fixed point of that morphism, namely $\Phi(X) = \text{Shift}^k(X)$. This means that $\Phi^{2k}(X) = \text{Shift}^{2k}(X)$ hence $\text{Shift}^k X$ is fixed by the conjugate morphism $\text{Shift}^k \Phi^{2k} \text{Shift}^{-k}$. Note that the pair of numbers $(\text{Shift}^k X)^\pm \cdot \infty$ is the translates of ξ^\pm by an element of $\text{PSL}_2(\mathbb{Z})$ so their Mahler exponents are all the same. Hence the result follows from Theorem 1.11. \square

Remark 2.10 (generalisations to infinite products). There are many ways of generalizing the previous result, to bridge the gap between geodesics or minimal geodesic laminations that are stably-expanding by isometries of \mathbf{M}' and all others.

First it is convenient to fix a projective class of measures on the minimal geodesic lamination Ξ' (that is a point in a finite dimensional real convex polytope associated to Ξ'). If Ξ' is stable and expanding under the action of an isometry, then this projective measure is unique. This enables to work in the piecewise linear space of measured geodesic laminations, and makes it easier to define the notion of stably-expanding fixed point by a mapping class.

Next, every measured minimal geodesic lamination Ξ' fills a certain subsurface $\mathbf{M}'[\Xi']$, and is a stably-expanding fixed point point by an infinite product of mapping classes that are pseudo-Anosov in restriction $\mathbf{M}'[\Xi']$.

Such an infinite product would yield an S -adic representation by substitutions of the associated symbolic space $X \subset \vec{E}(\mathcal{T}')$. However, this product may not be unique, and it is a subtle matter to define a procedure for producing a preferred sequence of (pseudo-Anosov) mapping classes with the desired lamination as fixed point.

On the other hand, one may start with special kinds of infinite products of mapping classes with a unique stably-expanding lamination Ξ' and discuss the diophantine approximation properties of its laves in terms of the complexity of the sequence defining the product (growth of exponents or stretch factors).

For instance, one may consider an alphabet of pseudo-Anosov elements $\varphi_1, \dots, \varphi_d$, and assume that the stretch factor increases under products, namely $\lambda(\varphi_i \varphi_j) > \max\{\lambda(\varphi_i), \lambda(\varphi_j)\}$.

Another example would be to consider two non-empty simple multiloops $\alpha = \sqcup_1^m \alpha_i$ and $\beta = \sqcup_1^n \beta_j$, and denote the associated Dehn multitwists by $D_\alpha = \prod_i^m D_{\alpha_i}$ and $D_\beta = \prod_j^n D_{\beta_j}$. Assume that the union $\alpha \cup \beta$ is filling \mathbf{M}' , in the sense that $\mathbf{M}' \setminus (\alpha \cup \beta)$ is homotopic to a disjoint union of discs with at most one puncture. In that case the product $D_\alpha D_\beta$ is pseudo-Anosov (this is the so called Penner construction for pseudo-Anosov mapping classes, see [Thu88, §6]). Now for a sequence $x \in (\mathbb{N}_{\geq 1})^{\mathbb{N}}$ let $D_x = D_\alpha^{x_0} D_\beta^{x_1} \dots$ be the infinite product of those Dehn multitwists. By contrast with the previous example, we may allow arbitrarily large powers of Dehn multitwists.

2.4 Questions and conjectures on the set of profinite simple numbers

Let us give a name to the (pairs of) numbers covered by Conjecture 0.3 (and Theorem 0.4).

Definition 2.11 (profinite simple). Define the subset $\tilde{\mathcal{S}}(\Gamma) \subset \mathcal{G}(\mathbf{HP})$ of *profinite simple geodesics* (for Γ) as consisting of those $\xi \in \mathcal{G}(\mathbf{HP})$ such that there exists a finite index subgroup $\Gamma' \subset \Gamma$ such that $\xi' = \text{mod } \Gamma' \in \mathcal{S}(\mathbf{M}')$. Define the subset $\tilde{\mathcal{S}}^+(\Gamma) \subset \mathbb{R} \mathbf{P}^1$ of *profinite simple numbers* (for Γ) as its image by the projection on (any) one of the two factors $\mathcal{G}(\mathbf{HP}) \rightarrow \mathbb{R} \mathbf{P}^1$.

We may define the subsets of *profinite congruence simple* geodesics and numbers $\mathcal{CS}^+(\Gamma)$, by restricting to congruence covers $\mathbf{M}(N) \rightarrow \mathbf{M}$ associated to the congruence subgroups $\Gamma(N) \subset \Gamma$ defined for $N \in \mathbb{N}_{>2}$ as the kernel of the reduction mod N morphism $\text{PSL}_2(\mathbb{Z}) \rightarrow \text{PSL}_2(\mathbb{Z}/N)$.

One may similarly define the subsets of *minimal* and *morphic* profinitely simple geodesics, as well as their intersections with the profinitely congruence simple geodesics, hence all the corresponding subsets of real numbers.

Remark 2.12 (Hausdorff dimension). It follows from [BS85] that for every finite cover $\mathbf{M}' \rightarrow \mathbf{M}$ we have $\dim_H \mathcal{G}(\mathbf{M}') = 0$. Thus $\tilde{\mathcal{S}}(\Gamma) \subset (\mathbb{R} \mathbf{P}^1)^2$ hence $\tilde{\mathcal{S}}^+(\Gamma) \subset \mathbb{R} \mathbf{P}^1$ have Hausdorff dimension 0.

Question 2.13 (continued fractions). A combinatorial characterisation of the continued fractions of profinite simple numbers may be given: they are all $\{L, R\}$ -sequences associated to lamination languages in trivalent graphs, which have mostly been characterised in [FZ08, LN13].

What if we restrict to profinite congruence simple numbers? Can we characterize describe the continued fraction expansions of ends of simple geodesics in the congruence cover $\mathbf{M}(N)$? (The case $N = 3$ boils down to the Markov and Sturmian sequences discussed in Section 3.)

In upcoming work, we will study in detail the combinatorics associated to simple geodesics in the normal congruence covers of genus 0 from Figure 3.

