Valuations on the Character Variety

Newton Polytopes and Residual Poisson bracket

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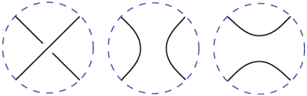
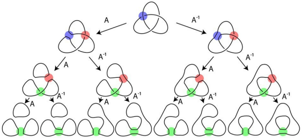
Orsay GTD : 06/06/2024





The $\mathrm{SL}_2(\mathbb{C})$-character variety of a finitely generated group π

Definition: Character variety $X(\pi)$ of a finitely generated group π

- *Representation variety* $\mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbb{C}))$, it admits an algebraic action $\mathrm{SL}_2(\mathbb{C})$ by conjugacy at the target.
- *Character variety* $X(\pi) = \mathrm{Hom}(\pi, \mathrm{SL}_2(\mathbb{C}))//\,\mathrm{SL}_2(\mathbb{C})$ is the algebraic quotient $= \mathrm{Spec}(\text{Invariant Functions})$
- For $\alpha \in \pi$, invariant function: $t_\alpha\colon \rho \mapsto \mathrm{Tr}(\rho(\alpha))$

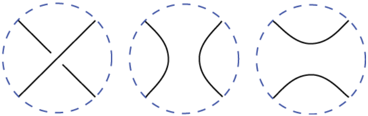
Theorem [CP17]: Presentation of the algebra $\mathbb{C}[X(\pi)]$ of characters

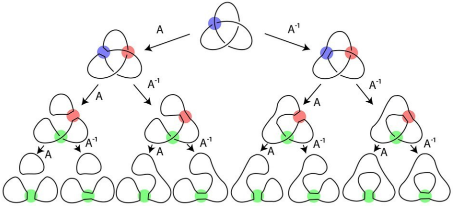
The algebra $\mathbb{C}[X(\pi)]$ of invariant functions has

- generators the $t_\alpha\colon [\rho] \mapsto \mathrm{Tr}(\rho(\alpha))$ for $\alpha \in \pi$
- ideal of relations generated by $t_1 - 2$ and $t_\alpha t_\beta - t_{\alpha\beta} - t_{\alpha\beta^{-1}}$

For π fundamental group of a closed oriented surface Σ

Multiloop $\cup\gamma_i$ with k self-intersections, apply trace relations to decompose:





$$(-t_\alpha)(-t_\beta)+(-t_{\alpha\beta})+(-t_{\alpha\beta^{-1}})=0$$

$$\boxed{\prod_{\text{loops}}(-t_{\gamma_i}) = (-1)^k \sum_{2^k \text{ states}} \prod_{\text{circles}} (-t_{\mu_j})}$$

Theorem [PS00]: Linear basis for the algebra $\mathbb{C}[X(\Sigma)]$

- *Multicurve* $\mu \subset \Sigma$: disjoint union of simples curves μ_j and $t_\mu = \prod t_{\mu_j}$.
- The t_μ for $\mu \in \mathrm{MC}$ form a linear basis of the algebra $\mathbb{C}[X(\Sigma)]$.

Question: what does the decomposition of t_γ look like ?

Example (Tchebychev): inside an immersed annulus Σ_0^2

- Fundamental group $\pi = \mathbb{Z}$ is free on one generator α

$$\mathbb{C}[X(\Sigma_0^2)] = \mathbb{C}[x]$$

- $\mathrm{Tr}(\alpha^n) = 2T_n(x/2)$ Tchecbychev polynomial of the first kind.

Example (Fricke): inside an embedded three holed sphere Σ_0^3

- Fundamental group $\pi = \langle a, b, c \mid abc = 1 \rangle$ is free on two generators,

$$\mathbb{C}[X(\Sigma_0^3)] = \mathbb{C}[t_a, t_b, t_c] = \mathbb{C}[x, y, z]$$

- Diagram computation: $\mathrm{Tr}([a, b]) = x^2 + y^2 + z^2 - xyz - 2$

Theorem [MS24]: Trace functions of multiloops are unitary

For all $\alpha_j \in \pi_1(\Sigma)$, the polynomial $\prod t_{\alpha_j} \in \bigoplus_{MC} \mathbb{Z} \cdot t_\mu$ is *unitary*.

