Topology of real planar algebraic curves

Combinatorics + enumeration of local & global configurations

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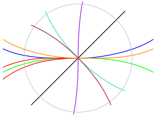
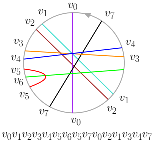
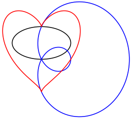
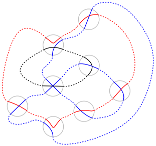
Conference in enumerative, real and birational geometry in Le Croisic.

9 June 2022

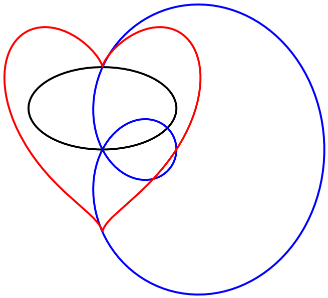








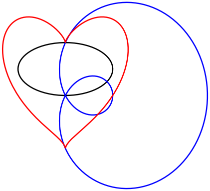
What is the topology of an algebraic curve in the plane ?

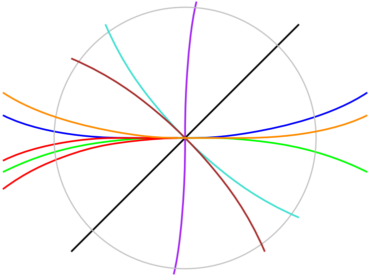


Algebraic curve in the real plane: $F(x, y) = 0$, $F \in \mathbb{R}[x, y]$.

- Singular points: $\partial_x F(0,0) = \partial_y F(0,0) = 0$.
- Connected components of the real locus of dimension 1.

What is the topology of an algebraic curve in the plane ?





We wish to understand:

- the combinatorics of local branches at singular points
- the isotopy class of the whole real locus

Road map

Topology and combinatorics of singularities

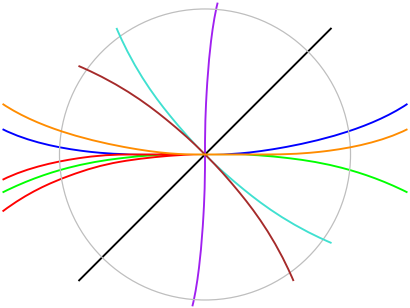
- Combinatorial invariants and their structure
- Enumerations of the combinatorial invariants

Topology and combinatorics of singular algebraic curves

Singularities of algebraic curves in the real plane

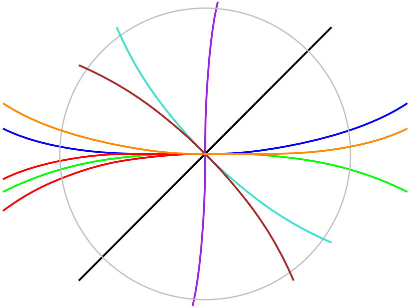
A convergent serie $F \in \mathbb{R}\{x, y\}$ with $F(0,0) = 0$ and $x \nmid F$.
Vanishes along a finite union of germs of injectively parametrized segments intersecting only at the origin:

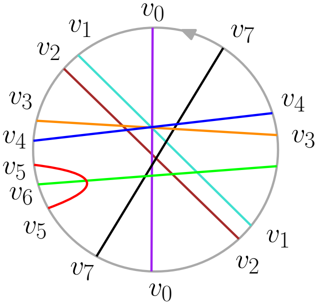
$$t \mapsto (\pm t^m, f(t)) \qquad f \in \mathbb{R}\{t\}.$$



In red: $x^5 + x^4 + 2yx^2 + y^2 = 0$

The chord diagram of the singularity

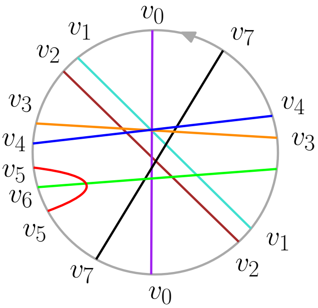




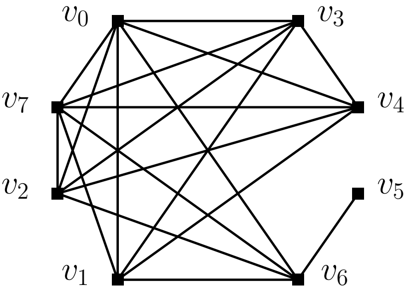
$$v_0 v_1 v_2 v_3 v_4 v_5 v_6 v_5 v_7 v_0 v_2 v_1 v_3 v_4 v_7$$

Chord diagram: encodes topology of the link of the singularity.

