The complexity of pinning loops in the plane

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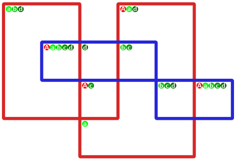
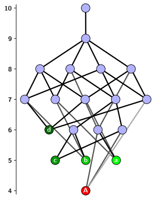
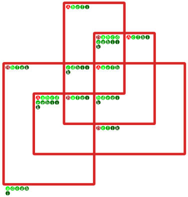
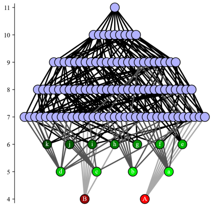
Geometry Lunch Seminar, 2024-10-30









Complexity of topological invariants

Recent results on the computational complexity of link invariants:

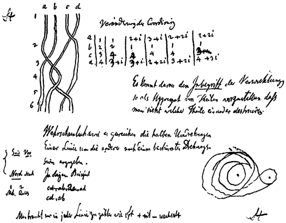
bounding the crossing number of a knot or link is NP-hard

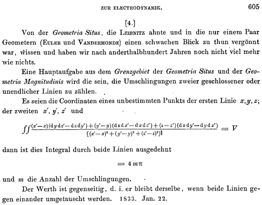
bounding the unknotting number of a knot or link is is NP-hard

bounding the genus of a knot in a general 3-manifold is NP-hard

bounding the genus of a knot in the sphere is in co-NP

computing a finite type invariant of degree d is in $O(n^d)$





ZUR ELECTRODYNAMIK. 605

[4.]

Von der *Geometria Situs*, die LEIBNITZ ahnte und in die nur einem Paar Geometern (EULER und VANDERMONDE) einen schwachen Blick zu thun vergönnt war, wissen und haben wir nach anderthalbhundert Jahren noch nicht viel mehr wie nichts.

Eine Hauptaufgabe aus dem *Grenzgebiet* der *Geometria Situs* und der *Geometria Magnitudinis* wird die sein, die Umschlingungen zweier geschlossener oder unendlicher Linien zu zählen.

Es seien die Coordinaten eines unbestimmten Punkts der ersten Linie x, y, z; der zweiten x', y', z' und

$$\iint \frac{(x'-x)(\mathrm{d}y\,\mathrm{d}z'-\mathrm{d}z\,\mathrm{d}y')+(y'-y)(\mathrm{d}z\,\mathrm{d}x'-\mathrm{d}x\,\mathrm{d}z')+(z-z')(\mathrm{d}x\,\mathrm{d}y'-\mathrm{d}y\,\mathrm{d}x')}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}} = V$$

dann ist dies Integral durch beide Linien ausgedehnt

$$= 4m\pi$$

und m die Anzahl der Umschlingungen.

Der Werth ist gegenseitig, d. i. er bleibt derselbe, wenn beide Linien gegen einander umgetauscht werden. 1833. Jan. 22.

Gauss discovers linking numbers.

Everything is contained in plane loops

Arnol'd [Arn00] in the Preface to Lecture 1:

A method for studying infinite dimensional configuration space is to compute the relative cohomology of their stratifications by singular loci. This is how Vassiliev introduced finite type invariants of knots $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3$. Arnol'd proposes an analogous study for immersions $\mathbb{S}^1 \looparrowright \mathbb{S}^2$, but...

> *The combinatorics of plane curves seems to be far more complicated than that of knot theory (which might be considered as a simplified "commutative" version of the combinatorics of plane curves and which is probably embedded in plane curves theory).*

Survey [Poé95] on extending immersions $\mathbb{S}^m \looparrowright \mathbb{S}^n$ to $\mathbb{D}^{m+1} \looparrowright \mathbb{S}^n$

Thom-Smale-Hirsch: solved using homotopy groups and obstruction theory when $m - n > 1$. Note: homotopy groups are commutative in degrees > 1.

Why self-intersection and pinning sets of multiloops ?