Question 2.14 (Lagrange spectrum). By Lemma 0.2, profinitely simple reals have finite Lagrange constant: can we describe the metric topology of the subsets $L(\mathcal{CS}^+(\Gamma)) \subset L(\mathcal{S}^+(\Gamma)) \subset L(\mathbb{R})$? What is their Hausdorff dimension? (Note: $\exists \mathcal{S} \subset \mathbb{R}$ with $\dim_H(\mathcal{S}) = 0$ but $\dim_H(L(\mathcal{S})) = 1$.)

Question 2.15 (Mahler spectra w_2, \hat{w}_2). Can we describe the metric topology of the Mahler spectra $w_2(\tilde{\mathcal{CS}}(\Gamma)) \subset w_2(\tilde{\mathcal{S}}(\Gamma))$ and $\hat{w}_2(\tilde{\mathcal{CS}}(\Gamma)) \subset \hat{w}_2(\tilde{\mathcal{S}}(\Gamma)) \subset [2, 1 + \phi]$?

We will recall in 3.19 the [BL05, Theorem 3.1] describing those spectra in restriction to numbers arising from simple geodesics in the modular torus \mathbf{M}' (associated to the derived group Γ') that pass through the Weierstrass point. This gives a (very) partial answer to the previous question, and some hints on how to refine it, and eventually address it.

3 Simple geodesics in the modular torus and abelian cover

Subsection 3.1 defines the modular torus \mathbf{M}' and its mapping class group $\mathrm{SL}_2(\mathbb{Z})$. Subsection 3.2 describes its universal abelian cover \mathbf{M}'' and its symmetries (the only original work in this Section). Subsection 3.3 uses the mapping class group action of $\mathrm{Mod}(\mathbf{M}')$ to find the continued fractions associated to simple geodesics in \mathbf{M}' , first observed by [Coh71], then revisited by [Haa87] and [Ser85], see also [Spr17, Spr24]. The last Subsection 3.4 surveys properties on the diophantine approximation properties of simple geodesics in the modular torus, from [ADQZ01, BL05, AB10].

3.1 The derived modular group and the modular torus

The abelianisation $\mathbb{Z}/2 * \mathbb{Z}/3 \rightarrow \mathbb{Z}/2 \times \mathbb{Z}/3$ of the modular group corresponds to the $\mathbb{Z}/6$ -Galois cover $\mathbf{M}' \rightarrow \mathbf{M}$ of the modular orbifold by the modular torus, which is a cusped torus whose fundamental group $\pi_1(\mathbf{M}') = \mathrm{PSL}_2(\mathbb{Z})'$ is freely generated by

$$A = [T, S^{-1}] = RL = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = [T^{-1}, S] = LR = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

(As an exercise, show that Γ' consists of all infinite order $C \in \Gamma$ with $\mathrm{Rad}(C) = 0 \pmod{6}$.)

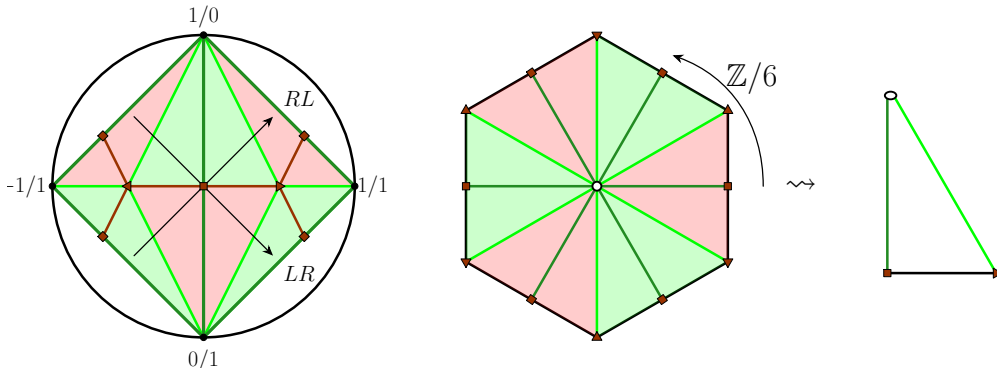


Figure 5: The free group $\mathrm{PSL}_2(\mathbb{Z})'$ acts on \mathbf{HP} with quotient a cusped torus \mathbf{M}' . The Galois group $\mathrm{PSL}_2(\mathbb{Z})/\mathrm{PSL}_2(\mathbb{Z})' = \mathbb{Z}/6$ acts on \mathbf{M}' with quotient \mathbf{M} .

The homotopy classes of loops in \mathbf{M}' correspond to the conjugacy classes in $\pi_1(\mathbf{M}')$, hence to the reduced cyclic words on $\{A, A^{-1}, B, B^{-1}\}$. For $n \in \mathbb{Z}^*$, the n -th power of the commutator $[A, B]$ corresponds to the loop winding n times around the cusp; every other non-trivial homotopy class contains a unique geodesic.

The cusp-compactification $\overline{\mathbf{M}'} = \mathbf{M}' \sqcup \partial \mathbf{M}' = \Gamma' \backslash (\mathbf{HP} \sqcup \mathbb{Q} \mathbf{P}^1)$ is homeomorphic to a torus with a marked point, and its fundamental group is the quotient of Γ' by the normal subgroup generated by $[A, B]$; that is $\pi_1(\overline{\mathbf{M}'}) = \Gamma'/\Gamma'' = H_1(\Gamma'; \mathbb{Z}) = \mathbb{Z}_A \oplus \mathbb{Z}_B$.

In \mathbf{M}' , the simple geodesics with both ends in the cusp are in bijection with the simple closed geodesics, and correspond to the non-trivial simple loops in $\overline{\mathbf{M}'}$. Hence, the abelianisation map yields a map $\Gamma' \bmod \mathit{conj} \rightarrow H_1(\mathbf{M}'; \mathbb{Z})$ that restricts to a bijection between classes associated to simple geodesics in \mathbf{M}' and primitive vectors of the lattice $H_1(\Gamma'; \mathbb{Z})$.