The interlace graph of the chord diagram

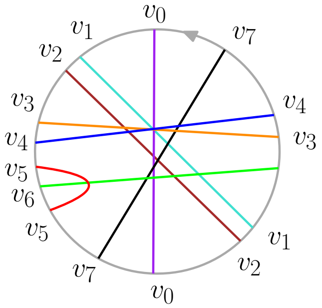


$v_0v_1v_2v_3v_4v_5v_6v_5v_7v_0v_2v_1v_3v_4v_7$

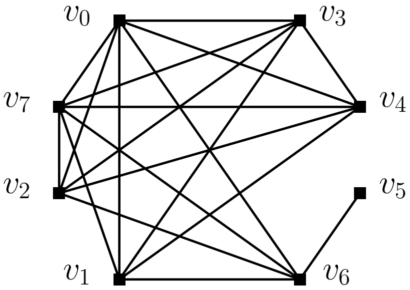


Interlace graph: encodes the linking matrix of the singularity.

The interlace graph of the chord diagram



$v_0v_1v_2v_3v_4v_5v_6v_5v_7v_0v_2v_1v_3v_4v_7$



Questions: combinatorics and enumeration.

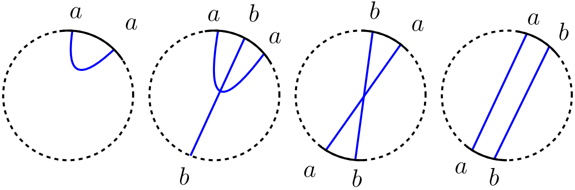
Which chord diagrams and graphs arise from singularities ?

How many diagrams are there for a given number of branches ?

Recursive description of analytic chord diagrams

Definition: *simplification* of a chord diagram.

Delete the letter "*a*" in a pattern consisting of:



an isolated chord, a fork, a pair of (true or false) twins.

Theorem [Ghy17]: analytic chord diagrams.

- The empty word is an analytic chord diagram.
- A non-empty diagram with no simplifications is not analytic.
- A non-empty diagram is analytic if and only if any of its simplications is analytic.

Blowups: intuition, geometry, terminology, algebra

To Blowup a point of smooth surface germ (S, o) consists in replacing o by its set of tangent directions in S.

Enables to distinguish first order behaviour in the vicinity of o.

There exists a morphism $\pi\colon (\hat{S}, E) \to (S, o)$ with $E \simeq \mathbb{P}^1$ inducing an isomorphism between $\hat{S} \setminus E$ and $S \setminus o$.

It is unique up to unique isomorphism above S.

The curve E is called the exceptional divisor.

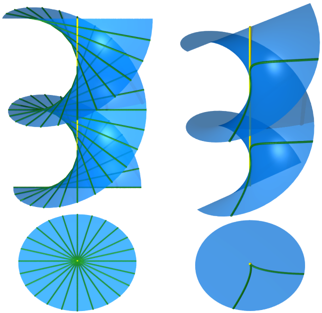
The germ $\hat{C} = \overline{\pi^{-1}(C \setminus \{o\})}$ is the strict transform of C.

In terms of the field of functions $\mathcal{O}_{S,o} \simeq \mathbb{R}\{x, y\}$ at o, the blowup is given by a change of variables $y \mapsto t = y/x$.

Blowing up a point in a real surface yields a Möbius band

Construction.

The graph of $(x, y) \in \mathbb{R}^2 \mapsto y/x \in \mathbb{RP}^1$ is the quadric $y = tx$.
Its closure in $\mathbb{R}^2 \times \mathbb{RP}^1$ is homeomorphic to a Möbius band.



