The unpinning game

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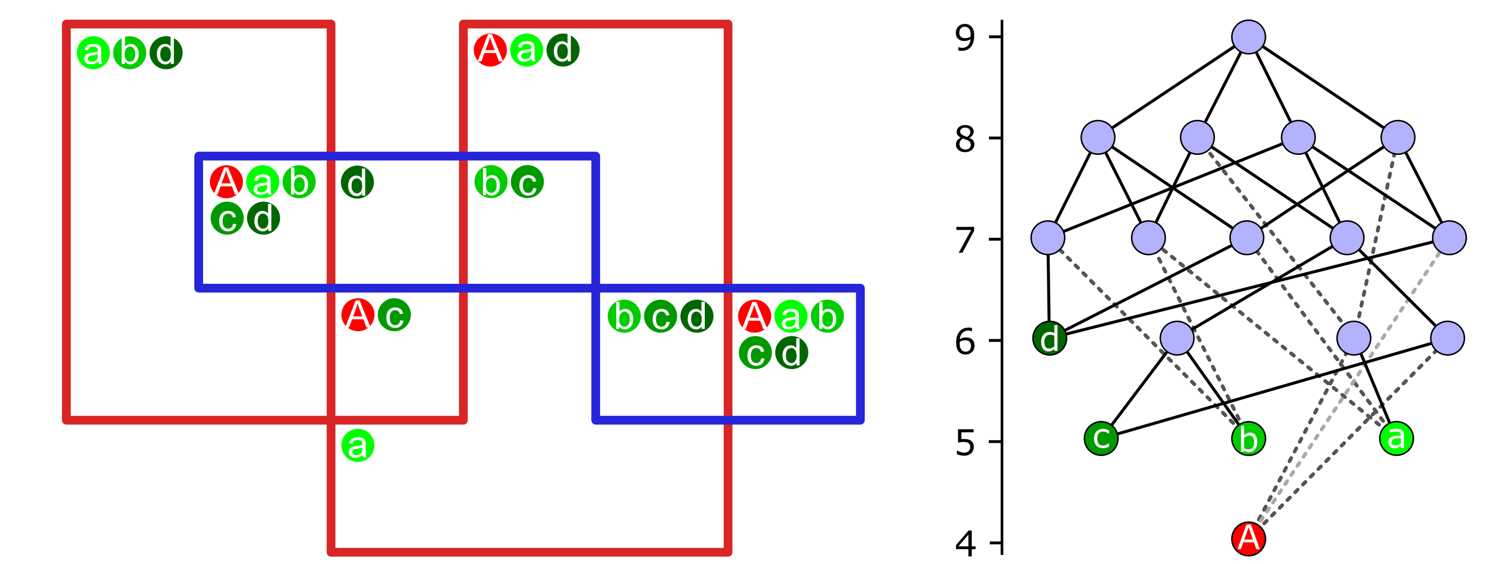
