Transcendance of simple geodesics in arithmetic surfaces

Dynamics of modular groups and diophantine approximation

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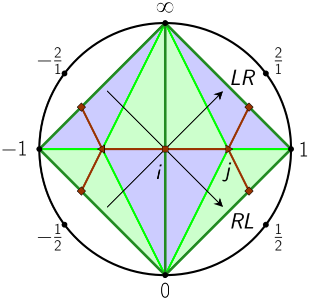
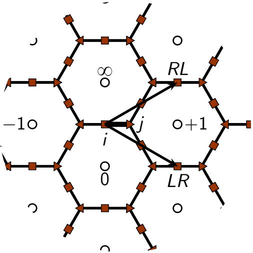
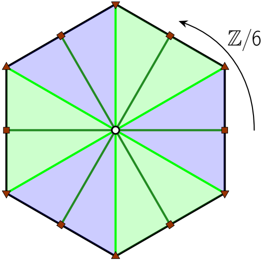


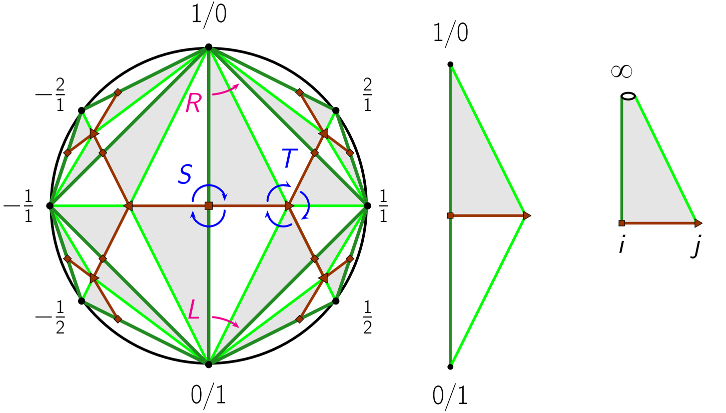




1. The modular group $\Gamma = \mathsf{PSL}_2(\mathbb{Z})$ and its orbifold $\mathbb{M} = \Gamma\backslash\mathbb{HP}$
2. The modular torus $\mathbb{M}' = \Gamma'\backslash\mathbb{HP}$ and its simple geodesics
3. Generalizing to other arithmetic surfaces
4. Periods of modular forms along simple geodesics

Modular group $\mathrm{PSL}_2(\mathbb{Z})$ acting on the hyperbolic plane $\mathbb{HP}$

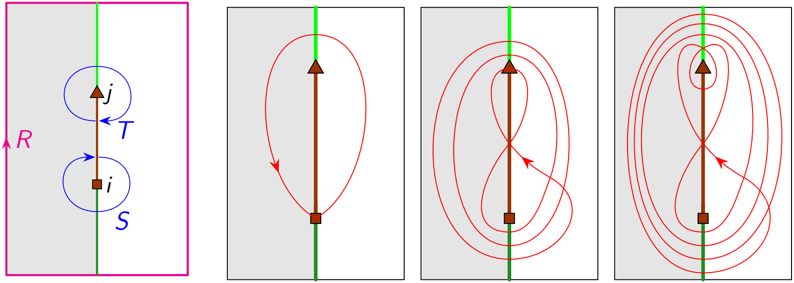
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$



Tiling $\mathbb{HP}$ under the action of the modular group $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$
by the (Farey) ideal triangulation together with its (Bass-Serre) dual tree.