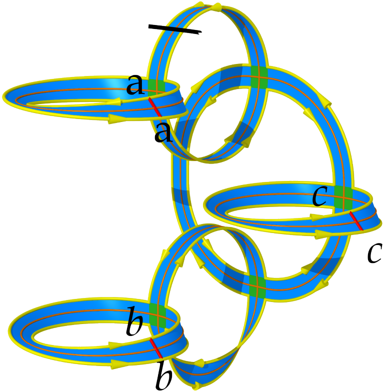
Blowing up a pencil of lines and a cusp.

Successive blowups resolve singularity yielding tree of $\mathbb{P}^1$

- To resolve a singular germ C one must distinguish higher order tangencies: successive blowups.
- $\exists$ morphism $\Psi\colon (\hat{S}, E) \to (S, o)$, isomorphism outside E, such that $\hat{C}$ is smooth and $\hat{C} \pitchfork E$.
- *The dual graph to the $\mathbb{P}^1 \hookrightarrow E$ is a rooted tree.*

  The surface $\hat{S}$ is a plumbing of their tubular neighbourhoods, which retracts by deformation onto E.

A tree of annuli and Möbius bands



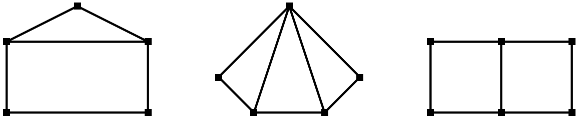
Over $\mathbb{R}$

- The surface $\hat{S}$ is a plumbing of annuli and Möbius bands
- Its connected boundary $\partial\hat{S}$ intersects $\hat{C}$ in pairs of points.

Which invariants come from singularities ?

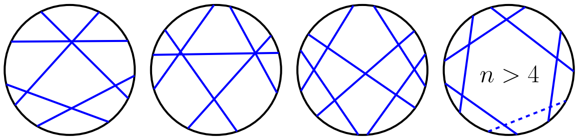
Theorem [GS20]: Interlace graphs of analytic chord diagrams.

- *Collapsible*: recursive description analogous to analytic diagrams (equivalence is non trivial !).
- No house, gem, domino or ($n > 4$)-cycle as induced subgraph.



Corollary [GS20]: characterisation of analytic chord diagrams.

The forbidden patterns for the partial order on chord diagrams are:



Road map

Topology and combinatorics of singularities

- Combinatorial invariants and their structure
- Enumerations of the combinatorial invariants

Topology and combinatorics of singular algebraic curves

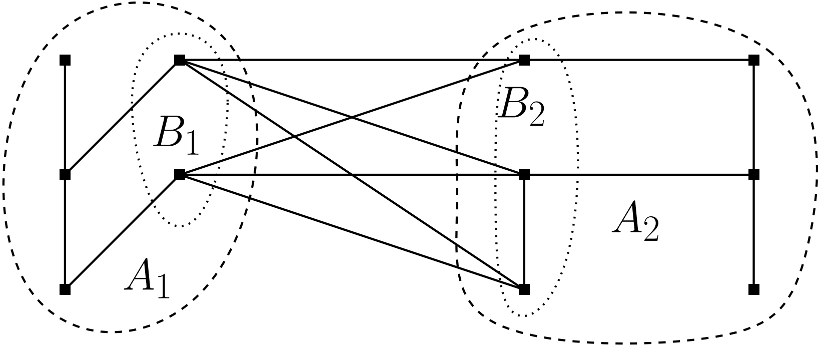
Split decomposition of graphs (Cunningham & Gioan-Paul)

Definition: Split of a graph $G = (V, E)$

A *bipartition* of its vertices $V = A_1 \cup A_2$ is *non-trivial* if $|A_k| > 1$.

