Quasi-characters of the modular group from linking numbers of modular knots

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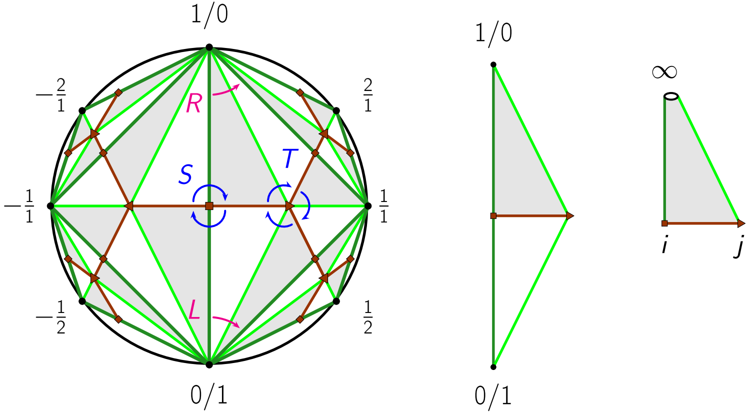
Modular group $\mathrm{PSL}_2(\mathbb{Z})$ acting on the hyperbolic plane $\mathbb{HP}$

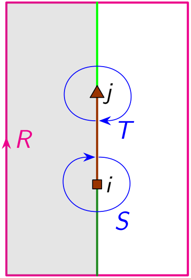
$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

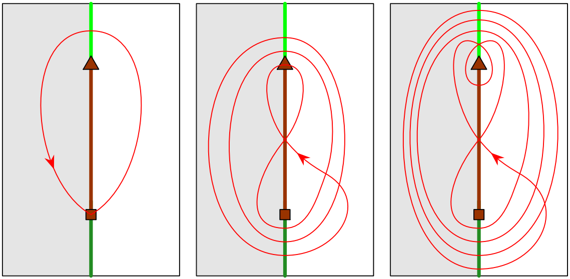


Tiling $\mathbb{HP}$ under the action of the modular group $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$

Loops in the modular orbifold $\mathbb{M} = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{HP}$

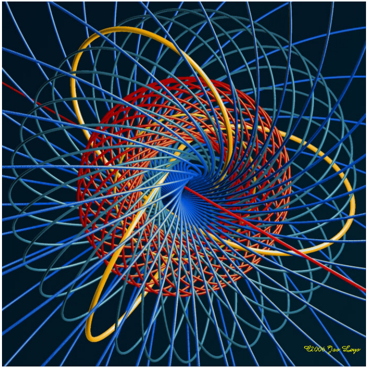
Free homotopy classes of oriented loops in $\mathbb{M}$	Conjugacy classes in $\pi_1(\mathbb{M}) = \mathrm{PSL}_2(\mathbb{Z})$
Around conic singularity i or j	Elliptic: S or $T^{\pm 1}$
Suround n times the cusp ∞	Parabolic: R^n, $n \in \mathbb{Z}$
$\exists!$ geodesic representative γ_A of length λ_A	Hyperbolic: $\mathrm{disc}(A) = \left(2 \sinh \frac{\lambda_A}{2}\right)^2$





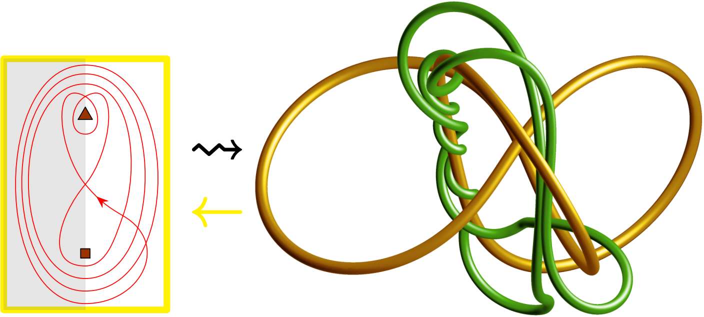
Unit tangent bundle $\mathbb{U}$ of the modular orbifold $\mathbb{M}$

$$
\begin{array}{ccc}
\mathsf{PSL}_2(\mathbb{R}) & \xrightarrow[\mathsf{PSL}_2(\mathbb{Z})]{} & \mathbb{U} \\
\mathbb{S}^1 \downarrow & & \downarrow \mathbb{S}^1 \\
\mathbb{HP} & \xrightarrow[\mathsf{PSL}_2(\mathbb{Z})]{} & \mathbb{M}
\end{array}
$$