Loops in the modular orbifold $\mathbb{M} = \mathsf{PSL}_2(\mathbb{Z})\backslash\mathbb{HP}$

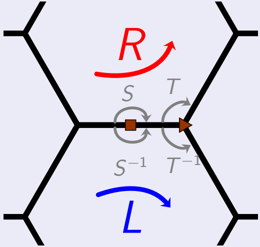
Free homotopy classes of oriented loops in $\mathbb{M}$	Conjugacy classes in $\pi_1(\mathbb{M}) = \mathsf{PSL}_2(\mathbb{Z})$
Around conic singularity i or j	Elliptic: S or $T^{\pm 1}$
Suround n times the cusp ∞	Parabolic: R^n, $n \in \mathbb{Z}$
$\exists!$ geodesic representative γ_A of length λ_A	Hyperbolic: $\lvert \mathrm{Tr}(A) \rvert = 2\cosh\left(\frac{1}{2}\lambda_A\right)$



Conjugacy classes and cyclic binary words

Euclidean monoid

The monoid $\mathrm{PSL}_2(\mathbb{N})$ is freely generated by $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$



Conjugacy class $[A]$ of an infinite order $A \in \mathrm{PSL}_2(\mathbb{Z})$:

$[A] \cap \mathrm{PSL}_2(\mathbb{N})$: cyclic permutations of an $L\&R$-word $\neq \varnothing$.

Class is primitive $\iff$ cyclic word is primitive.

Class is hyperbolic $\iff$ $\#L > 0$ and $\#R > 0$.

Continued fractions and geodesics in $\mathbb{M}$

Every $\gamma \in \mathbb{R}_{\geqslant 1}$ has a *unique* Euclidean continued fraction expansion

$$\lfloor c_0, c_1, \dots \rfloor = c_0 + \frac{1}{c_1 + \dots} = R^{c_0} L^{c_1} \cdots \cdot (\infty) \quad \text{with} \quad c_j \in \mathbb{N}_{\geqslant 1}.$$

The sequence (c_j) is finite if and only if γ is rational; otherwise, it is:

periodic $\iff$ γ is quadratic, and fixed by $R^{c_0} \dots L^{c_{2p-1}} \in \mathrm{PSL}_2(\mathbb{N})$.

bounded $\iff$ γ is quadratic or transcendental ? (Conjecture [Sha92, §4].)

The axis of $A = R^{a_0} \dots L^{a_{2n-1}} \subset R \cdot \mathrm{PSL}_2(\mathbb{N}) \cdot L$ is $(\alpha_-, \alpha_+) \subset \mathbb{HP}$ with

$$\alpha^+ = \lfloor \overline{a_0, \dots, a_{2n-1}} \rfloor \qquad -1/\alpha^- = \lfloor \overline{a_{2n-1}, \dots, a_0} \rfloor$$

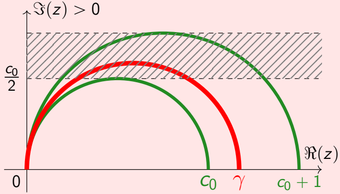
For all $-1/\alpha_- = \lfloor a_{-1}, a_{-2}, \dots \rfloor$ and $\alpha_+ = \lfloor a_0, a_1, a_2, \dots \rfloor$ in $\mathbb{R}_{>1}$, the geodesic $(\alpha_-, \alpha_+) \subset \mathbb{HP}$ intersects $\triangle$ according to the sequence a_n.

Lagrange spectrum and heights of geodesics

The size of c_n measures the depth of the n^{th} excursion to the cusp:

Horoball $\{\Im(z) > h\} \subset \mathbb{HP}$ at height $h \geqslant 1$ projects to $\mathbb{B}(h) \subset \mathbb{M}$ with area $1/h$.
The geodesic $(0, \gamma) \in \mathbb{HP}$ projected in $\mathbb{M}$ penetrates $\mathbb{B}(h)$ each time $n \in \mathbb{N}$ satisfies:

$$\lfloor 0, c_{n-1}, \ldots, c_0 \rfloor + \lfloor c_n, c_{n+1}, \ldots \rfloor \geqslant 2h.$$



The Lagrange constant $\mathscr{L}(\gamma)$ is the asymptotic height of $(0, \gamma)$:

$$\mathscr{L}(\gamma) = \limsup_n(\lfloor 0, c_{n-1}, \ldots, c_0 \rfloor + \lfloor c_n, c_{n+1}, \ldots \rfloor)$$
$$\mathscr{L}(\gamma) = \sup\{L \geqslant 0 \colon |\alpha - \tfrac{p}{q}| < \tfrac{1}{Lq^2} \text{ for infinitely many } p, q \in \mathbb{N}\}$$