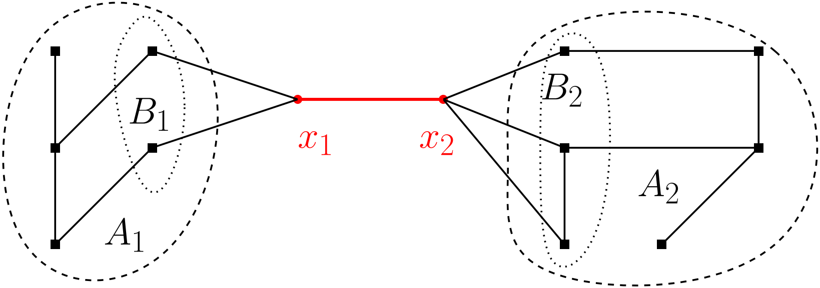
A *split* is a non-trivial bipartition with $B_1 \subset A_1$, $B_2 \subset A_2$ such that

$$Edges(A_1, A_2) = Clique(B_1, B_2)$$



Split decomposition of graphs (Cunningham & Gioan-Paul)

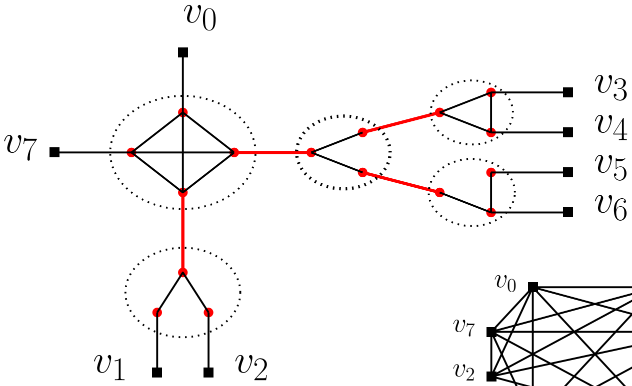
Factorise by adding control vertices: obtain a *graph-labelled-tree*, with *internal nodes* labelled by the graphs induced by $A_k \cup \{x_k\}$.

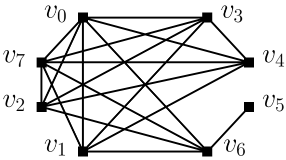


Continue until all nodes are labelled by graphs which are *prime* (no splits) or *degenerate* (all non-trivial bipartitions = splits).

Split decomposition of graphs (Cunningham & Gioan-Paul)

This may look like:

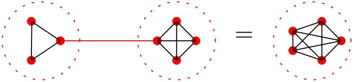


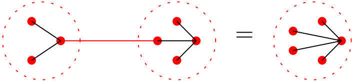


Here all nodes are labelled by degenerate graphs: *star & clique*.

Split decomposition of graphs (Cunningham & Gioan-Paul)

A graph-labelled-tree is *reduced* if no clique-join or star-join:



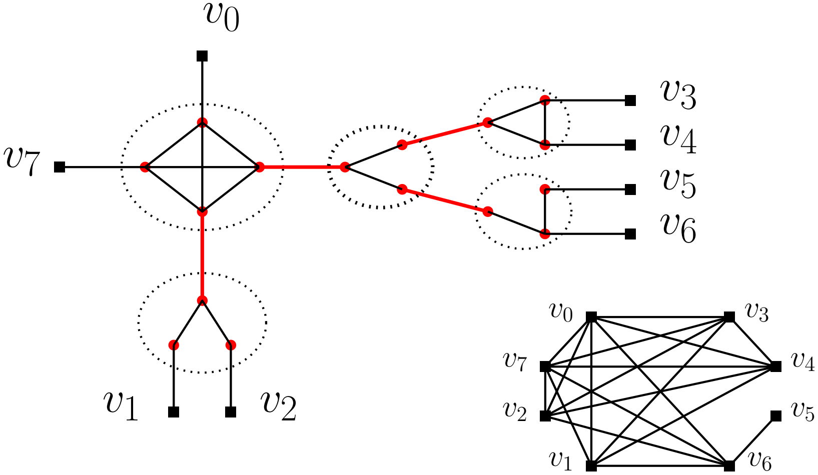


Cunningham's split decomposition theorem after Gioan-Paul:

Every connected graph admits a unique factorisation into a reduced graph-labelled-tree.