A (whimsical) physical motivation: Neumann-Coto [NC01]

In a surface $\mathbb{F}$, an immersion $\gamma \colon \sqcup_1^s \mathbb{S}^1 \looparrowright \mathbb{F}$ is isotopic to a collection of shortest geodesics for a complete Riemannian metric on $\mathbb{F}$ if and only if it is taut (realizes the minimal number of double-points in its homotopy class).

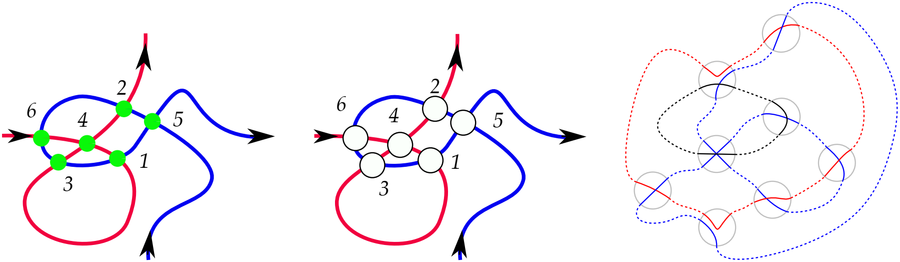
Where do punctures come into the picture ?

For multiloops in the sphere, we will study the relation between self-intersection numbers and punctures to define pinning numbers. Applications to the bounded cohomology of braid groups and Homeo($\mathbb{S}^2$).

Using multiloops as a computation scheme

Vaughan Jones: planar algebras

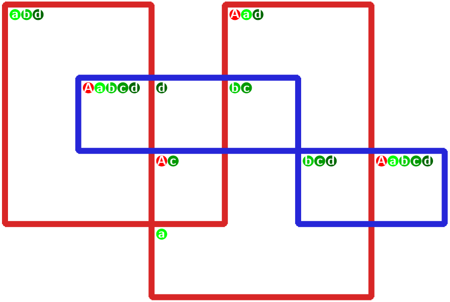
The algebra of tangles and multiloops in punctured spheres encapsulate efficiently the essence (and complexity) of many computational schemes.



Computing with planar algebras.

Multiloops in punctured spheres

Disjoint union of circles $\sqcup_1^s \mathbb{S}^1$. Punctured sphere $\mathbb{S}^2_P = \mathbb{S}^2 \setminus P$.
A multiloop with s strands is a generic immersion $\gamma\colon \sqcup_1^s \mathbb{S}^1 \looparrowright \mathbb{S}^2_P$ up to orientation preserving diffeomorphisms of source and the target.
Here generic means all multiple points are transverse double-points.
The number of double-points is denote by $\#\gamma$.
The regions R are the connected components of $\mathbb{S}^2 \setminus \mathrm{im}(\gamma)$.

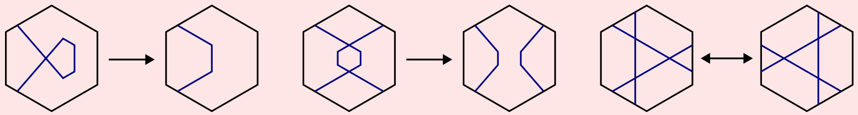


The multiloop 10^2_{16} has $\#\gamma = 8$. Various sets of punctures.

Multicurves, self-intersection, taut multiloops

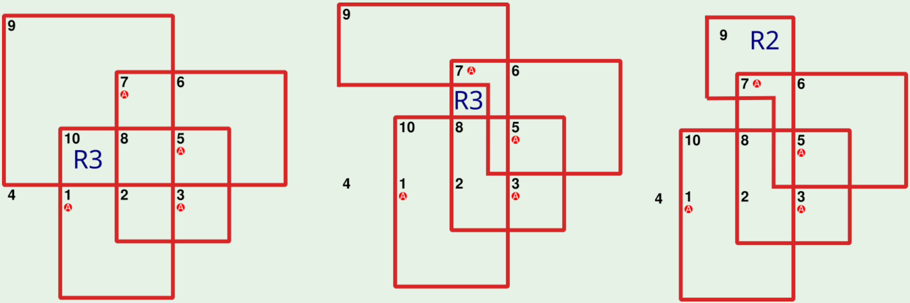
A multicurve is a homotopy class of multiloops.
The self-intersection $\mathrm{si}(\gamma)$ of a multicurve is the minimal number of double-points among its generic representatives.
A multiloop γ is taut when $\#\gamma = \mathrm{si}(\gamma)$, namely when it has the minimal number of double-points in its homotopy class.