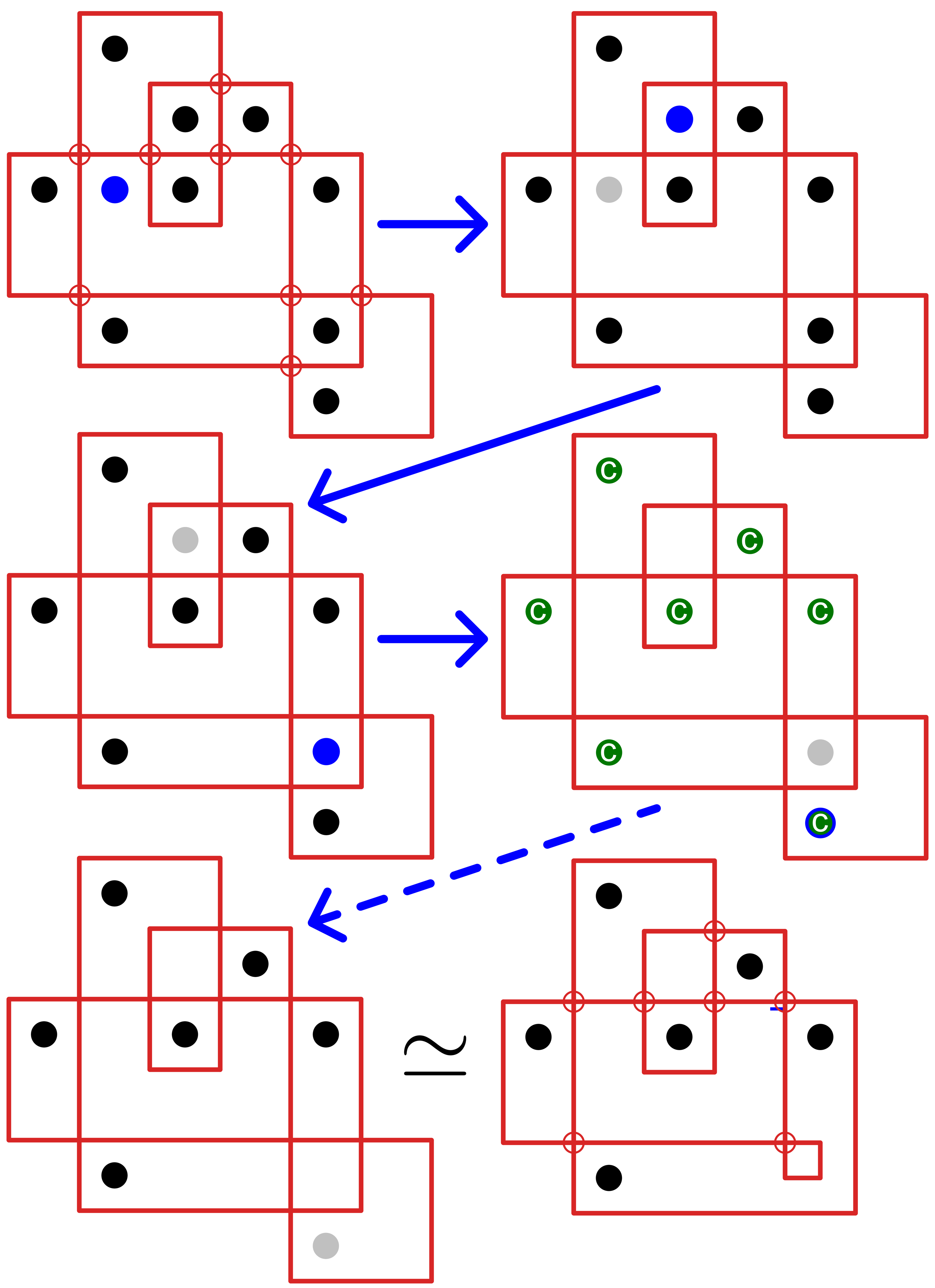
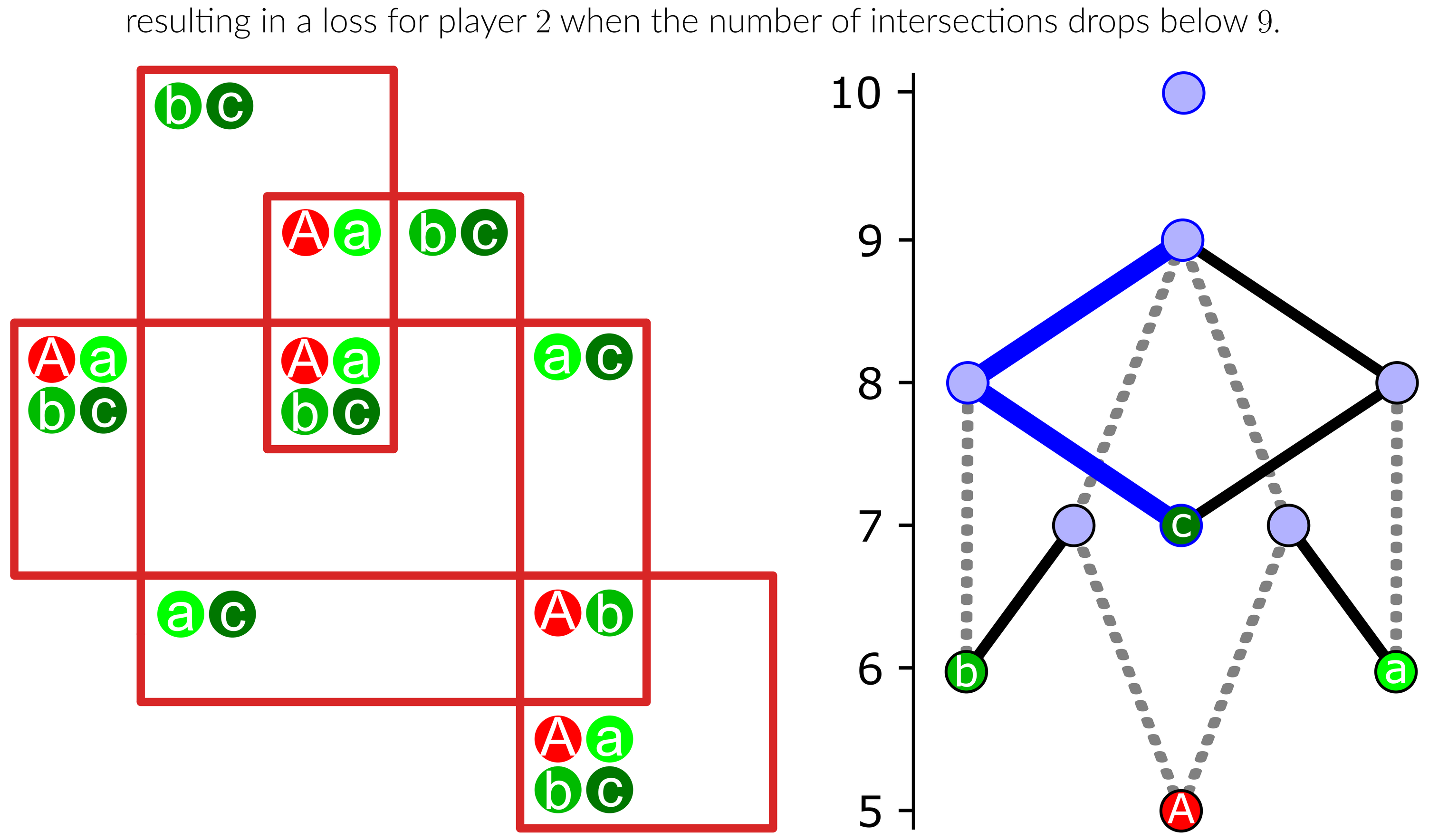