More precisely the abelianisation map yields a map $(\Gamma' \bmod \mathit{conj})^2 \rightarrow H_1(\mathbf{M}'; \mathbb{Z})^2$ that restricts to a bijection from independent conjugacy classes of bases of the free group Γ' to bases of the free abelian group $H_1(\Gamma'; \mathbb{Z})$, and those correspond to pairs of simple closed geodesics in \mathbf{M}' having one intersection point (equivalently, they are represented by pairs of elements in Γ' whose commutator corresponds to the loop winding once around the cusp).

The mapping class group $\text{Mod}(\overline{\mathbf{M}'})$ is identified by the Dehn-Nielsen-Baer [FM12, Theorem 8.1] with the outer automorphism group $\text{Out}(\pi_1(\overline{\mathbf{M}'}) = \text{GL}(\mathbb{Z}_A \oplus \mathbb{Z}_B)$, and contains the orientation preserving mapping class group $\text{Mod}^o(\overline{\mathbf{M}'}) = \text{SL}(\mathbb{Z}_A \oplus \mathbb{Z}_B)$ with index 2, the quotient being generated by $D_J: (A, B) \mapsto (B, A)$. The group $\text{Mod}^o(\overline{\mathbf{M}'}) = \text{SL}(\mathbb{Z}_A \oplus \mathbb{Z}_B)$ is generated by the positive and negative Dehn twists $D_R^{\pm 1}$ and $D_L^{\pm 1}$ along the simple loops A and B :

$$\begin{aligned} D_R: (A, B) &\mapsto (A, AB) & D_R^{-1}: (A, B) &\mapsto (A, A^{-1}B) \\ D_L: (A, B) &\mapsto (BA, B) & D_L^{-1}: (A, B) &\mapsto (B^{-1}A, B) \end{aligned}$$

and its relation are generated by $D_L D_R^{-1} D_L = D_R^{-1} D_L D_R^{-1}$. It contains the elements $D_T = D_L D_R^{-1}$ and $D_S = D_L D_R^{-1} D_L$ whose common power $D_S^2 = D_T^3 = D_{-1}$ has order 2.

The group $\text{Mod}(\mathbf{M}')$ of diffeotopy classes of diffeomorphisms of \mathbf{M}' fixing the cusp fits into Birman's short exact sequence $1 \rightarrow \pi_1(\mathbf{M}') \rightarrow \text{Mod}(\mathbf{M}') \rightarrow \text{Mod}(\overline{\mathbf{M}'}) \rightarrow 1$, and the associated map $\text{Mod}(\overline{\mathbf{M}'}) \rightarrow \text{Out}(\pi_1(\mathbf{M}'))$ is an isomorphism. Restricting to orientation preserving classes yields $1 \rightarrow \mathbb{Z}_A * \mathbb{Z}_B \rightarrow \text{Map}^+(\mathbf{M}') \rightarrow \text{SL}(\mathbb{Z}_A \oplus \mathbb{Z}_B) \rightarrow 1$.

3.2 The second derived modular group and hexpunctured plane

The abelianisation $\Gamma' \rightarrow \Gamma'/\Gamma''$ associated to Hurwitz' map $\pi_1(\mathbf{M}') \rightarrow H_1(\mathbf{M}'; \mathbb{Z})$, given by $\mathbb{Z}_A * \mathbb{Z}_B \rightarrow \mathbb{Z}_A \times \mathbb{Z}_B$, corresponds to the universal abelian cover $\mathbf{M}'' \rightarrow \mathbf{M}'$ with Galois group $\Gamma'/\Gamma'' = H_1(\mathbf{M}'; \mathbb{Z})$. The Jacobian integration map of \mathbf{M}' based at the cusp $\infty \in \partial\mathbf{M}'$ lifts to the total space \mathbf{M}'' and identifies it with the lattice-punctured plane $H_1(\mathbf{M}'; \mathbb{R}) \setminus H_1(\mathbf{M}'; \mathbb{Z})$.

The kernel of the abelianization of $\pi_1(\mathbf{M}') = \Gamma'$ yields the fundamental group $\pi_1(\mathbf{M}'') = \Gamma''$, which is freely generated by an infinite set of elements (whose conjugacy classes in Γ' are) indexed by $H_1(\mathbf{M}'; \mathbb{Z}) = \Gamma'/\Gamma''$. For example, Γ'' is freely generated by the set of commutator-conjugates $\{(A^m B^n) \cdot [A, B] \cdot (A^m B^n)^{-1} : (m, n) \in \mathbb{Z}^2\}$ and by the set of commutators $\{[A^m, B^n] : (m, n) \in \mathbb{Z}^2\}$.

The quotient $\mathbf{HP} \rightarrow \Gamma'' \backslash \mathbf{HP}$ maps the ideal triangulation Δ with vertices $\Gamma/\langle R \rangle \subset \partial\mathbf{HP}$ to an ideal triangulation Δ'' of \mathbf{M}'' with vertices $\Gamma'' \backslash \Gamma/\langle R \rangle$, and the trivalent tree $\mathcal{T} \subset \mathbf{HP}$ dual to Δ to a trivalent graph $\mathcal{H} \subset \mathbf{M}''$ dual to Δ . Hence the inclusion $\mathcal{H} \subset \mathbf{M}''$ is a homotopy equivalence, and the complementary regions in $\mathbf{M}'' \setminus \mathcal{H}$ are punctured regular hexagons (for the hyperbolic metric).