What values can it take? Bounding size and pattern-complexity of (c_j)

$\mathscr{L}(\gamma) = 0 \iff \gamma \in \mathbb{Q}$

For $\alpha \notin \mathbb{Q}$ we have $\mathscr{L}(\gamma) \geqslant \limsup_n (\lfloor 0, 1, \ldots, 1 \rfloor + \lfloor 1, 1, \ldots \rfloor) = \sqrt{5}$

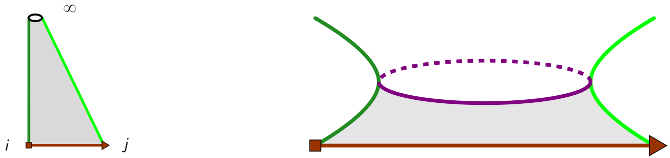
If $\mathscr{L}(\gamma) \leqslant 3$ then (c_j) is $\lesssim 2$ and has low complexity patterns...

Aparte: Deforming the representation $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R})$

Faithful discrete $\rho_q \colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R})$ mod $\mathrm{PSL}_2(\mathbb{R})$-conjugacy

or complete hyperbolic metrics on $\mathbb{M}$, are parametrized by $q \in \mathbb{R}^*$:

$$L_q = \begin{pmatrix} q & 0 \\ 1 & q^{-1} \end{pmatrix} \qquad R_q = \begin{pmatrix} q & 1 \\ 0 & q^{-1} \end{pmatrix}.$$



The orbifold $\mathbb{M} = \mathbb{M}_1$ and its deformation $\mathbb{M}_q$ with $q = (2\sinh\frac{\lambda}{2})^2$

The universal representation $\rho_q \colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}[q^{\pm 1}])$ is Burau

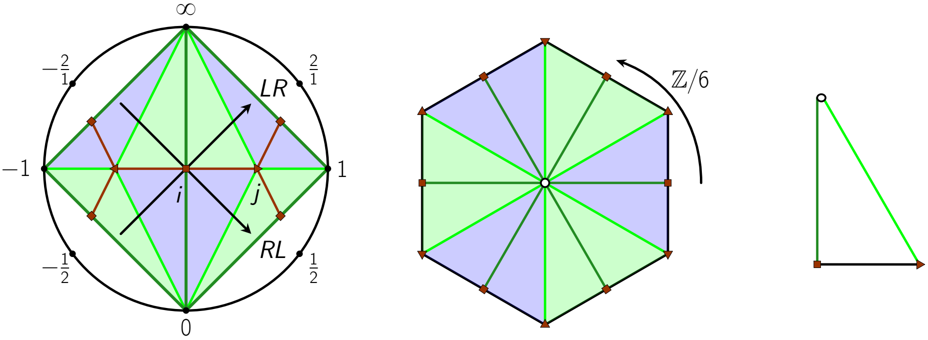
The fixed points of $A_q = \rho_q(A)$ are the q-deformed quadratic numbers $\alpha_q \in \mathbb{Z}(q, q^{-1})[\sqrt{\mathrm{disc}(A_q)}]$.

The modular torus and its fundamental group

The abelianisation $\mathbb{Z}/2 * \mathbb{Z}/3 \to \mathbb{Z}/2 \times \mathbb{Z}/3$ of the modular group corresponds the Galois cover of $\mathbb{M}$ by a punctured torus $\mathbb{M}'$.