The multiloop 10^2_{16} and part of its pinning ideal.



A sequence of moves in the unpinning game associated to the loop 11^1_{34} resulting in a loss for player 2 when the number of intersections drops below 9.



Left: A visualization of all minimal winning game states A, a, b, and c.
Right: The game sequence represented as a path in part of the pinning ideal.

Definitions

- A ***multiloop*** in a smooth oriented surface Σ is an isotopy class of generic immersions

$$\gamma\colon \sqcup_1^s S^1 \looparrowright \Sigma$$

When it has $s = 1$ ***strand***, we call it a ***loop***. All figures here have $\Sigma \in \{\mathbb{R}^2, S^2\}$.
- Its set of ***regions*** $\mathrm{R}(\gamma)$ consists of the connected components of $\Sigma \setminus \mathrm{im}(\gamma)$.
- Its ***number of double points*** is a nonnegative integer denoted by $\#\gamma$.
- For a set of regions $P \in \mathcal{P}(\mathrm{R}(\gamma))$, we define the ***self-intersection*** function

$$si_P(\gamma) = \min\{\#\gamma' : \gamma' \text{ is freely homotopic to } \gamma \text{ in } \Sigma \setminus P\}.$$

- We say that P is ***pinning*** γ when $si_P(\gamma) = \#\gamma$. For instance $\mathrm{R}(\gamma)$ is pinning.
- In the power set of all regions $(\mathcal{P}(\mathrm{R}(\gamma)), \subset)$, the pinning sets form a subposet which is closed under supersets: ***the pinning ideal*** $(\mathcal{PI}, \subset)$.
- The ***pinning number*** $\varpi(\gamma)$ is the minimal cardinal of a pinning set.
- Note: a ***minimal*** pinning set may have cardinality exceeding $\varpi(\gamma)$.
- To a loop $\gamma : S^1 \looparrowright \Sigma$ we associate a ***mobidisc formula*** in conjunctive normal form (CNF) on the set of variables $\mathrm{R}(\gamma)$ whose disjunctions correspond to certain subsets called ***mobidiscs*** associated to the immersed monogons and bigons of γ.

Theorems

The pinning number is NPC: *The problem whose input is a loop $\gamma\colon S^1 \looparrowright \mathbb{R}^2$ with an integer k and whose objective is to decide whether $\varpi(\gamma) \leq k$ is NP-complete.*

- Why NP? We compute $si_P(\gamma)$ in polynomial time using ideas of Birman–Series concerning the action of the free group on its boundary.
- Why NP-hard? We reduce the vertex cover problem for plane graphs to the pinning number problem for plane loops.