Split decomposition of graphs (Cunningham & Gioan-Paul)

Conversely, we recover the *accessibility graph* of the leaves:



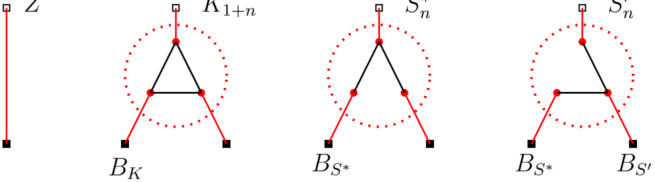
Collapsible graphs are (star & clique)-labelled-trees

Corollary: characterisation collapsible graphs as *SK-graphs*.

A connected collapsible graph is the accessibility graph of a unique reduced graph-labelled-tree whose nodes are all degenerate.

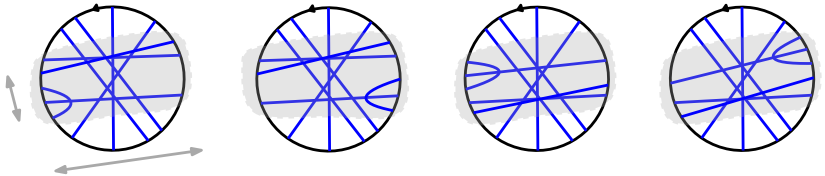
Operad structure (context-free grammar)

Consider SK-trees with leaves labelled by $\{0, \ldots, n\}$, compose them by branching 0-leaf of children on $>$ 0-leaves of parents.



Operad of connected analytic chord diagrams with marking

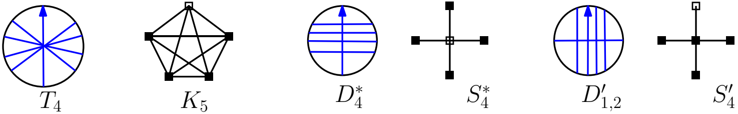
- Lift the *SK*-tree factorisation of bushes (unlabeled but rooted) to *connected rooted chord diagrams*.
- A bush yields several chord diagrams differing by *mutations*:



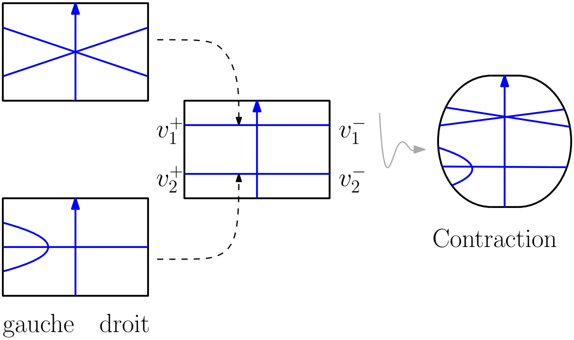
- A rooted diagram has a left hand side and a right hand side.
- A rooted tree induces a partial order on its nodes.

Operad of connected analytic chord diagrams with marking

Decorate internal nodes of a tree by the connected chord diagrams:



They are unlabelled but rooted. Retain left & right while inserting.



Context-free grammar

Lemma [Sim22]: unique factorisation of rooted diagrams.

The connected and rooted analytic chord diagrams correspond bijectively to the rooted trees whose nodes are degenerate chordiag.

Equations for the grammar of connected rooted diagrams.

$$C(z) = \frac{1}{2}\left(C_T + C_{D^*} + C_{D'}\right)$$

$$C_T(z) = z + \sum_{n>1} C_{D^*}^n + \sum_{k+l>0} C_{D'} C_{D^*}^{k+l}$$

$$C_{D^*}(z) = z + \sum_{n>1} C_T^n + \sum_{k+l>0} C_{D'} C_{D^*}^{k+l}$$

$$C_{D'}(z) = z + \sum_{n>1} C_{D^*}^n + \sum_{n>1} C_T^n$$

Counting analytic chord diagrams: linear and connected

Proposition [Sim22]: Analytic chordiag, marked & connected.

Algebraic generating series: $2C^3 + (z+2)C^2 + (2z-1)C + z = 0$.