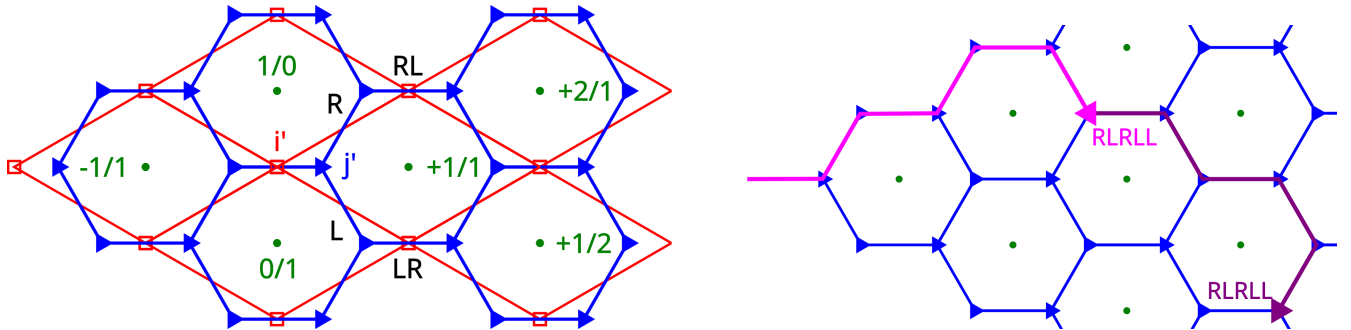


Figure 6: Action of Γ/Γ'' on the honeycomb graph \mathcal{H} with base arc $[i', j']$, and cusps $\Gamma'' \backslash \Gamma/\langle R \rangle$. Paths encoded by $RLRL$ and $(RLRL)^2$ in the honeycomb graph \mathcal{H} .

Let us describe the Galois action of Γ/Γ'' on the metabelian cover $\mathbf{M}'' \rightarrow \mathbf{M}$ combining the action of Γ'/Γ'' on $\mathbf{M}'' \rightarrow \mathbf{M}'$ with the action of Γ/Γ' on $\mathbf{M}' \rightarrow \mathbf{M}$. (Recall that Galois actions are given by left multiplication whereas monodromy actions are given by right multiplication: these coincide on the identity coset, but otherwise are related by conjugacy by elements of the coset.)

The group Γ/Γ'' acts freely transitively on the arcs of \mathcal{H} . More precisely the generators S and T act by rotations of order 2 and 3 around the mid-edges and vertices of \mathcal{H} . The subgroup Γ'/Γ'' generated by $A = RL$ and $B = LR$ acts by translation of \mathcal{H} with fundamental domain a tripod formed by three adjacent edges.

Consequently the action of $C \in \Gamma$ on \mathcal{H} sends the base arc to another arc whose angle is determined by the coset $C \bmod \Gamma'$ with a representative $C' \in \pm\{\mathbf{1}, L, R, S, T, T^{-1}\}$, and whose base point is determined by $C(C')^{-1} \bmod \Gamma''$. In particular Γ'/Γ'' identifies with the arcs of \mathcal{H} that are parallel to the base arc.

This description leads to the following Theorem observed in [Sim22, Section 3.2].

Theorem 3.1 (hexagonal symmetries). *The group Γ/Γ'' is the semi-direct product*

$$\Gamma/\Gamma'' = \Gamma'/\Gamma'' \rtimes \Gamma/\Gamma'$$

where the action of the quotient $\Gamma/\Gamma' = \mathbb{Z}/2 \times \mathbb{Z}/3$ by outer-automorphisms (given by conjugacy) on the kernel $\Gamma'/\Gamma'' = \mathbb{Z}_A \oplus \mathbb{Z}_B$ is generated by

$$\text{AD}_S(A, B) = (A^{-1}, B^{-1}) \quad \text{AD}_T(A, B) = (B^{-1}, AB^{-1}).$$

This represents Γ/Γ'' as the affine isometry group of the oriented hexagonal lattice $H_1(\Gamma'; \mathbb{Z}) = \mathbb{Z}_A \oplus \mathbb{Z}_B$ with $\text{angle}(A, B) = \frac{2\pi}{6}$, where the translation action of Γ'/Γ'' is by \mathbb{Z}^2 -translation while the outer-automorphism action of Γ/Γ' is by $\mathbb{Z}/6$ -rotation.

Finally, the involutions $K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ generating $\Gamma^\pm/\Gamma = \mathbb{Z}/2$ act as reflections of $H_1(\Gamma'; \mathbb{Z}) = \mathbb{Z}_A \oplus \mathbb{Z}_B$ across the horizontal and vertical axes, namely:

$$\text{AD}_J(A, B) = (B, A) \quad \text{AD}_K(A, B) = (B^{-1}, A^{-1})$$

This describes Γ^\pm/Γ'' as a semi-direct product $(\Gamma/\Gamma'') \rtimes (\Gamma^\pm/\Gamma) = (\Gamma'/\Gamma'') \rtimes (\Gamma^\pm/\Gamma')$ and represents it as the group of affine isometries of the hexagonal lattice $H_1(\mathbf{M}'; \mathbb{Z})$.

The Jacobian map yields a conformal diffeomorphism sending the hyperbolic metric on \mathbf{M}' to the Euclidean metric on $(H_1(\mathbf{M}'; \mathbb{R}) \setminus H_1(\mathbf{M}'; \mathbb{Z})) \bmod H_1(\mathbf{M}'; \mathbb{Z})$. In \mathbf{M}' , a simple Euclidean geodesic is homotopic to a unique simple hyperbolic geodesic: their lifts in \mathbf{M}'' are homotopic relative to their ends, so they cut the edge of the triangulation Δ'' dual to \mathcal{H} according to the same combinatorics.

3.3 Combinatorics of simple geodesics in the modular torus

The aim of this section is to recall the description of simple geodesics $\xi' \subset \mathbf{M}'$ in terms of the continued fraction expansions of their endpoints $\xi^\pm \in \mathbb{R}\mathbf{P}^1$. The method is to lift these geodesics to simple geodesics $\xi'' \subset \mathbf{M}''$, which intersect Δ'' according to the same combinatorics as (segments of) straight lines in the Euclidean plane $H_1(\mathbf{M}'; \mathbb{R})$, and use the action of $\text{Mod}^o(\mathbf{M}') = \text{SL}_2(\mathbb{Z}_A \oplus \mathbb{Z}_B)$.

There are seven *topological types* of simple geodesics ξ' in \mathbf{M}' , depending on the asymptotic behaviour of its ends, which could either escape to the cusp or else accumulate on a minimal geodesic lamination, either periodic or without periodic leaves (see [Haa86, Lemma 5.1]); the types are: cusp-to-cusp, cusp-to-periodic, periodic-to-cusp, periodic, cusp-to-aperiodic, aperiodic-to-cusp, aperiodic. The action of $\text{Mod}^o(\mathbf{M}')$ preserves simple geodesics and their topological type.