The mobidisc CNF: *The pinning sets of a loop γ correspond to the satisfying assignments of its mobidisc formula. The mobidisc formula is computable in polynomial time.*

- We show that a loop is pinned if and only if all of its mobidiscs are pinned using ideas of Hass–Scott about combinatorial curve shortening flow. This reduces the pinning problem to a satisfiability problem equivalent to hypergraph vertex cover.
- The computability of the mobidisc formula relies on characterizations of curves that bound an immersed disk of Blank, Frisch, Shor–Van Wyk.

The unpinning game

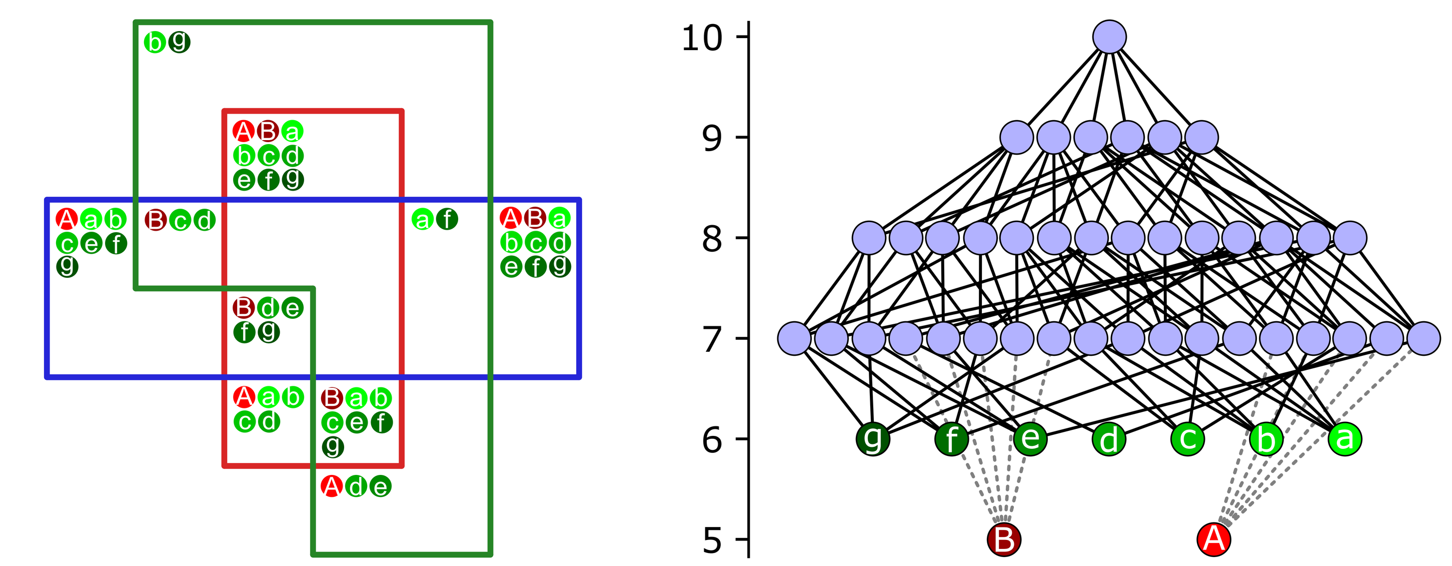
To a multiloop $\gamma : \sqcup_1^s S^1 \looparrowright \Sigma$ we associate an impartial combinatorial game for two players called the ***unpinning game***.

The initial configuration has a pin in each region; two players take turns removing pins so as to leave a pinning set after their turn; the first player with no legal move loses.