Primitive modular geodesics in $\mathbb{M}$ lift to modular knots in $\mathbb{U}$

Indefinite classes in $BQF(\mathbb{Z})$	Hyperbolic classes in $\mathrm{PSL}_2(\mathbb{Z})$	Modular geodesics in $\mathbb{M}$	Periodic orbits in $\mathbb{U}$
discriminant form	trace form	intersection form	linking form



The primitive modular geodesic γ_A lift to the modular knot k_A

Conjugacy classes and cyclic binary words

Euclidean monoid

The monoid $\mathrm{PSL}_2(\mathbb{N})$ is freely generated by $L = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$

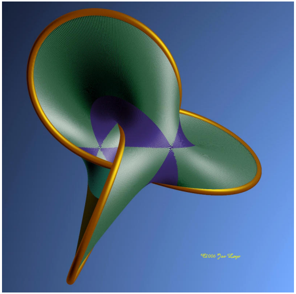
Conjugacy class $[A]$ of an infinite order $A \in \mathrm{PSL}_2(\mathbb{Z})$:

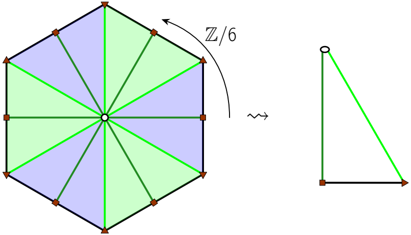
- $[A] \cap \mathrm{PSL}_2(\mathbb{N})$: cyclic permutations of an $L\&R$-word $\neq \emptyset$.
- Class is primitive $\iff$ cyclic word is primitive.
- Class is hyperbolic $\iff$ $\#L > 0$ and $\#R > 0$.

Rademacher cocycle of $A \in \mathrm{PSL}_2(\mathbb{N})$ is $\mathrm{Rad}(A) = \#R - \#L$

- Cocycle for the Eisenstein series E_2 (logarithm of Dedekind function η)
- Signature defect of the torus bundle with monodromy A
- Special value at 0 of the Shimizu L-function for $L_A(s) = \sum \frac{\mathrm{sign}\, Q_A(m,n)}{|Q_A(m,n)|^s}$

Linking the trefoil recovers the Rademacher cocycle

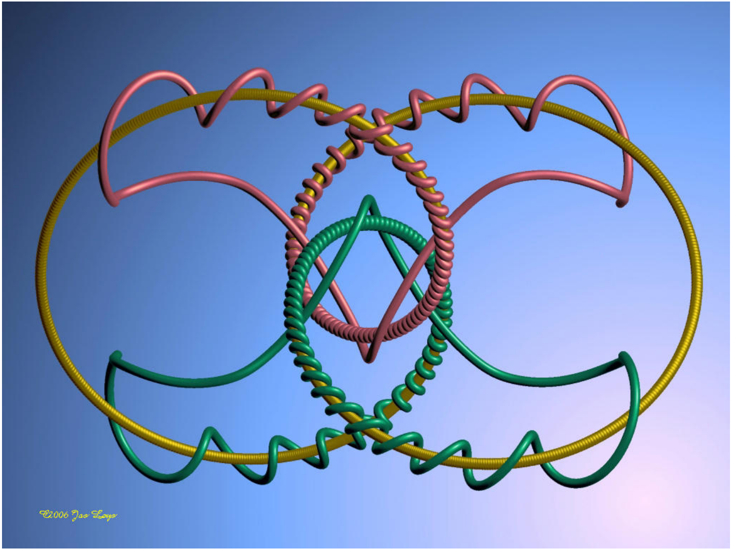




Theorem [Ghy07]:

For hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$ we have $\mathrm{Rad}(A) = \mathrm{lk}(\text{trefoil}, k_A)$.
The function $\mathrm{Rad}\colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathbb{Z}$ is a homogeneous quasi-morphism...

What about the linking of modular knots ?



Quasimorphisms and quasicharacters of a group Γ

Quasimorphism and homogeneous quasimorphisms

A function $\phi\colon \Gamma \to \mathbb{R}$ is a *quasimorphism* when it has a bounded derivative:

$$d\phi\colon \Gamma \times \Gamma \to \mathbb{R} \qquad d\phi(A,B) = \phi(B) - \phi(AB) + \phi(A).$$

It is homogeneous when $\phi(A^n) = n\phi(A)$ for all $A \in \Gamma$ and $n \in \mathbb{Z}$.