Asymptotic: $C_n \sim c_0\, n^{-\frac{3}{2}}\, \gamma^{-n}$ with $13 < \gamma^{-1} < 14$:

$$\gamma = \frac{1}{12}\left(49 - \frac{433}{\sqrt[3]{24407 - 1272\sqrt{318}}} - \sqrt[3]{24407 - 1272\sqrt{318}}\right)$$

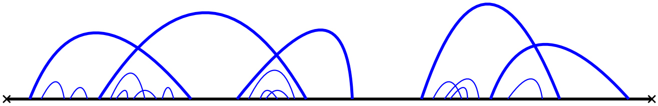
Closed formula by Lagrange inversion:

$$C_n = \frac{1}{n}\sum_{k=0}^{n-1}\binom{n-1+k}{n-1}\binom{2n+k}{n-1-k}2^k$$

First terms of the sequence $(C_n)_{n\in\mathbb{N}}$:

$1, 4, 27, 226, 2116, 21218, 222851, 2420134, 26954622, 306203536$.

Counting analytic chord diagrams: linear



Theorem [Sim22] Analytic chord diagrams with marking.

The generating series of linear analytic diagrams is algebraic:
$(z^3 + z^2)A^6 - z^2 A^5 - 4zA^4 + (8z+2)A^3 - (4z+6)A^2 + 6A - 2 = 0.$

Asymptotic: $A_n \sim a_0\, n^{-\frac{3}{2}}\, \alpha^{-n}$ where $15 < \alpha^{-1} < 16$.
First terms of the sequence $(A_n)_{n\in\mathbb{N}}$ (checked algorithmically):
$1, 1, 3, 15, 105, 923, 9417, 105815, 1267681, 15875631, 205301361$

Proposition [Sim22]: Analytic chord diagrams $\tilde{A}_n \sim \frac{A_n}{n}$

Road map

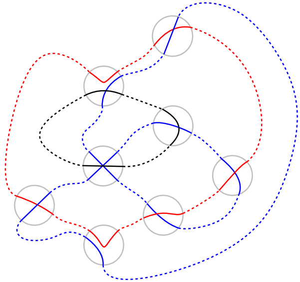
Topology and combinatorics of singularities

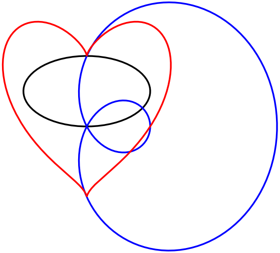
Topology and combinatorics of singular algebraic curves

- Global topology: which obstructions ?
- Enumeration of configurations in $\mathbb{S}^2$ and $\mathbb{RP}^2$

Question of global configurations

Globally, a real algebraic curve consists in a collection of singularities connected by smooth arcs.





Definition of a combinatorial curve in S

A map of S with vertices decorated by chord diagrams.

Formulation with three permutations on the set of rays.

No global obstructions in the analytic category

Theorem [Sim22]: Analytic combinatorial curves

Every combinatorial curve with analytic chord diagrams arises from a real analytic curve.

Proof sketch.

- Realise the combinatorial curve by $\mathscr{C}^\infty$-smooth arcs connecting analytic singularities.
- Blowup surface S at singular points to find smooth curve in $\hat{S}$.
- Construct a $\mathscr{C}^1$-approximation by analytic smooth curve : Fourier series and Chebychev polynomials, then orthogonal projection (Kollár) and local-to-global (Cartan $H^1(S;\mathcal{O}) = 0$).
- Implode approximating curve.

□

Homological obstructions to algebraic approximations

Definition: Algebraic approximation of $Y \subset X$.

In a real algebraic variety X, a closed and $\mathscr{C}^\infty$-smooth submanifold $i\colon Y \hookrightarrow X$ admits an algebraic approximation if every neighbourhood of i in $\mathscr{C}^\infty(X, Y)$ contains a map $h\colon Y \to X$ such that $h(Y)$ is Zariski closed and non-singular.

Homology of a real algebraic surface $H_1^{alg}(S; \mathbb{Z}/2)$.