If two multiloops in $\mathbb{S}^2_P$ are homotopic, then they are related by a sequence of isotopies and Reidemeister moves $R1, R2, R3$.



How to know when a multiloop is taut ?

Try combinations of $R1, R2, R3$ until finding γ' with $\#\gamma' < \#\gamma$...



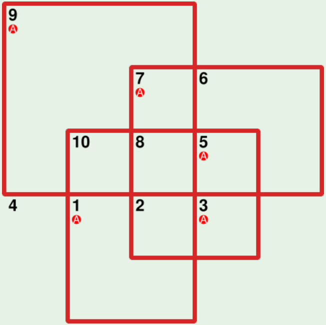
Can we order and bound the sequences of Rk-moves ? We won't insist...

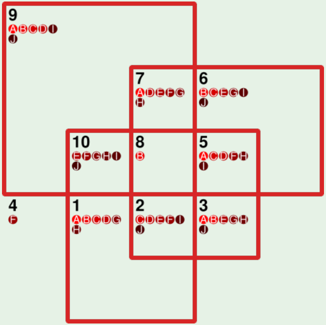
Can we construct an algorithm computing $\mathrm{si}(\gamma)$ to compare with $\#\gamma$?

How to know when a multiloop is taut ?

Pinning sets

Which puncture sets $P \subset R$ will yield a loop $\gamma\colon \mathbb{S}^1 \looparrowright \mathbb{S}^2_P$ that is taut ?

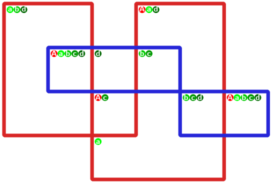


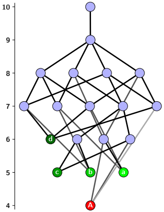


The set $P = R$ of all regions works. We can try deleting one pin at a time...

Pinning sets, pinning number, pinning ideal

Consider a multiloop $\gamma\colon \sqcup_1^s \mathbb{S}^1 \looparrowright \mathbb{S}^2$ with regions R.
A subset of regions $P \subset R$ is pinning when $\gamma_P\colon \sqcup_1^s \mathbb{S}^1 \looparrowright \mathbb{S}^2_P$ is taut.
The pinning sets of γ form its pinning ideal: a subposet $\mathcal{PI} \subset \mathcal{P}(R)$ absorbant under union and containing R.
The pinning number $\varpi(\gamma)$ is the minimum cardinal of pinning sets.
A pinning set optimal if it has the minimum cardinal, in particular it must be minimal (with respect to inclusion).





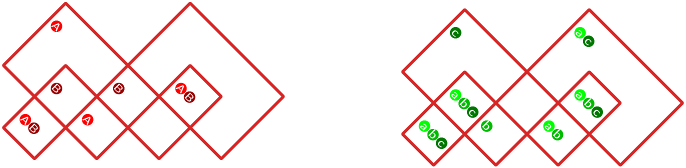
The multiloop 10^2_{16} and its pinning semi-lattice obtained from unions of minimal pinning sets (in red or green), together with the whole set R.

Main questions:

Given a multiloop $\gamma\colon \sqcup_1^s \mathbb{S}^1 \looparrowright \mathbb{S}^2$, how (efficiently) can we:

- Construct minimal or optimal pinning sets?
- Find which regions belong to every pinning set?
- Compute the pinning number $\varpi(\gamma)$?

What can be the shape of the pinning ideal of a (multi)loop?
What are its statistics for certain random models?



The loop 9_5^1 has 2 optimal pinning sets, and 3 other minimal pinning sets.