Valuations and simple valuations

Strategy to study decomposition of functions in the linear basis

- Define “monomial” valuations with respect to the linear basis MC
- Define the Newton set of f as the “extremal points” in its support

Definition [MS21]: Valuations on $\mathbb{C}[X(\Sigma)]$ centred at infinity

A valuation is $v\colon \mathbb{C}[X] \to \{-\infty\} \cup \mathbb{R}_+$ satisfying for all f, g:

$$v(f) = -\infty \iff f = 0$$

$$v(fg) = v(f) + v(g)$$

$$v(f+g) \leq \max\{v(f), v(g)\}$$

Weak topology: pointwize convergence of the $v(f)$ for $f \in \mathbb{C}[X]$.

Definition [MS21]: Simple valuation (“monomial” w.r.t. linear basis)

A valuation $v\colon \mathbb{C}[X(\Sigma)] \to \{-\infty\} \cup \mathbb{R}_+$ is *simple* when for all $f = \sum m_\mu t_\mu$:

$$v(f) = \max\{v(t_\mu) \mid m_\mu \neq 0\}$$

Simple valuations are measured laminations

Theorem [MS21]: Simple valuations are the completion of $\mathbb{Q}\mathrm{MC}$

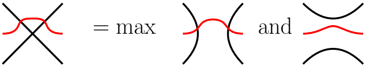
For $\lambda \in \mathrm{MC}$, there exists a unique simple valuation v_λ such that

$$\forall \alpha \in \pi_1(\Sigma): \quad v_\lambda(t_\alpha) = i(\lambda, \alpha)$$

The set of simple valuations ML is equal to the completion of $\mathbb{Q}\mathrm{MC}$.

$\rightarrow$ Well defined by D. Thurston intersection formula:

$$i(\lambda,\alpha) = \bigvee_{\mu} \sum_{\mu_j} i(\lambda,\mu_j) = \max\{i(\lambda,\mu) \mid \mathrm{states}\,\mu\} = v_\lambda(t_\alpha)$$



Morphism $v(fg) = v(f) + v(g)$ deduced from integrality of $\bigoplus_{n\in\mathbb{N}} F_n/F_{n-1}$ where $F_n = \mathrm{Span}\{t_\alpha \mid \alpha \in \pi_1(\Sigma),\ i(\lambda,\alpha) \leq n\}$.

$\leftarrow$ Bass-Serre tree of $\mathrm{SL}_2(\mathbb{C}(X), v)$, Morgan-Otal Skora domination

Most simple valuations are strict

Thurston-Masur volume on the space ML

The topological space ML admits (a PL-structure of dim $6g-6$ and) a unique Mod(Σ)-invariant Borelian measure up to scaling.

Defined on open subsets $U \subset \mathrm{ML}$ by: $\quad \mathrm{Vol}(U) = \lim_{r \to \infty} \frac{\mathrm{Card}(r \cdot U \cap \mathrm{MC})}{r^{6g-6}}$

Definition [MS21]: Strict valuations (implies simple and positive)

A valuation $v\colon \mathbb{C}[X(\Sigma)] \to \{-\infty\} \cup \mathbb{R}_+$ is *strict* when for all $\mu, \nu \in \mathrm{MC}$:

$$\mu \neq \nu \implies v(t_\mu) \neq v(t_\nu)$$

This implies in particular that it is simple, and that $v(t_\mu) > 0$ for all $\mu \neq \emptyset$.

Proposition [MS21]: Most simple valuations are strict.

The set of strict valuations has full measure in ML.

Newton set of a function

Definitions: Support, Extremal multicurve, Newton Set

The *support* of $f = \sum m_\mu t_\mu \in \mathbb{C}[X(\Sigma)]$ is $\mathrm{Supp}(f) = \{\mu \in \mathrm{MC}, m_\mu \neq 0\}$.

- A multicurve $\mu \in \mathrm{Supp}(f)$ is *extremal* in f if there exists a multicurve λ such that $i(\lambda, \mu) > i(\lambda, \nu)$ for all $\nu \in \mathrm{Supp}(f)$ distinct from μ.