A punctured torus with $\pi_1(\mathbb{M}') = \mathsf{PSL}_2(\mathbb{Z})' = \mathbb{Z}_X * \mathbb{Z}_Y$ where:

$$X = [T^{-1}, S] = LR = \left(\begin{smallmatrix}1 & 1\\ 1 & 2\end{smallmatrix}\right) \quad \text{and} \quad Y = [T, S^{-1}] = RL = \left(\begin{smallmatrix}2 & 1\\ 1 & 1\end{smallmatrix}\right)$$

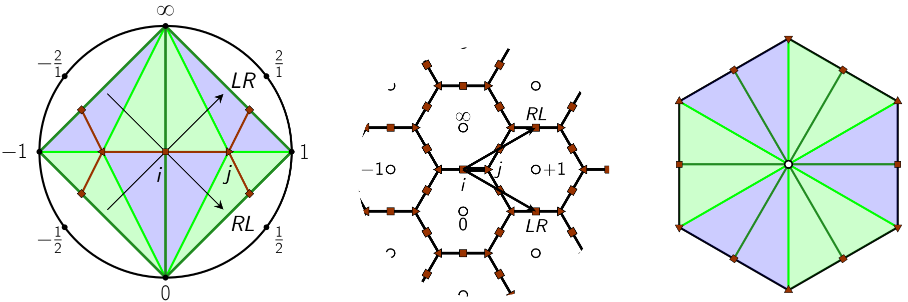


$$\mathbb{HP} \xrightarrow{\Gamma' \simeq F_2} \mathbb{M}' \xrightarrow{\Gamma/\Gamma' \simeq \mathbb{Z}/6} \mathbb{M}$$

The universal abelian cover of the modular torus

Hurwicz: $\pi_1(\mathbb{M}') \to H_1(\mathbb{M}';\mathbb{Z})$ = Abel: $\mathbb{Z}_X * \mathbb{Z}_Y \to \mathbb{Z}_X \oplus \mathbb{Z}_Y$ corresponds to a Galois cover of $\mathbb{M}'$ by a lattice-punctured-plane $\mathbb{M}''$.

The Jacobi integration map of $\mathbb{M}'$ based at the cusp $\infty \in \partial\mathbb{M}'$ yields an identification $\mathbb{M}'' \to H_1(\mathbb{M}';\mathbb{R})\backslash H_1(\mathbb{M}';\mathbb{Z})$

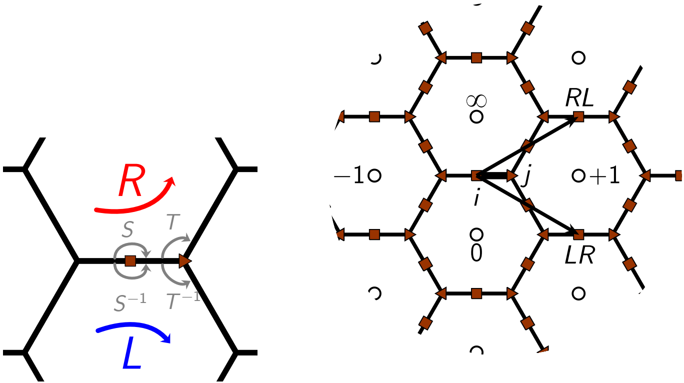


$$\mathbb{HP} \xrightarrow{\Gamma'' \simeq F_\infty} \mathbb{M}'' \xrightarrow{\Gamma''/\Gamma' \simeq \mathbb{Z}^2} \mathbb{M}'$$

The hexagonal group of isometries

The short exact sequence $\Gamma'/\Gamma'' \rightarrowtail \Gamma/\Gamma'' \twoheadrightarrow \Gamma/\Gamma'$ is split

The isometry group $\Gamma/\Gamma'' = (\mathbb{Z}_X \oplus \mathbb{Z}_Y) \rtimes ((\mathbb{Z}/2)_{D_S} \times (\mathbb{Z}/3)_{D_T})$ of $\mathbb{M}''$ acts like the crystallographic group of the hexagonal lattice.



Action of Γ/Γ'' on the hexagonal graph $\mathcal{H}$ and cusps $\Gamma''\backslash\Gamma/\langle R\rangle$.