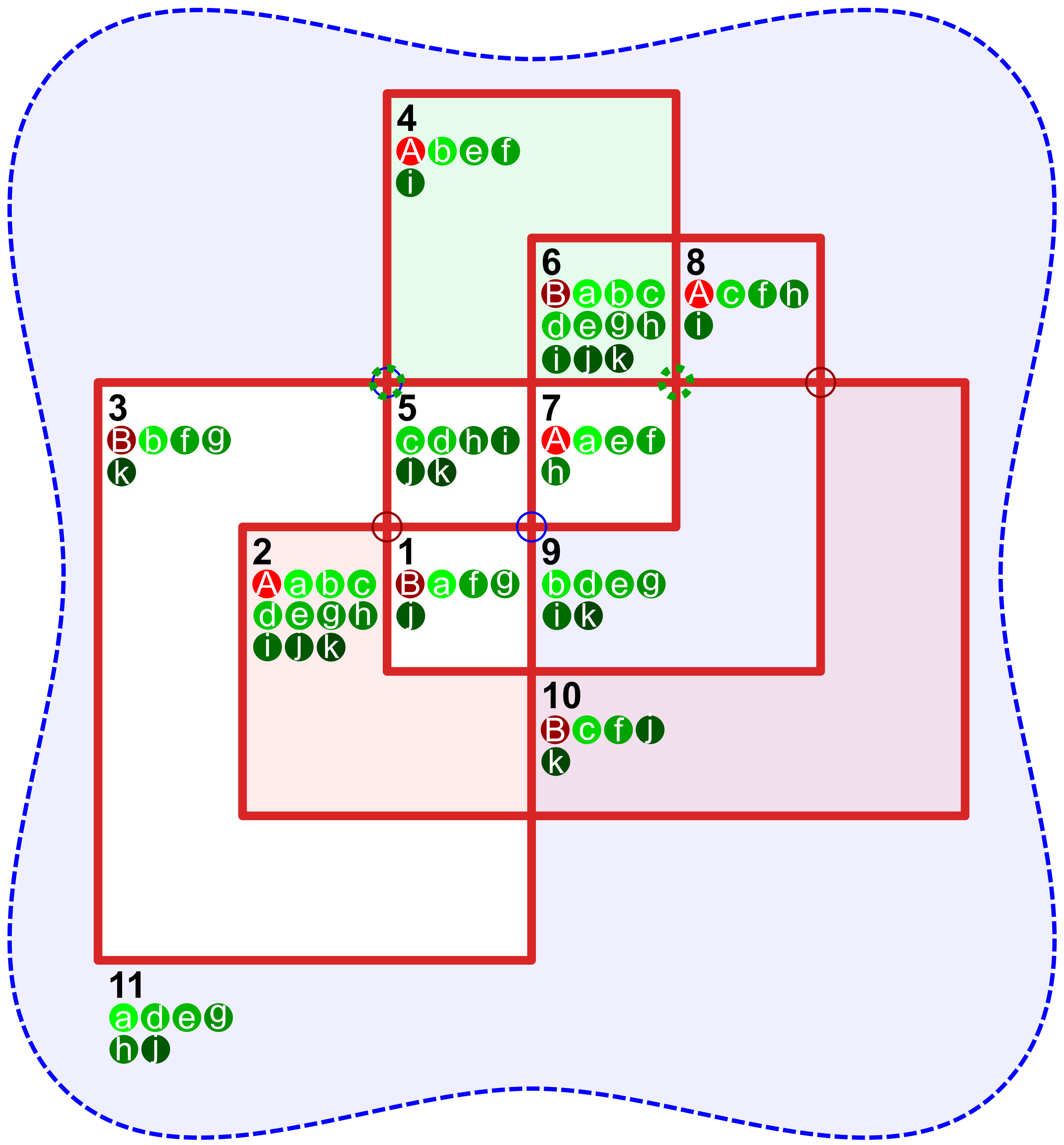
This game has an associated Grundy number that measures its complexity and uniquely characterizes it up to game equivalence (by the Sprague–Grundy theorem).

Questions

1. How efficiently can we decide winning positions and compute winning strategies? Is the associated decision problem PSPACE-complete?
2. What are the Grundy numbers associated to multiloops with s strands in genus g?
3. Construct infinite families of multiloops yielding increasingly hard unpinning games (for instance whose Grundy numbers grow fast with the number of double points).