The geodesics on \mathbf{M}' correspond to sequences over the symmetric set of generators $\{A^{\pm 1}, B^{\pm 1}\}$ of the fundamental group Γ' , indexed by an interval of \mathbb{Z} which is considered up to shift of indices

(which acts as cyclic permutation on finite words). For simple geodesics, this sequence involves at most two letters: up to the action of the finite order mapping classes $\text{AD}_S, \text{AD}_T \in \text{Out}(\Gamma')$ by $\text{AD}_S(A, B) = (A^{-1}, B^{-1}), \text{AD}_T(A, B) = (B^{-1}, AB^{-1})$, we may assume that the alphabet is $\{A, B\}$.

Let us first recall from [Fog02, Chapter 6] and [Aig13] the definitions of Christoffel words and Cohn matrices to associated them Markov rational and quadratic real numbers.

Definition 3.2 (Christoffel word, Cohn matrix, Markov numbers). To a rational slope $\sigma \in [1, \infty]$ with continued fraction expansion $\sigma = [c_0; c_1, \dots, c_{2k-1}]$, we associate the following data.

The (lower) *Christoffel word* in $\{A, B\}^*$ is obtained by applying the composition of substitutions $D_\sigma = D_R^{c_0} \dots D_L^{c_{2k-1}}$ to the seed A . This yields a matrix in $\Gamma' \cap \text{PSL}_2(\mathbb{N})$, whose factorisation in $\{L, R\}^*$ is obtained from the substitutions $(A, B) \mapsto (RL, LR)$, that turns out to be conjugate to a unique symmetric matrix in $R\text{PSL}_2(\mathbb{N})L$, which we call the *Cohn matrix* $C_\sigma \in \Gamma' \cap \text{PSL}_2(\mathbb{N})$. We thus define the *Markov rational* $\rho_\sigma = C_\sigma \cdot \infty \in \mathbb{Q}_{>1}$ and the *pair of Markov quadratic conjugates* as its (repulsive, attractive) fixed points: $(\xi_\sigma^-, \xi_\sigma^+) = (C_\sigma^{-\infty} \cdot \infty, C_\sigma^{+\infty} \cdot \infty) \in (-1, 0) \times (1, \infty)$.

Remark 3.3 (cyclically primitive and symmetric). Since $C_\sigma \in R\text{PSL}_2(\mathbb{N})L$ is cyclically primitive and symmetric, the continued fraction expansion of ρ_σ is a palindrome which is the minimal period of the purely periodic number ξ_σ^+ , whose Galois conjugate satisfies $\xi_\sigma^- = -1/\xi_\sigma^+$.

Remark 3.4 (all rational slopes). We may extend Definition 3.2 to all slopes in $\mathbb{Q}_{\geq 0}$ using the action of $J: \sigma \mapsto 1/\sigma$ hence of $D_{J\sigma} = D_J D_\sigma$, so that $C_{J\sigma}$ is obtained from C_σ by applying the substitution $(A, B) \mapsto (B, A)$ namely $\text{AD}_J: (R, L) \mapsto (L, R)$, which by \dagger -symmetry amounts to reversing the order of the $\{L, R\}$ word and inverting $\rho_\sigma, \xi_\sigma^\pm$ (so for $\sigma \in [0, 1)$ we have $\rho_\sigma, \xi_\sigma^+, -1/\xi_\sigma^- \in (0, 1)$).

One could further extend Definition 3.2 to all slopes in $\mathbb{Q}\mathbf{P}^1$ using the action of $S: (0, \infty] \ni \sigma \mapsto -1/\sigma \in (-\infty, 0]$ so that $D_{S\sigma} = D_S D_\sigma$ hence $C_{S\sigma}$ is obtained from C_σ by applying the substitution $(A, B) \mapsto (B, A^{-1})$ (that is $D_S: (A, B) \mapsto (A^{-1}BA, A^{-1})$ followed by a cyclic permutation), and we may similarly define the associated Markov numbers.

Remark 3.5 (slope is frequency of C_σ). The slope $\sigma \in \mathbb{Q}\mathbf{P}^1$ is read from C_σ by considering its abelianisation $(m, n) \in H_1(\mathbf{M}'; \mathbb{Z})$: if $C_\sigma \equiv A^m B^n \pmod{\Gamma''}$ then $\sigma = m/n \in \mathbb{Q}\mathbf{P}^1$.

Example 3.6 (from slopes to Markov numbers). Let us choose a slope $\sigma \in \mathbb{Q}_{\geq 0}$ and compute the corresponding Christoffel word C_σ , and quadratic pair $(\xi_\sigma^-, \xi_\sigma^+)$.

For $\sigma = \infty = []$, apply nothing to $A = RL$ and cyclically permute to find $C_\sigma = RL$, yielding $\rho_\sigma = 2/1$ and $\xi_\sigma^\pm = [(1, 1,)^\mathbb{N}] = \frac{1}{2}(1 \pm \sqrt{5})$.

For $\sigma = 1/1 = [0, 1]$, apply $D_R^0 D_L^1$ to A to find BA and cyclically permute to obtain $C_\sigma = RRLL$, yielding $\rho_\sigma = 5/2$ and $\xi_\sigma^\pm = [(2, 2,)^\mathbb{N}] = 1 \pm \sqrt{2}$.

For $n \in \mathbb{N}$ and $\sigma = 1+1/n = [1, n]$, apply $D_R^1 D_L^n$ to A to find $(AB)^n A$ and cyclically permute to obtain $C_\sigma = RL(RRLL)^{n-1} RL$ thus $\rho_\sigma = [1, 1, (2, 2,)^{n-1}, 1, 1]$ and $\xi_\sigma^+ = [(1, 1, (2, 2,)^{n-1}, 1, 1)^\mathbb{N}]$.