Overview of our main complexity results

Computing si and check optimal solutions in P-steps

We construct a polynomial algorithm which given a multiloop $\gamma\colon \sqcup_1^s \mathbb{S}^1 \looparrowright \mathbb{S}^2_P$ computes its self intersection-number $\mathrm{si}(\gamma)$.

Thus in P-steps we can: check if a given set $P \subset R$ is pinning, if so minimal and optimal; and we can find a minimal pinning set.

We implemented it to build an online catalog of pinning data for multiloops.

The Pinning number is NPC:

The following LooPiNum decision problem is NPC:

Input: A loop $\gamma\colon \mathbb{S}^1 \looparrowright \mathbb{S}^2$ and an integer $k \in \mathbb{N}$.

Output: Is $\varpi(\gamma) \leq k$?

Proof: we will construct two reductions:

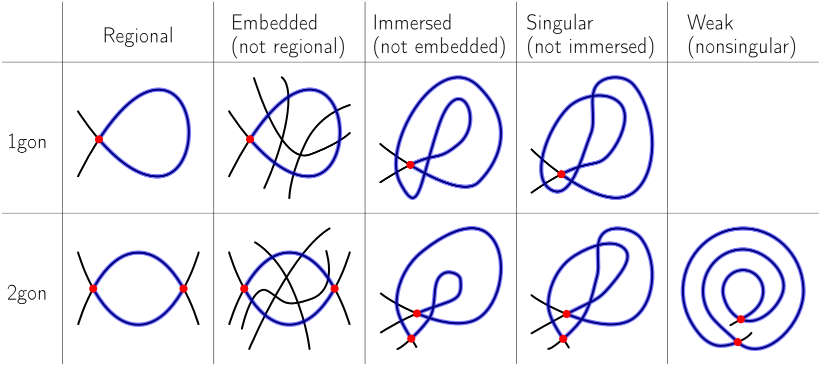
NP-easy: reduction from LooPiNum to hypergraph-vertex-cover.

NP-hard: reduction from graph-vertex-cover to LooPiNum.

Singular monorbigons

Singular monorbigons of a loop $\gamma\colon \mathbb{S} \looparrowright \mathbb{S}^2_P$:

A singular monogon is a non-trivial closed interval $I \subset \mathbb{S}^1$ such that $\gamma(\partial I) = \{x\}$ and $\gamma(I) \subset \mathbb{S}^2_P$ is null-homotopic.
A singular bigon is a disjoint union of non-trivial closed intervals $I \sqcup J \subset \mathbb{S}^1$ such that $\gamma(\partial I) = \{x, y\} = \gamma(\partial J)$ and $\gamma(I \sqcup J) \subset \mathbb{S}^2_P$ is null-homotopic.
It is embedded when γ is injective on the interior of $K = I$ or $K = I \sqcup J$.



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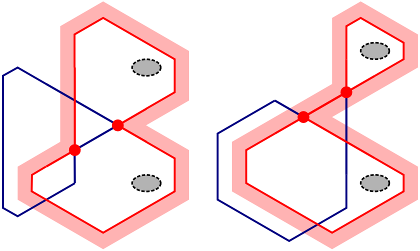


A singular (embedded) bigon $R3$-moves into a non-singular (weak) bigon.

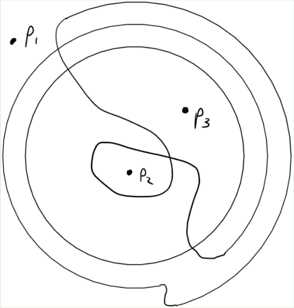
Hass-Scott characterise tautness with singular monorbigons

Hass-Scott [HS85, Theorems 2.7 and 4.7]

2.7: If $\#\gamma > 0$ but $\mathrm{si}(\gamma) = 0$, then γ has an embedded monorbigon.
4.2: If $\#\gamma > \mathrm{si}(\gamma)$, then γ has a singular monorbigon.