- Subcomplex $C_*^{alg}(S; \mathbb{Z}/2)$ of algebraic chains in $C_*^{cell}(S; \mathbb{Z}/2)$
  Algebraic homology $H_*^{alg}(S; \mathbb{Z}/2)$
- The $H_1^{alg}(S; \mathbb{Z}/2)$ is the subvector space of $H_1^{top}(S; \mathbb{Z}/2)$ generated by algebraic cycles (as every 0-cycle is algebraic).

Algebraic approximation theorem of Nash-Tognoli.

In a real algebraic surface X, a compact smooth hypersurface Y admits an algebraic approximation $\iff$ $[Y] \in H_1^{alg}(X; \mathbb{Z}/2)$.

Global obstructions in the algebraic category are homological

Theorem [Sim22]: Algebraic combinatorial curves

Every combinatorial curve with analytic chord diagrams and algebraic fundamental class arises from a real algebraic curve.

Proof sketch.

- Realise the combinatorial curve by $\mathscr{C}^\infty$-smooth arcs connecting analytic singularities.
- Blowup $C \subset S$ at singular points to find smooth $\hat{C} \hookrightarrow \hat{S}$.
- Notice that $H_1^{alg}(\hat{S}; \mathbb{Z}/2) = H_1^{alg}(\hat{S}; \mathbb{Z}/2) + \mathbb{Z}/2 \cdot [E]$ so the smooth compact $\hat{C}$ has $[\hat{C}] \in H_1^{alg}(\hat{S}; \mathbb{Z}/2)$ and Nash-Tognoli provides an algebraic approximation.
- Implode approximating curve.

□

Road map

Topology and combinatorics of singularities

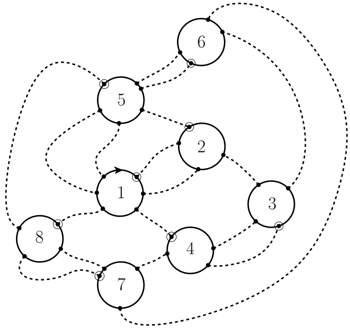
Topology and combinatorics of singular algebraic curves

- Global topology: which obstructions ?
- Enumeration of configurations in $\mathbb{S}^2$ and $\mathbb{RP}^2$

Enumeration of algebraic combinatorial curves in $\mathbb{S}^2$

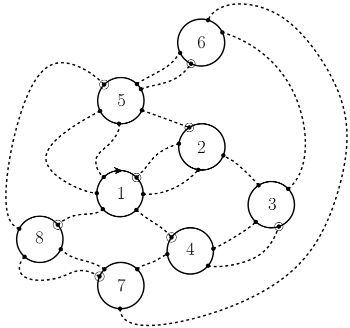
Definition of a slicing.

Surface $P_k = \mathbb{S}^2 \setminus \bigsqcup_{l=1}^{s} \mathbb{D}^2$ with $2k = (2k_1, \ldots, 2k_s) \in (2\mathbb{N}^*)^s$ indicating decorated points on boundary components J_l.



A slicing of $P_{(6,4,4,4,6,4,4,4)}$.

Enumeration of algebraic combinatorial curves in $\mathbb{S}^2$



A slicing of $P_{(6,4,4,4,6,4,4,4)}$.

Tutte counts slicings P_k with one marked point per boundary:

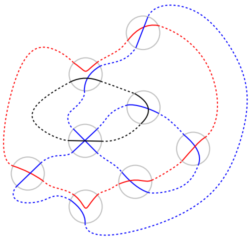
$$\frac{(c-1)!}{(c-s-2)!} \prod_{v=1}^{s} k_v \binom{2k_v}{k_v}.$$

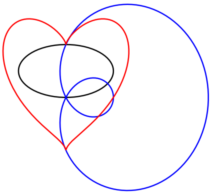
Enumeration of algebraic combinatorial curves in $\mathbb{S}^2$

Theorem [Sim22]: Algebraic combinatorial curves in $\mathbb{S}^2$

The number of algebraic combinatorial curves of $\mathbb{S}^2$ with s indexed vertices decorated by rooted chord diagrams of sizes $k_1, \ldots, k_v$ is:

$$\frac{(c-1)!}{(c-s-2)!} \prod_{v=1}^{s} k_v \binom{2k_v}{k_v} A_{k_v}.$$





Bounding the number of algebraic curves in $\mathbb{RP}^2$

Proposition [Sim22]: majoration by the number of edges.