For $n \in \mathbb{N}$ and $\sigma = n+1 = [n, 1]$, apply $D_R^n D_L^1$ to A to find $BA^n A = (LR)(RL)^{n+1}$ and cyclically permute to obtain $C_\sigma = RR(RL)^n LL$ thus $\rho_\sigma = [2, 1, \dots, 1, 2]$ and $\xi_\sigma^+ = [(2, 1, \dots, 1, 2)^\mathbb{N}]$.

For $n \in \mathbb{N}_{>0}$ and $\sigma = 1/n = [0, n]$, apply $D_R^0 D_L^n$ to A to find $B^n A = (LR)^n RL$ and cyclically permute to obtain $C_\sigma = LL(RL)^{n-1} RR$ thus $\rho_\sigma = [0; 2, 1, \dots, 1, 2]$ and $\xi_\sigma^+ = [0; (2, 1, \dots, 1, 2)^\mathbb{N}]$.

The next proposition is explained in [Spr24, §14].

Proposition 3.7 (simple geodesics in \mathbf{M}' with rational slope). Consider a primitive vector $(m, n) \in H_1(\mathbf{M}'; \mathbb{Z}) = \mathbb{Z}_A + \mathbb{Z}_B$ and denote the associated slope $\sigma = m/n \in \mathbb{Q}\mathbf{P}^1$.

The simple closed geodesic homologous to $A^m B^n$ is $\xi_\sigma^i \in \mathcal{G}(\mathbf{M}')$, namely the projection mod Γ' of the geodesic $\xi_\sigma = (\xi_\sigma^-, \xi_\sigma^+) \in \mathcal{G}(\mathbf{HP})$.

The unique simple geodesic from cusp-to-cusp that does not intersect ξ'_σ is the projection mod Γ' of the geodesic $(0, \rho_\sigma) \in \mathcal{G}(\mathbf{HP})$.

There are two more simple geodesics avoiding ξ'_σ , their past arise from the cusp and future accumulate on ξ'_σ : those are the projections mod Γ' of the geodesics $(0, \xi_\sigma^+)$ and (∞, ξ_σ^-) in $\mathcal{G}(\mathbf{HP})$.

These 4 geodesics of slope σ are disjoint.

Every simple geodesic of the type cusp-to-cusp or cusp-to-periodic or periodic arises as such, in a unique way (namely distinct slopes give rise to distinct quadruples), and any two such geodesics with different slope have intersections.

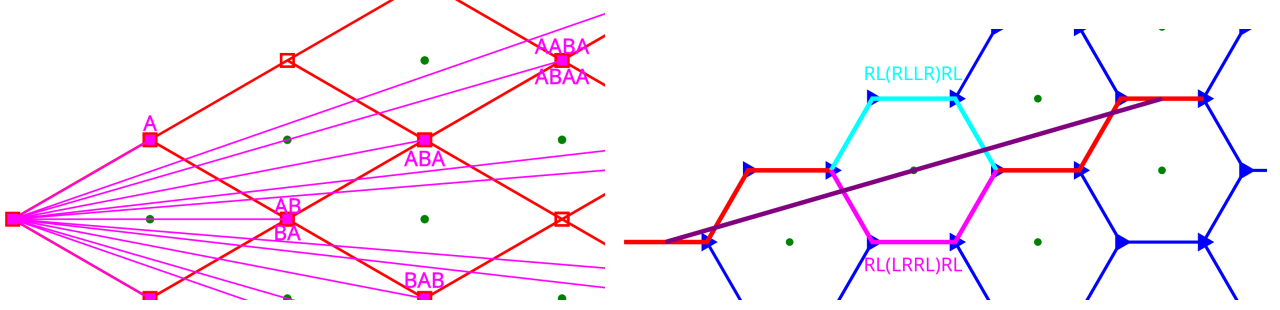


Figure 7: Simple closed geodesics in \mathbf{M}' lift to lines of rational slope in $H_1(\mathbf{M}'; \mathbb{R})$ through i' . The corresponding combinatorial hex-paths are unique only up to cyclic permutation.

We now recall the notion of Sturmian sequences from [Fog02, Chapter 6] to associate them pair of real numbers (which are transcendental as we will recall 3.18).

Definition 3.8 (Sturmian sequences and numbers). Define the set of *Sturmian parameters* as the (slope, intercepts) $(\sigma, \tau) \in (0, 1) \times [0, 1)$ such that $\sigma \notin \mathbb{Q}$ and $\tau \notin (1/2 + \mathbb{Z}) + \sigma(1/2 + \mathbb{Z})$.

To such Sturmian parameters, is associated the (lower) *Sturmian sequence* $X_{\sigma, \tau} \in \{A, B\}^{\mathbb{Z}}$ by

$$\forall n \in \mathbb{Z}: \quad X_{\sigma, \tau}[n] = A^{r_{\sigma, \tau}(n)} B \quad \text{where} \quad r_{\sigma, \tau}(n) = \lfloor (n+1)\sigma + \tau \rfloor - \lfloor n\sigma + \tau \rfloor \in \{0, 1\}$$

encoding the \mathbb{Z} -orbit of τ under addition by σ_1 for the partition $\mathbb{R} \bmod \mathbb{Z} = [-\sigma_1, 1 - \sigma_1)$ in two intervals $[-\sigma_1, 1 - 2\sigma_1) \sqcup [1 - 2\sigma_1, 1 - \sigma_1)$, as $r_{\sigma, \tau}(n)$ is the n -th return time to the left interval.

We may thus define the *pair of conjugate Sturmian numbers* $(\xi_{\sigma, \tau}^-, \xi_{\sigma, \tau}^+) \in (-\infty, -1) \times (0, 1)$ so that $-1/\xi_{\sigma, \tau}^-$ and $1/\xi_{\sigma, \tau}^+$ are the fixed points of the infinite products of the generators $A, B \in \Gamma'$:

$$-1/\xi_{\sigma, \tau}^- = X_{\sigma, \tau}[-1] \cdot X_{\sigma, \tau}[-2] \cdots \infty \quad \xi_{\sigma, \tau}^+ = X_{\sigma, \tau}[0] \cdot X_{\sigma, \tau}[1] \cdots \infty$$

whose continued fraction expansions in $\{1, 2\}^{\mathbb{N}}$ can be recovered from $X \in \{A, B\}^{\mathbb{Z}}$ by applying the morphisms $(A, B) \mapsto (RL, LR)$ to find two sequences in $\{L, R\}^{\mathbb{Z}}$ and grouping exponents.