  Equivalently, there exists a strict $v \in \mathrm{ML}$ such that $v(t_\mu) = v(f)$.
- The *Newton set* $\Delta(f)$ of f is the set of extremal multicurves in f.
- The dual Newton polytope is $\Delta^*(f) = \{v \in \mathrm{ML} \mid v(f) \leq 1\}$.

Theorem [MS24]: Trace functions of multiloops are unitary

For all $\alpha_j \in \pi_1(\Sigma)$, the polynomial $f = \prod t_{\alpha_j} \in \mathbb{C}[X(\Sigma)]$ is *unitary*:

$$\forall \mu \in \Delta(f): \quad m_\mu = \pm 1$$

(Proof: Show that for a strict valuation v and multiloop α with smoothings α_-, α_+ at an intersection we have $v(t_{\alpha_-}) \neq v(t_{\alpha_+})$.)

Residual value of a function at a strict valuation

Extend $v \in \mathrm{ML}$ to $v \colon \mathbb{C}(X) \to \{-\infty\} \cup \mathbb{R}$ by $v(f/g) = v(f) - v(g)$.

- *Group of values* $\Lambda_v = v(\mathbb{C}(X))$, and *rational rank* $\dim \mathbb{Q} \otimes \Lambda_v$.
- The *transcendence degree* of its *residue field* $k_v = \mathcal{O}_v / \mathcal{M}_v$.

Abhyankar inequality: $\mathrm{rat.rk}(v) + \mathrm{tr.deg}(k_v) \leq \dim(X) = 6g - 6$

Proposition [MS24]: $\mathrm{strict} \iff \mathrm{tr.deg} = 0 \iff \mathrm{rat.rk} = 6g - 6$

For a simple valuation $v \in \mathrm{ML}$ the following properties are equivalent:

strict, that is $\forall \mu, \nu \in \mathrm{MC} : \mu \neq \nu \implies v(t_\mu) \neq v(t_\nu)$

minimal transcendence degree: $\mathrm{tr.deg}(k_v) = 0$, or $k_v = \mathbb{C}$.

maximal rational rank: $\mathrm{rat.rk}(v) = 6g - 6 = \dim(X) = \dim(\mathrm{ML})$.

Definition: residual value at a strict valuation $v \in \mathrm{ML}$ of $f \in \mathcal{O}_v$

The *residual value* $f_v \in \mathbb{C}$ is defined as $(f \bmod \mathcal{M}_v) \in k_v$.
It equals the coefficient m_μ of t_μ for $\mu \in \Delta(f)$ such that $v(f) = v(t_\mu)$.

Quest: study the structure constants for multiplication

The linear basis of MC is not monomial: $i(\mu,\nu) \neq 0 \implies t_\mu t_\nu \neq t_\xi$.
What are the structure constants $c_{\mu\nu}^{\xi}$ for multiplication ?

$$\mathbb{C}[X(\Sigma)] = \bigoplus_{\mu \in \mathrm{MC}} \mathbb{C} \cdot t_\mu \qquad t_\mu t_\nu = \sum_{\xi \in \mathrm{MC}} c_{\mu\nu}^{\xi} t_\xi$$

Example [FG00]: In the torus Σ_1 with the bracelet basis

- Fundamental group $\pi = \langle a, b \mid [a,b] = 1 \rangle \simeq \mathbb{Z}^2$ is abelian.
- Characters $\simeq$ representations $a \mapsto \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}$ and $b \mapsto \begin{pmatrix} y & 0 \\ 0 & 1/y \end{pmatrix}$.
- Triangular change of basis from multicurves to bracelets $T_{p,q} = \mathrm{Tr}(a^p b^q)$ for $p \wedge q = 1$ and $T_{np,nq} = Tcheb_n(T_{p,q})$:

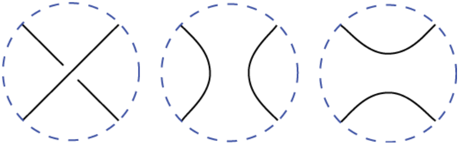
$$\mathbb{C}[X(\Sigma_1)] = \mathbb{C}\left[x^{\pm 1}, y^{\pm 1}\right]^{\sigma} = \bigoplus_{\mathbb{Z}^2/\pm 1} \mathbb{C} \cdot T_{p,q}$$