Just for loops: no generalisation to multiloops

A multiloop $\gamma\colon \mathbb{S}^1 \sqcup \mathbb{S}^1 \looparrowright \mathbb{S}^2 \setminus \{p_1, p_2, p_3\}$ which is *not taut*, but with *no singular monorbigons* (generalised to multiloop):

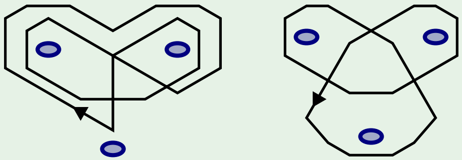


Winding numbers help to pin loops

Corollary: winding numbers yield sufficient conditions for pinning sets

Consider a loop $\gamma\colon \mathbb{S}^1 \looparrowright \mathbb{S}^2$ and a set of regions $P \subset R$. If P is not pinning then there exists a singular monorbigon α of γ such that for every pair of regions $o, \infty \in P$ we have $\mathrm{lk}(\alpha, [o] - [\infty]) = 0$.

The converse is false: the commutator loop in $\mathbb{S}^2 \setminus \{0, 1, \infty\}$



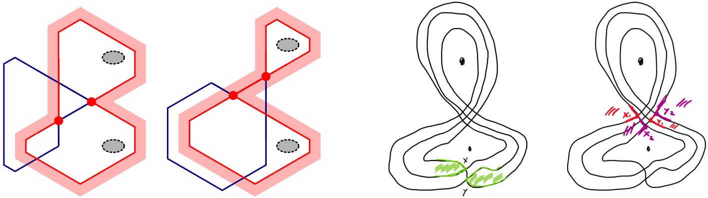
Strategy

To characterise pinning sets of loops in terms of singular monorbigons and linking numbers, we will restrict to the class of immersed monorbigons... These will define the obvious collections of regions to pin: the mobidiscs.

Immersed monorbigons and the Mobidisc theorem

Definition: For a loop $\gamma\colon \mathbb{S} \looparrowright \mathbb{S}^2_P$,

A singular monorbigon $K \subset \mathbb{S}^1$ is immersed when there exists $\iota\colon \mathbb{D} \looparrowright \mathbb{S}^2_P$ such that the restriction $\gamma\colon K \looparrowright \mathbb{S}^2_P$ factors through $\iota\colon \partial\mathbb{D} \looparrowright \mathbb{S}^2_P$.
A mobidisc consists of the regions in the image $\iota(\mathbb{D})$ of such an immersion.
Denote by $\mathrm{MoB}(\gamma) \subset \mathcal{P}(R)$ the set of all mobidiscs.



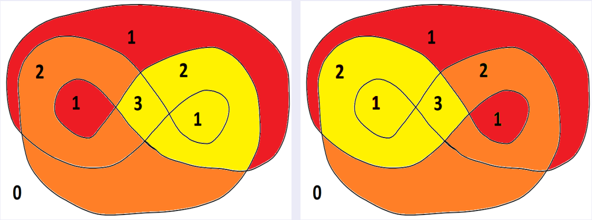
The MoBiDisc Theorem [SS24]

For a loop $\gamma\colon \mathbb{S}^1 \looparrowright \mathbb{S}^2$ and a collection of regions $P \subset R$.
The loop $\gamma\colon \mathbb{S}^1 \looparrowright \mathbb{S}^2_P$ is taut if and only if it has no immersed monorbigons.
Thus $P \in \mathcal{P}(R)$ is pinning if and only if it intersects every $D \in \mathrm{MoB}(\gamma)$.

Computing MoBiDiscs in polynomial time

Non-unique immersed discs but unique mobidisc

An immersed monorbigon may bound *several non-isotopic immersed discs*:



but they all define *the same mobidisc*: its regions are distinguished by

$$(p, q) \in R \times R \mapsto \operatorname{sign} \mathrm{lk}(\gamma(K), [p] - [q]) \in \{0, \pm 1\}.$$

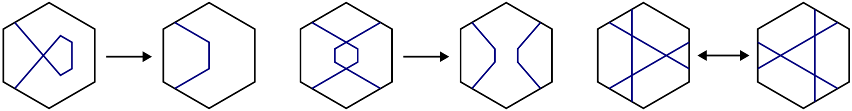
Polynomial computation of the set of mobidiscs

Given a loop $\gamma \colon \mathbb{S}^1 \looparrowright \mathbb{S}^2$, relying on [Bla67, Fri10] and linking numbers, we can compute $\mathrm{MoB}(\gamma) \subset \mathcal{P}(R)$ time $O(\mathrm{Card}(R)^4)$.