When $t = 0$, the sequence $X_{\sigma, 0}$ and numbers $\xi_{\sigma, 0}^\pm$ are often called *characteristic Sturmian*.

Remark 3.9 (upper Sturmian sequence). One may also define the upper Sturmian sequence $X_{\sigma, \tau}$ using a ceiling in the definition of $r_{\sigma, \tau}(n)$, but the $\{L, R\}$ -encoding of $X_{\sigma, \tau}$ will only differ by Shift.

Remark 3.10 (all irrational slopes). We may extend Definition 3.8 to all positive irrational slopes $\sigma \in \mathbb{R}_{>0}$ by the action of $J: (\sigma, \tau) \mapsto (1/\sigma, 1 - \tau)$ which changes $X_{\sigma, \tau}$ by applying the substitution $D_J: (A, B) \mapsto (B, A)$. We may further extend Definition 3.8 to all irrational slopes $\sigma \in \mathbb{R}$ by the action of $S: (\sigma, \tau) \mapsto (-1/\sigma, -\tau)$ which changes $X_{\sigma, \tau}$ by applying the substitution $D_S: (A, B) \mapsto (B, B^{-1}A^{-1}B)$, that is $(A, B) \mapsto (B, A^{-1})$ followed by a cyclic permutation.

Hence we may similarly define the associated Sturmian numbers.

Remark 3.11 (From Ostrowski to S -adic expansion). Consider Sturmian parameters $(\sigma, \tau) \in (0, 1) \times [0, 1)$ and write $\sigma = [0; s_1, \dots]$. The *Ostrowski expansion* of τ in base σ is the unique $t \in \mathbb{N}^{\mathbb{N}}$ satisfying for all $n \in \mathbb{N}$ that $0 \leq t_n \leq s_n$ and $t_{n+1} = s_{n+1} \implies t_n = 0$, such that:

$$\tau = \sum_{n=0}^{\infty} \frac{t_n}{\sigma_n} = t_1 \cdot [0; s_1, \dots] + t_2 \cdot [0; s_2, \dots] + \dots$$

With the sequences $s \in (\mathbb{N}_{\geq 1})^{\mathbb{N}_{\geq 1}}$ and $t \in (\mathbb{N}_{\geq 1})^{\mathbb{N}_{\geq 1}}$, one may write $\xi_{\sigma, \tau} \in \{A, B\}^{\mathbb{Z}}$ as the attractive fixed point under an infinite composition of homeomorphisms of $\{A, B\}^{\mathbb{Z}}$, namely the Shift and the extensions of D_L, D_R :

$$X_{\sigma, \tau} = \text{Shift}^{t_1} \circ D_L^{s_1} \circ \text{Shift}^{t_2} \circ D_R^{s_2} \circ \text{Shift}^{t_3} \circ D_L^{s_3} \circ \text{Shift}^{t_4} \circ D_R^{s_4} \circ \dots (A^{\mathbb{Z}})$$

In particular $X_{\sigma, \tau}$ is morphic (that is the image by a morphism of $(\{A, B\}, \text{Shift})$ by the fixed point of a single morphism of the monoid $\{A, B\}^*$), if and only if both s, t are eventually periodic, which means that σ is quadratic and $\tau \in \mathbb{Q}(\sigma)$ [BHZ06, Theorem 2.19 and Proposition 2.11].

Note that in the characteristic Sturmian case $t = 0$ one may forget the Shift^{t_k} . It follows that a Sturmian number $\xi_{\sigma, \tau}^+$ is a limit of Markov quadratic numbers if and only if $\tau = 0$.

Remark 3.12 (cutting sequence and frequencies). Behold the Euclidean plane $H_1(\mathbf{M}'; \mathbb{R})$ with coordinates (A, B) punctured along the rhombic lattice translated by $(1/2, 1/2)$, the line $\alpha = \sigma\beta + \tau$ starting from the point $-(1/2, 1/2) + (0, \tau)$ is homotopic (relative its endpoints) to a broken line in the Cayley graph of $H_1(\mathbf{M}'; \mathbb{Z})$, and this yields the Sturmian sequence $X_{\sigma, \tau} \in \{A, B\}^{\mathbb{Z}}$.

The sequence $X_{\sigma, \tau} \in \{A, B\}$ admits frequencies: for any sequence of subwords $X_{\sigma, \tau}([k, k+l])$ of growing length $l \in \mathbb{N}$, the ratio between the numbers of A 's and B 's converges to the slope σ .

Remark 3.13 (characterization by factor complexity). Over any alphabet, a sequence X is:

- periodic $\iff \exists n \in \mathbb{N}: \text{fac}_n(X) \leq n$ (see [MH38], and [CN10, §4.3]).
- Sturmian $\iff \forall n \in \mathbb{N}: \text{fac}_n(X) = n + 1$ (see [Fog02, Chapter 6])

Remark 3.14 (recovering parameters from symbolics and geometry). The following data determine one-other:

- the Shift-orbit of the sequence $X_{\sigma, \tau} \in \{A, B\}^{\mathbb{Z}}$
- the slope $\sigma \in \mathbb{R}_{>0}$ and the class of the intercept $\tau \in \mathbb{R} \bmod (\mathbb{Z} + \sigma\mathbb{Z})$
- the Γ' -orbit of $(\xi_{\sigma, \tau}^-, \xi_{\sigma, \tau}^+) \in \mathcal{G}(\mathbf{HP})$, that is the simple geodesic $\xi_{\sigma, \tau} \in \mathcal{S}(\mathbf{M}')$

Proposition 3.15 (simple geodesics on \mathbf{M}' of irrational slope). *For an irrational slope $\sigma \in \mathbb{R} \setminus \mathbb{Q}$ there is a minimal geodesic lamination $\Xi'_\sigma \subset \mathbf{M}'$ consisting of the set of simple geodesics $\xi_{\sigma, \tau}^l \in \mathcal{S}(\mathbf{M}')$ for all $\tau \in \mathbb{R} \bmod \mathbb{Z} + \sigma\mathbb{Z}$ such that $\tau \notin (1/2 + \mathbb{Z}) + \sigma(1/2 + \mathbb{Z})$. These are all the non-empty minimal geodesic lamination with no closed leaves in \mathbf{M}' .*

Remark 3.16 (limit of Markov if and only if characteristic). A Sturmian number $\xi_{\sigma, \tau}^+$ is a limit of Markov quadratic numbers ξ_σ^+ if and only if it is characteristic ($\tau = 0 \bmod \mathbb{Z}$). Geometrically, those are the leaves of minimal geodesic laminations that pass through one of the three Weierstrass points of \mathbf{M}' , namely $\{i, Li, Ri\} \bmod \Gamma$.