Quasi-characters

The $\mathbb{R}$-vector space $QMH(\Gamma)$ has weak* topology (pointwize convergence).

- Semi-norm $\|d\phi\|_\infty$ with $\ker \|d\phi\|_\infty = H^1(\Gamma;\mathbb{R}) \subset QMH(\Gamma)$
- Banach space of quasicharacters $\mathcal{QC}(\Gamma) = QMH(\Gamma)/H^1(\Gamma)$

For a topological $\mathbb{R}$-vector space V, a sequence $(x_n)_{n\in\mathbb{N}}$ is a *basis* when

$$\forall v \in V, \quad \exists!\,(c_n(v)) \in \mathbb{R}^{\mathbb{N}}, \quad v = \sum c_n(v)\cdot x_n,$$

and a *Schauder basis* when the coefficient functionals c_n are continuous.

Shauder basis of $\mathcal{QC}(\mathrm{PSL}_2(\mathbb{Z}))$ from linking numbers

Proposition [Sim22]: Quasi-morphisms from linking numbers

For all $A \in \mathrm{PSL}_2(\mathbb{Z})$, let $\mathrm{C}_A(B) = \frac{1}{2}\left(\mathrm{lk}(A, B) - \mathrm{lk}(A^{-1}, B)\right)$.
The function $C_A \colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathbb{Z}$ is a homogeneous quasi-morphism.
It is trivial if and only if A is conjugate to A^{-1}.

The set $\mathcal{P}$ of primitive infinite order conjugacy classes in $\mathrm{PSL}_2(\mathbb{Z})$,
subset $\mathcal{P}_0$ of those which are stable under inversion.
Partition $\mathcal{P} \setminus \mathcal{P}_0 = \mathcal{P}_- \sqcup \mathcal{P}_+$ in two subsets in bijection by inversion.
We may choose $R \in \mathcal{P}_+$, and note that $\mathrm{C}_R = \mathrm{lk}(\mathit{trefoil}, \cdot)$.

Theorem [Sim22]: Shauder basis for $\mathcal{QC}(\mathrm{PSL}_2(\mathbb{Z}))$

The collection of $\mathrm{C}_A \in \mathcal{QC}(\Gamma; \mathbb{R})$ for $A \in \mathcal{P}_+$ is a Shauder basis:

- $\forall f \in \mathcal{QC}(\mathrm{PSL}_2(\mathbb{Z}); \mathbb{R}), \quad \exists!\, (c_A(f))_A \in \mathbb{R}^{\mathcal{P}_+}, \quad f = \sum_A c_A(f) \cdot \mathrm{C}_A$
- The period coefficients $c_A \colon f \mapsto c_A(f)$ are continuous

$\Longrightarrow$ Fourier theory of quasi-characters, with a natural basis of *cosigns*.

Link equivalence

To prove the non-triviality and linear independance of the C_A for $A \in \mathcal{P}_+$, we were led to show the non-degeneracy of the linking form.

Lemma [Sim22] : The linking pairing is non-degenerate

If hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ are link equivalent, then they are conjugate. (Link equivalence: $\forall X \in \mathrm{PSL}_2(\mathbb{Z})$, $\mathrm{lk}(A, X) = \mathrm{lk}(B, X)$.)

Linking recovers intersection

For hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ we have $\frac{1}{2} I(A, B) = \mathrm{lk}(A, B) + \mathrm{lk}(A^{-1}, B)$.

Intersection from modular cocycles

For hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$, [DIT17] construct a modular function whose symbol recovers $I(A, \cdot) = 2(\mathrm{lk}(A, \cdot) + \mathrm{lk}(A^{-1}, \cdot))$.
What about the cosign $\mathrm{C}_A(\cdot) = \mathrm{lk}(A, \cdot) - \mathrm{lk}(A^{-1}, \cdot)$?

Bibliography

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Happy 80'th monodromy around the sun to Nicolas Katz !

> *Do you think you are exactly the same person you were half your lifetime ago? If not, it is almost certainly because you are aware, at some level, of your personal monodromy.*

(There is a unique geodesic of discriminant 80, namely $R^{16}L^{1}$.)