Action of $\mathrm{Map}(\mathbb{M}')$ on loops and bases of $\pi_1(\mathbb{M}')$

Loops, simple loops and bases

Loops in $\mathbb{M}'$ = conjugacy classes of $\mathbb{Z}_X * \mathbb{Z}_Y$ = cyclic words on X, Y.

Simple loops in $\mathbb{M}'$ = primitive vectors in $H_1(\Gamma'; \mathbb{Z}) = \mathbb{Z}_X \oplus \mathbb{Z}_Y$.

Bases of $\Gamma' = \mathbb{Z}_X * \mathbb{Z}_Y$ correspond to bases of $H_1(\Gamma'; \mathbb{Z}) = \mathbb{Z}_X \oplus \mathbb{Z}_Y$, hence to pairs of simple loops in $\mathbb{M}'$ with one intersection point (their commutator in Γ' yields the loop circling once around the cusp).

The mapping class group $\mathrm{Map}(\mathbb{M}') = \mathrm{Out}(\mathbb{Z}_X * \mathbb{Z}_Y) = \mathrm{GL}(\mathbb{Z}_X \oplus \mathbb{Z}_Y)$

contains the positive mapping class group $\mathrm{Map}^+(\mathbb{M}') = \mathrm{SL}(\mathbb{Z}_X \oplus \mathbb{Z}_Y)$ (with index 2 and cokernel generated by $(X, Y) \mapsto (Y, X)$), which is generated by the positive Dehn twists along the simple loops X and Y:

$$D_X \colon (X, Y) \mapsto (X, YX) \qquad D_Y \colon (X, Y) \mapsto (XY, Y)$$

(whose relations are generated by the braid $D_Y D_X^{-1} D_Y = D_X^{-1} D_Y D_X^{-1}$).
The substitution monoid $\mathrm{SL}_2(\mathbb{N}_X \oplus \mathbb{N}_Y)$ freely generated by D_X, D_Y acts freely transitively on the set of oriented bases of the monoid $\mathbb{N}_X \oplus \mathbb{N}_Y$.

Action of $\mathrm{Map}(\mathbb{M}')$ on simple geodesics of $\mathbb{M}'$

Which cyclic words on $\{X, Y\}$ correspond to simple loops in $\mathbb{M}'$?

For a primitive vector $(p, q) \in \mathbb{Z}^2$, determine the cyclic word in $X \& Y$ associated to the unique simple geodesic of $\mathbb{M}'$ homological to $X^p Y^q$.

Act by D_S, D_J so that $p \geqslant q \geqslant 0$, expand $\frac{p}{q} = \lfloor c_0, \ldots, c_{2n+1} \rfloor$.

As $C = R^{c_0} \ldots L^{c_{2n+1}}$ sends $\frac{1}{0}, \frac{0}{1}$ to the last convergents $\frac{p}{q}, \frac{p'}{q'}$:
let $D_C = D_Y^{c_0} \ldots D_X^{c_k}$ act by substitution on the basis (Y, X)
to find the cyclic words of the simple loops in that basis.

Simple geodesics in $\mathbb{M}'$ from Sturmian sequences on $\{LR, RL\}^{\mathbb{Z}}$

The simple loops of $\mathbb{M}'$ are, up to $\mathbb{Z}/6$-rotations, the projections of axes $(\alpha_-, \alpha_+) \subset \mathbb{HP}$ with $-1/\alpha_- = \lfloor a_{-1}, \ldots \rfloor$ and $\alpha_+ = \lfloor a_0, \ldots \rfloor$ in $\mathbb{R}_{>1}$ such that (a_n) is a Sturmian sequence on $\{1, 2\}$.