3.4 Diophantine approximation of simple geodesics in \mathbf{M}'

Transcendence of Sturmian continued fractions

We finally mention the initial motivation of this work,

Context 3.17. Fix an irrational slope $\sigma \in \mathbb{R}_{>0}$, consider its continued fraction expansion $\sigma = [s_0, s_1, \dots]$, and denote $\lambda_\sigma = \limsup_k [s_k; s_{k-1}, \dots, s_1, s_0]$.

Let $\tau \in (0, 1)$ such that $\tau \notin (1/2 + \mathbb{Z}) + \sigma(1/2 + \mathbb{Z})$ so as to obtain the Sturmian sequence $X_{\sigma, \tau} \in \{A, B\}^{\mathbb{Z}} \subset \{L, R\}^{\mathbb{Z}}$ and real numbers $\xi_{\sigma, \tau}^\pm \in \mathbb{R}$.

The transcendence of $\xi_{\sigma, \tau}^\pm$ was initially proved in [ADQZ01, Proposition 3 and Theorem 7]: their strategy consists of showing that a Sturmian number is very well approximated by quadratic numbers, well enough to satisfy the hypotheses in the following of Schmidt [Sch67]: for a real irrational $\xi^+ \in \mathbb{R} \setminus \mathbb{Q}$, if there is real $\epsilon > 3$ and infinitely many quadratic irrationals ξ_k^+ such that $|\xi^+ - \xi_k^+| < H(\xi_k)^{-3-\epsilon}$, then ξ^+ must be transcendental. Let us mention another proof of transcendence using the fact that they begin with infinitely many palindromes ([AA07, §2]). The following [AB10, Theorem 2.2.1] gives the state of the art on their Mahler measures.

Theorem 3.18 (transcendence of Sturmian numbers). *The numbers $\xi_{\sigma, \tau}^\pm$ are transcendental, and more precisely: if s is unbounded then $w_2(\xi_{\sigma, \tau}^\pm) = \infty$ (Mahler type U_2), and if s is bounded then $\exists c \in \mathbb{R}_{>0}, \forall n \in \mathbb{N}_{\geq 1}: w_n(\xi^+) \leq \exp(c(\log 3d)^3(\log \log 3d)^2)$ (Mahler class S or T).*

[BL05, Theorem 3.1] computes the Mahler measures for characteristic Sturmian numbers.

Theorem 3.19 (transcendence measures of characteristic Sturmian numbers). *For an irrational $\sigma \in \mathbb{R}_{>0}$ with $\sigma = [s_0, s_1, \dots]$, denote $\lambda_\sigma = \limsup_k [s_k; s_{k-1}, \dots, s_1, s_0]$.*

The characteristic Sturmian number $\xi_{\sigma, 0}^+$ has transcendence measures:

$$\hat{w}_2(\xi_{\sigma, 0}^+) = 1 + \frac{2}{\lambda_\sigma} \quad w_2(\xi_{\sigma, 0}^+) = 1 + 2\lambda_\sigma$$

Remark 3.20 (Mahler spectra of all Sturmian). The Cassaigne spectrum $\sigma \mapsto \lambda_\sigma$ appears in various related settings and its metric topology still work in progress (see [Cas99, CT22, KS24]). Its understanding is equivalent to that of the Mahler w_2, \hat{w}_2 spectra of all Sturmian numbers.

Lagrange constants of Markov quadratics and Sturmian transcendentals

The following Theorem is due to [Haa87] building on work of Cohn [Coh71], where one can also learn the connection with the arithmetic of Markov quadratic forms. A recent geometric proof in the spirit of our previous discussion can be found in [Spr17].

Theorem 3.21 (Lagrange constants of Markov and Sturmian numbers). *For $\xi^+ \in \mathbb{R} \setminus \mathbb{Q}$:*

- $L(\xi^+) > 3 \iff \text{PGL}_2(\mathbb{Z}) \cdot \xi^+$ contains a Markov quadratic
- $L(\xi^+) = 3 \iff \text{PGL}_2(\mathbb{Z}) \cdot \xi^+$ contains a Sturmian transcendental

In those cases, the orbit $\text{PSL}_2(\mathbb{Z}) \cdot \xi^+$ contains a unique Markov or Sturmian number up to the action of S and of the Shift on $\{R, L\}^{\mathbb{N}}$. For a Markov or Sturmian number $\xi^+ \in \mathbb{R}$, its conjugate is the unique $\xi^- \in \mathbb{R} \setminus \{\xi^+\}$ such that $C(\xi^-, \xi^+) = L(\xi^+)$, and the simple geodesic $\xi' \subset \mathbf{M}'$ has length given by $C(\xi^-, \xi^+) = 3 \coth\left(\frac{1}{2}\ell_{\mathbf{M}'}(\xi')\right)$.

Remark 3.22 (worst approximable rationals). [Spr24] proposes to define the Cohn constant of $r \in \mathbb{Q}$ as $C(\infty, r) = \sup\{q^2|r - p/q|: (p, q) \in (\mathbb{Z} \times \mathbb{N}^*), r \neq p/q\}$ and characterises those rationals with $C(r) > 3$ as the set of Markov rationals $C_\sigma \cdot \infty$ and their companions $C_\sigma^n \cdot \{0, \infty\}$ for $n \in \mathbb{N}_{\geq 2}$.

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