- Product to sum: $T_{p,q} \cdot T_{r,s} = T_{p+r,q+s} + T_{p-r,q-s}$

The Luo products are extremal multicurves

Definition [Luo10]: Luo product of multicurves

For $\mu, \nu \in \mathrm{MC}$, define $L_\mu(\nu)$ from $\mu \cup \nu$ by smoothing intersections with left turns as we travel along segments of μ which meet segments of ν.



$\mu \cup \nu$ $\quad$ $L_\mu(\nu)$ $\quad$ $L_\nu(\mu)$

Proposition [MS24]: Luo products are extremal multicurves of $t_\mu t_\nu$

For all $\mu, \nu \in \mathrm{MC}$ such that $i(\mu, \nu) > 0$, the Luo products $L_\mu(\nu)$ and $L_\nu(\mu)$ are distinct, and both belong to $\Delta(t_\mu t_\nu)$, with coefficients $(-1)^{i(\mu,\nu)}$.

Poisson algebra structure on $\mathbb{C}[X(\Sigma)]$

Theorem [Gol86]: Poisson bracket on $\mathbb{C}[X(\Sigma)]$

The Atiyah-Bott-Weil-Petersson-Goldman symplectic structure on X defines a Poisson bracket on $\mathbb{C}[X(\Sigma)]$. For $\alpha, \beta \in \pi_1(S)$ it is given by

$$\{t_\alpha, t_\beta\} = \sum_{p \in \alpha \cap \beta} \epsilon_p \left(t_{\alpha_p \beta_p} - t_{\alpha_p \beta_p^{-1}} \right)$$

where the sum ranges over all intersection points p between transverse representatives for $\alpha \cup \beta$ and ϵ_p is the sign of such an intersection, while α_p, β_p denote the homotopy classes of α, β based at p.

$$\{t_\alpha, t_\beta\} = \sum_\xi w_\xi t_\xi = \sum_\xi \left(\sum_{\sigma_\xi} \prod_p \sigma_\xi(p) \right) t_\xi \qquad \text{(PB-state-sum)}$$

where $w_\xi = \sum_{\sigma_\xi} \prod_p \sigma_\xi(p)$ is the sum over the smoothings $\sigma_\xi \colon \alpha \cap \beta \to \{\pm 1\}$ of $\alpha \cup \beta$ yielding the multiloop ξ.

Newton set of the Poisson bracket

Corollary [MS24]: "$\Delta(\{f,g\}) \subset \Delta(fg)$"

For $f, g \in \mathbb{C}[X]$, we have $v(\{f,g\}) \leq v(fg)$ for all $v \in \mathrm{ML}$.
This property amounts to the inverse inclusion of the dual polytopes:

$$\Delta^*(\{f,g\}) \supset \Delta^*(fg)$$

Hence the Goldman Poisson bracket induces a residual Poisson bracket at any strict valuation v. (This endows $T_v\mathrm{ML}$ with a symplectic structure...)

Proof: Apply unitarity of $t_\alpha t_\beta$ and (PB-state-sum) formula for $\{t_\alpha, t_\beta\}$.

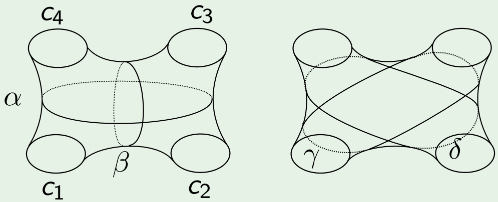
Corollary [MS24]: Luo products are extremal multicurves of $\{t_\mu, t_\nu\}$

For all $\mu, \nu \in \mathrm{MC}$ such that $i(\mu,\nu) > 0$, the Luo products $L_\mu(\nu)$ and $L_\nu(\mu)$ are distinct, and both belong to $\Delta(\{t_\mu, t_\nu\})$, with coefficients $\pm i(\mu,\nu)$.