Diophantine approximation of (simple) geodesics in $\mathbb{M}'$

The Markov-Cohn constant of geodesic $(\alpha_-, \alpha_+) \subset \mathbb{HP}$ mod $\mathrm{PSL}_2(\mathbb{Z})$

$\mathscr{C}(\alpha_-, \alpha_+)$ is the infimum of $2h \geqslant 2$ such that (α_-, α_+) intersects $\mathbb{B}(h)$.
When $-1/\alpha_- = \lfloor a_{-1}, a_{-2}, \dots \rfloor$ and $\alpha_+ = \lfloor a_0, a_1, a_2, \dots \rfloor$ we have:

$$\mathscr{C}(\alpha_-, \alpha_+) = \sup_n(\lfloor 0, a_{n-1}, a_{n-2}, \dots \rfloor + \lfloor a_n, a_{n+1}, \dots \rfloor).$$

Simple loops in $\mathbb{M}'$ (Haas [Haa87] building on Cohn [Coh71])

For distinct $\alpha_-, \alpha_+ \in \mathbb{RP}^1$ not both rational, the following are equivalent:

- The geodesic $(\alpha_-, \alpha_+) \subset \mathbb{M}'$ is simple (and closed).
- The sequence (a_j) is Sturmian on $\{1, 2\}$ (and periodic).
- The cusp height $\mathscr{C}(\alpha_-, \alpha_+)$ is $\in [\sqrt{5}, 3]$ (and < 3).

For such simple $(\alpha_-, \alpha_+) \subset \mathbb{M}'$ we have $\mathscr{C}(\alpha_-, \alpha_+) = 3\coth\left(\frac{1}{2}\ell_{\mathbb{M}'}(\alpha)\right)$:

- If $\mathscr{C} < 3$ then $\alpha_\pm$ are conjugate quadratic roots of Markov forms.
- If $\mathscr{C} = 3$ then $\alpha_\pm$ are transcendental ([ADQZ01]).

Geometry of the unicity conjecture for the Markov spectrum

The trace relation for all $A, B \in \mathrm{SL}_2$, denoting $C_\pm = AB^{\pm 1}$

⁜ $\mathrm{Tr}(A)\,\mathrm{Tr}(B) = \mathrm{Tr}(C_+) + \mathrm{Tr}(C_-)$ by Cayley-Hamilton

$\odot$ $\mathrm{Tr}[A, B] = \mathrm{Tr}(A)^2 + \mathrm{Tr}(B)^2 + \mathrm{Tr}(C_\pm)^2 - \mathrm{Tr}(A)\,\mathrm{Tr}(B)\,\mathrm{Tr}(C_\pm) - 2$

Holed tori = Fuchsian groups $\langle A, B\rangle$ with $\mathrm{Tr}[A, B] \leqslant -2$.

Bases of *punctured* hyperbolic tori yield solutions of Markov cubic

For a basis (α, β) of $\pi_1(\mathbb{M}')$ namely simple loops with $\mathrm{i}(\alpha, \beta) = 1$, the loop $\gamma = \alpha\beta = D_\beta(\alpha)$ yields bases (α, γ) & (γ, β) and $[\alpha, \beta] = \odot$.

The traces (a, b, c) of superbases (α, β, γ) satisfy $a^2 + b^2 + c^2 = abc$.

$D_S, D_T \in \mathrm{SL}_2(\mathbb{Z}_X \oplus \mathbb{Z}_Y)$ change (a, b, c) to $(b, a, ab - c), (c, b, a)$ and the orbit of $(3, 3, 3)$ yields *all* integral solutions to Markov cubic.

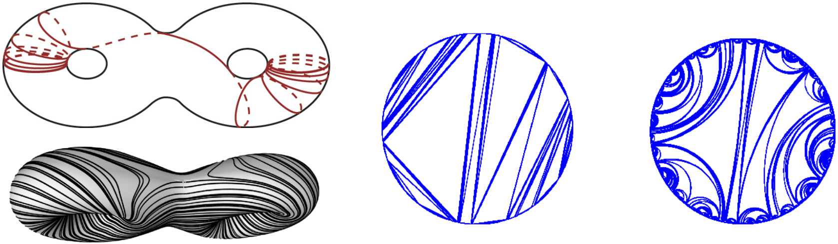
Markov conjectures unicity length/height spectrum simple loops $\mathbb{M}'$

Simplicity of the q-variable spectrum is known.