The number $Calc_{\mathbb{S}^2}(c)$ of rooted algebraic combinatorial curves of the sphere with c edges satisfies: $Calc_{\mathbb{S}^2}(c) \leq c^3 \rho^c$ for $\rho < 134$.

Theorem [Sim22]: domination by the algebraic degree.

The number $Cal_{\mathbb{RP}^2}(d)$ of rooted combinatorial curves realised by algebraic curves of degree d in the projective plane $\mathbb{RP}^2$ satisfies:

$$Cal_{\mathbb{RP}^2}(d) = o\left(12^{d^2}\right).$$

Comparing with Kharlamov-Orevkov estimation

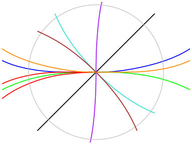
The number of *algebraic isotopy classes* of (maybe disconnected) smooth algebraic curves in $\mathbb{RP}^2$ with degree d is asymptotically contained between expressions of the form $\exp\left(Cd^2 + o(d^2)\right)$.

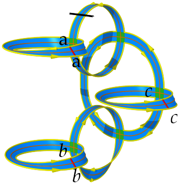
Summary: Real analytic singularities

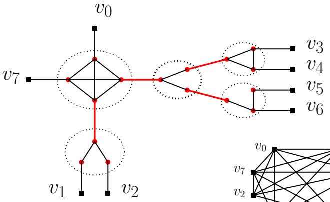
Between analytic or algebraic singularities and algebraic or analytic combinatorics

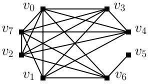
Topology of singularities: chord diagrams & interlace graphs.

- Structure: blowup and algebraic combinatorics.
- Enumeration: Split decomposition, analytic combinatorics.









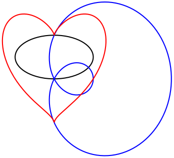
$$(z^3 + z^2)A^6 - z^2A^5 - 4zA^4 + (8z + 2)A^3 - (4z + 6)A^2 + 6A - 2 = 0$$

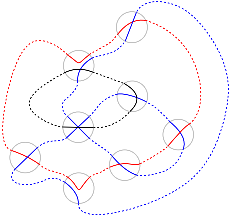
Summary : Real algebraic singular curves in a surface

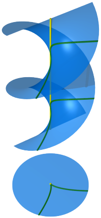
Between analytic or algebraic singularities and algebraic or analytic combinatorics

Global topology: combinatorial curve.

- Structure: blowup and analytic or algebraic approximation.
- Enumeration: Tutte's enumeration & local configurations.



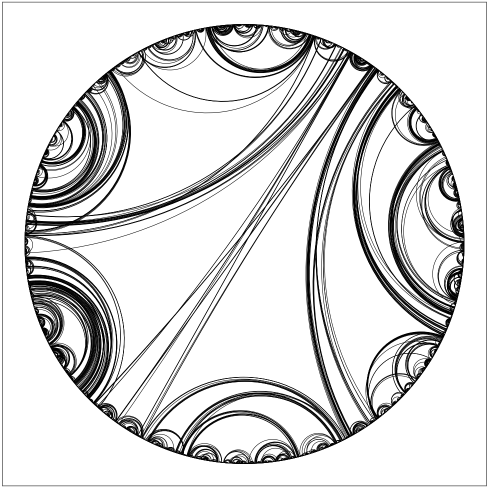




Nash-Tognoli $H_1^{alg} \subset H_1$

$$\frac{(c-1)!}{(c-s-2)!} \prod_{v=1}^{s} k_v \binom{2k_v}{k_v} A_{k_v}$$

Merci d'avoir écouté...



A big connected analytic cord diagram (M. Maazoun, 14/12/18).

… et bon appétit !

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