Proof: The (PB-state-sum) formula implies $L_\mu(\nu), L_\nu(\mu) \in \Delta(\{t_\mu, t_\nu\})$.

Poisson algebra structure on $\mathbb{C}[X(\Sigma_0^4)]$

Example: Product and Poisson bracket of $\alpha, \beta \subset \Sigma_0^4$ with $i(\alpha, \beta) = 2$



The Luo product are $L_\alpha(\beta) = \delta$ and $L_\beta(\alpha) = \gamma$ and

$$t_\alpha t_\beta = t_{c_1} t_{c_3} + t_{c_2} t_{c_4} - t_\gamma - t_\delta \qquad \{t_\alpha, t_\beta\} = 2t_\delta - 2t_\gamma$$

$$\Delta(t_\alpha t_\beta) = \{c_1 \cup c_3, c_2 \cup c_4, \gamma, \delta\} \qquad \Delta(\{t_\alpha, t_\beta\}) = \{\gamma, \delta\}$$

The Newton set of $t_\alpha t_\beta$ decomposes ML into 4 domains where $i(\lambda, \alpha \cup \beta)$ equals the intersection of λ with $c_1 \cup c_3$ or $c_2 \cup c_4$ or γ or δ respectively. In the interior of these domains $\{t_\alpha, t_\beta\}$ has residual values $0, 0, -2, 2$.

Mirzakhani asymptotics as volumes of Newton Polytopes*

Topological interpretation of $\mathrm{Vol}\,\Delta^*(t_\alpha)$.

For a multiloop α, can we give a topological interpretation for the Thurston-Masur volume $\mathrm{Vol}\,\Delta^*(t_\alpha)$?
It vanishes unless α is *filling*, meaning it intersects every simple curve, in which case for every other filling multiloop β we have:

$$\lim_{r\to\infty} \frac{\mathrm{Card}\{\varphi \in \mathrm{Mod}(S) \mid i(\lambda, \varphi(\alpha)) \le r\}}{r^{6g-6}} = \frac{\mathrm{Vol}\,\Delta^*(t_\beta)\,\mathrm{Vol}\,\Delta^*(t_\alpha)}{m_g}$$

Computation in terms of elementary cones in ML indexed by $\Delta(f)$

Identification between measured laminations and simple valuations implies

$$\forall f \in \mathbb{C}[X(\Sigma)]: \quad \Delta^*(f) = \bigcap_{\mu\in\mathrm{Supp}(f)} \Delta^*(t_\mu) = \bigcap_{\mu\in\Delta(f)} \Delta^*(t_\mu)$$

The $\Delta^*(t_\mu)$ are described by explicit sets of linear inequalities in any PL chart of ML, and the volume of their intersection is computable.

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Thank you for your attention and feel free to ask (m)any questions !

Teichmüller space embeds in real locus of character variety

1. The Teichmüller space of Σ is the space of complex structures on Σ.
2. By the uniformisation theorem, every complex structure on Σ is conformal to a unique hyperbolic structure.
3. A hyperbolic structure on Σ is uniquely determined by its holonomy representation $\rho\colon \pi_1(\Sigma) \to \mathsf{PSL}_2(\mathbb{R})$, well defined up to conjugacy.
4. These correspond to the Fuchsian representations, or by Milnor-Wood inequalities to those with extremal euler class $\pm\chi = \pm(2-2g)$.
5. If $\rho\colon \pi_1(\Sigma) \to \mathsf{PSL}_2(\mathbb{R})$ has even euler class then it lifts to $\mathsf{SL}_2(\mathbb{R})$, so there are 2×2^{2g} copies of Teichmüller space in $X(\Sigma)$.
6. Teichmüller space of Σ is Zariski dense in the character variety $X(\Sigma)$ (as Fuchsian representations form open subset of $\mathrm{Hom}(\pi, \mathsf{SL}_2(\mathbb{R}))$, which quotient to open subset of $X(\pi, \mathsf{SL}_2(\mathbb{R}))$.)
7. Trace function of loop $\leftrightarrow$ length of the unique geodesic:

$$t_\alpha([\rho]) = 2\cosh(l_\alpha(m)/2)$$