Generalizing to other arithmetic surfaces (ongoing work)

Conjecture: quadratic-transcendent dichotomy in arithmetic surfaces

In an arithmetic surface $S = \Gamma \backslash \mathbb{HP}$ consider a simple geodesic.
If it is not asymptotic to a cusp or to a closed geodesic
then any of its lifts in $\mathbb{HP}$ has ends in $\mathbb{RP}^1$ that are transcendent.



Strategy and philosophy of the proof

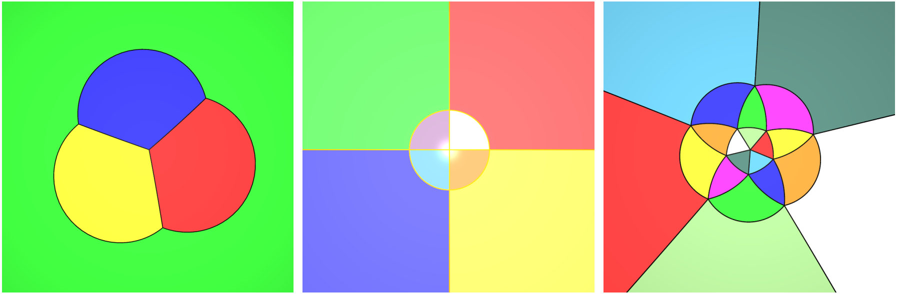
Topological simplicity leads (via $\mathrm{Map}(S)$ symmetries or dynamics) to
low symbolic complexity leads (via Schmidt subspace dichotomy) to
arithmetic rigidity (deg $\leqslant 2$ over invariant trace field or transcendent)

The example of congruence subgroups

Questions: the simple real quadratic numbers of level N

Which real quadratic numbers arise from end-lifts of simple closed geodesics in the congruence cover $\mathbb{M}_n$?

The congruence subgroups $\Gamma(n)$ for $n = 3, 4, 5$ yield the platonic covers $\mathbb{M}_n = \mathbb{CP}^1$ minus regular tetrahedron, octahedron, icosahedron and $\mathrm{Map}(\mathbb{M}_n, \partial\mathbb{M}_n) = \mathrm{Braid}_n(\mathbb{CP}^1)$ for $n = 3, 5, 11 = \dim H_1(\mathbb{M}_n, \mathbb{Z})$.



Stereographic projections of the platonic triangulations

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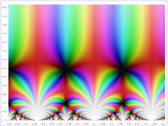
Values, limits and periods of the modular $\mathrm{j}\colon \mathbb{HP} \to \mathbb{C}$

Unique Γ-modular holomorphic function $\mathrm{j}\colon \mathbb{HP} \to \mathbb{C}$ uniformizing $\mathbb{M}$



$$\mathrm{j}(\infty) = \infty^1 \quad \mathrm{j}(j) = 0 \quad \mathrm{j}(i) = 12^3$$

The quadratic transcendence dichotomy (Schneider [Sch49])

For $\tau \in \mathbb{HP}$: $\mathrm{j}(\tau)$ is algebraic if and only if τ is complex quadratic.

Fundamental theorem in class field theory ∩ complex multiplication

For quadratic $\tau \in \mathbb{HP}$, the extension $\mathbb{Q}(\tau)(\mathrm{j}(\tau))/\mathbb{Q}(\tau)$ is unramified abelian, depending only on the order $\mathcal{O}$ of the lattice $\mathbb{Z}[\tau]$, with Galois group $\mathrm{Cl}(\mathcal{O})$. This describes all unramified abelian extensions of complex quadratic fields.

Kronecker Jugendtraum : abelian extensions of real quadratic fields ?

Study limits of j at $\gamma \in \mathbb{RP}^1$ and cycle integrals along $\alpha \subset \mathbb{M}$?