Outline of the proof

Theorem [HS94]: curve shortening flow

If two loops in $\mathbb{S}^2_P$ are homotopic, then they are related by a sequence of isotopies and shortening Reidemeister moves $R1, R2, R3$.



Lemma: extending immersed monorbigons across R moves

Consider a loop β and a shortening-Reidemeister move to a loop α.
If α has an immersed monorbigon then β has an immersed monorbigon.

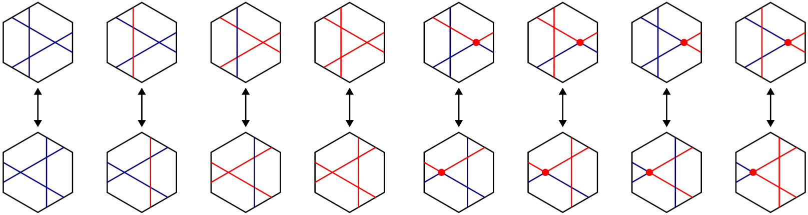
Proof: Chasing mobidiscs accross $R3$ moves by case analysis:

Easy cases: the move restricts to a regular isotopy of the monorbigon.

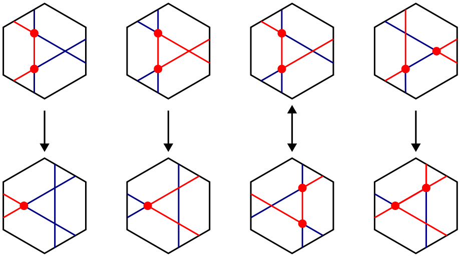
Medium cases : tweak local structure to find immersed monorbigon.

Hard cases: global constructions and arguments to find new mobidisc.

Proof of the Lemma



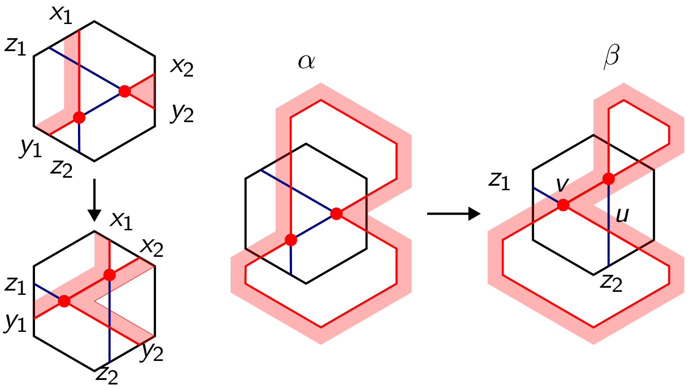
Configurations of type $R3(0,j)$. Configurations of type $R3(1,j)$.



Configurations of type $R3(2,j)$.

Proof of the Lemma: hardest case

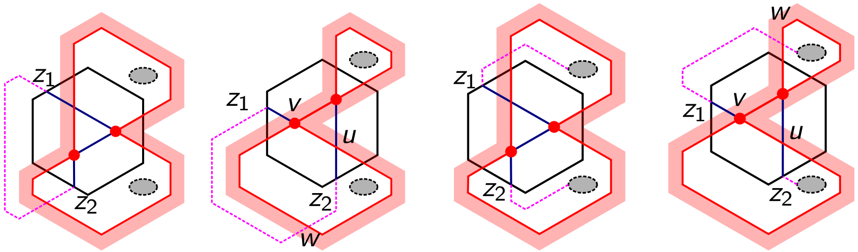
Last case: the singular bigon dissappears into a week bigon, so we must construct another singular monorbigon which is immersed.



The move $R3(2,2)D$ leads to a weak bigon.