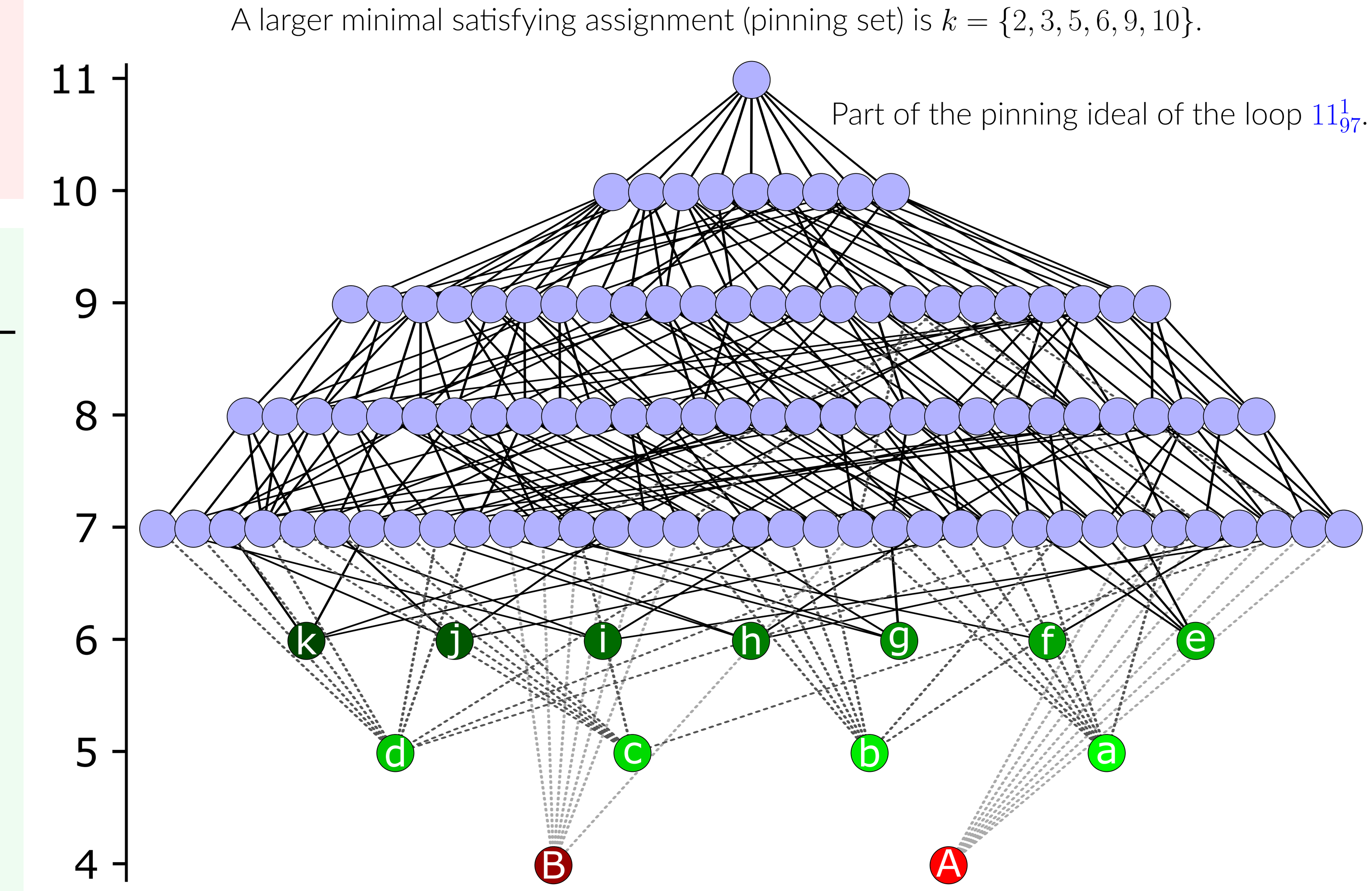
The multiloop 12^3_{56} and part of its pinning ideal.



The loop 11^1_{97} in S^2. Its associated mobidisc formula is

$$(1 \vee 2) \wedge (1 \vee 4 \vee 5) \wedge (1 \vee 5 \vee 7 \vee 9) \wedge (1 \vee 8 \vee 9) \wedge (2 \vee 3) \wedge (\mathbf{2} \vee \mathbf{10}) \wedge (3 \vee 4 \vee 5 \vee 11) \wedge (3 \vee 5 \vee 7)$$
$$\wedge (3 \vee 8 \vee 11) \wedge (\mathbf{4} \vee \mathbf{6}) \wedge (4 \vee 10 \vee 11) \wedge (6 \vee 7) \wedge (6 \vee 8) \wedge (7 \vee 9 \vee 10) \wedge (\mathbf{8} \vee \mathbf{9} \vee \mathbf{10} \vee \mathbf{11}).$$

A satisfying assignment (pinning set) of smallest cardinality is $A = \{2, 4, 7, 8\}$.
A larger minimal satisfying assignment (pinning set) is $k = \{2, 3, 5, 6, 9, 10\}$.



Part of the pinning ideal of the loop 11^1_{97}.

References

- Prepublication : https://arxiv.org/abs/2405.16216
- Index of multiloops with pinning data: https://christopherlloyd.github.io/LooPindex/