Proof of the Lemma: hardest case

Last case: the singular bigon dissappears into a week bigon, so we must construct another singular monorbigon which is immersed.
Subcase analysis: the construction and proof use orientability of the sphere, Theorems 2.7,& 4.2 of Hass-Scott, and more case analysis.



Finding an immersed monorbigon in cases x and y.

Experimentations: couter-examples and patterns

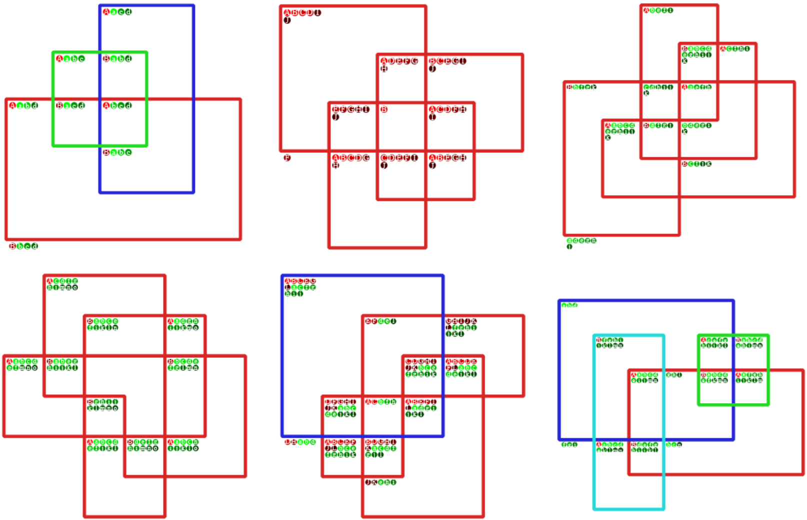
An online catalog: https://christopherlloyd.github.io/LooPindex/

We constructed a data basis of $> 10^3$ multiloops with their pinning data. This enabled us to find counter-examples to various naïve conjectures:

- invariance under Reidemeister moves or mutations
- monotonicity properties of pinning numbers and their averages
- Some distinct loops have isomorphic pinning semi-lattices

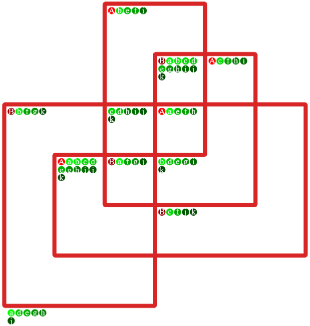
but many patterns are to be understood...

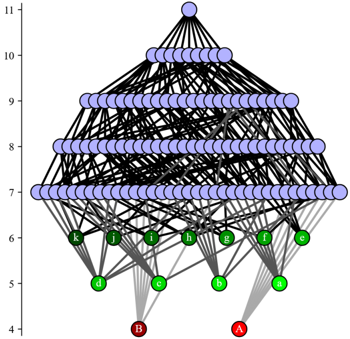
Experimentations: couter-examples and patterns



Multiloops with regions of degree > 2...

Experimentations: couter-examples and patterns





... have a rich pinning semi-lattice !

Further directions of research

Complexity of the pinning number on special instances

1 Find interesting classes of loops for which the `LooPiNum` problem is `P`.
(Add local or global restrictions of topological or geometrical nature.)

2 Generalise the MoBiDisc theorem to certain classes of multiloops.
(This requires extending some of the results by Hass-Scott.)

What do pinning quantities say about plane loops ?

1 Which are related to other invariants of plane loops ?
(Relations with Arnold's cohomological invariants J^-, J^+, St ?)

2 How do they vary under Reidemeister moves ?
(Study the minimal pinning number over an $R3$-equivalence classes.)

Applications to bounded cohomology of braid groups and Homeo($\mathbb{S}^2$)

For $\phi \in$ Homeo($\mathbb{S}^2$): pinning numbers of multiloops $\phi(\gamma) \cup \gamma$ can measure "braiding index" or "distance" to $\mathcal{B}_p$, translation length on curve graphs...

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