Periods of Dedekind $\eta^4(z)dz$ or the Abel Jacobi map on $\mathbb{M}'$

Primitive of the abelian differential and cusp compactification

Abelian du on $\mathbb{M}'$ lifts on $\mathbb{HP}$ to $C\eta^4(z)dz$ where $C = \frac{2^{10/3}}{3^{3/4}} \frac{\pi^{5/2}}{\Gamma(1/3)^3}$.

$\eta^4(6\tau) = q\prod_1^\infty (1-q^{6n})^4 = \sum \psi(n)\exp(i2\pi n\tau)$ is [LMF24, 32.2.a.a] the unique normalised cusp eigenform for the group $\Gamma_0(36)$.

The primitive $\operatorname{hexp}(\tau) = \int_\infty^\tau C\eta^4(z)dz = \frac{12C}{i\pi}\sum_1^\infty \frac{\psi(n)}{n}\cdot\exp\left(\frac{i\pi}{12}n\tau\right)$ yields $\operatorname{hexp}\colon \mathbb{HP} \to \mathbb{C}\backslash\Lambda$ uniformizing $H_1(\mathbb{M}';\mathbb{R})\backslash H_1(\mathbb{M}';\mathbb{Z}) = \mathbb{M}''$.

Cusp compactification $\partial\operatorname{hexp}\mathbb{QP}^1 \to \Lambda = H_1(\mathbb{M}';\mathbb{Z}) = \Gamma'/\Gamma''$.

Periods of the abelian differential

For coprime $a, c \in \mathbb{Z}$, there exists $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'$: let $A \equiv X^m Y^n \mod \Gamma''$.
The limit of the improper integral and conditionally convergent series:

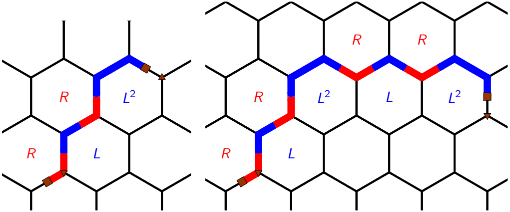
$$\partial\operatorname{hexp}(\tfrac{a}{c}) = \int_\infty^{\frac{a}{c}} C\eta^4(z)dz = \frac{12C}{i\pi}\sum_{n=1}^{\infty}\frac{\psi(n)}{n}\cdot\exp\left(\frac{i\pi}{12}\frac{a}{c}n\right)$$

is $\quad \partial\operatorname{hexp}(\frac{a}{c}) = |\omega_0|\left(m\exp(-\frac{i\pi}{6}) + n\exp(+\frac{i\pi}{6})\right) \quad$ where $\quad |\omega_0| = \frac{2\pi^{1/2}}{3^{1/4}}$.

The Radial compactification Shexp: $\mathscr{R} \to \mathbb{S}H_1(\mathbb{M}'; \mathbb{R})$

We define the radial compactification Shexp: $\mathscr{R} \to \mathbb{R}/2\pi\mathbb{Z}$

$\mathscr{R}$ = the set of $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ such that as $\tau \in \mathbb{HP}$ converges to $\alpha \in \partial\mathbb{HP}$, the argument arg hexp$(\tau) \in \mathbb{R}/(2\pi\mathbb{Z})$ converges to Shexp(α), namely the geodesic hexp$(i, \alpha) \subset \mathbb{M}''$, following the $L\&R$-cf-expansion of α, escapes in a definite direction which defines Shexp$(\alpha) \in \mathbb{S}H_1(\mathbb{M}'; \mathbb{R})$.



Shexp(α) recovers the slope of the parallel Sturmian sequences to α

Shexp: $\mathscr{R} \to \mathbb{R}/2\pi\mathbb{Z}$ restricts to continuous surjection $\mathscr{S} \to \mathbb{R}/2\pi\mathbb{Z}$

Continued fraction expansion for hexp(λ)... and Shexp(γ) ?

The modular function $\lambda \colon \mathbb{HP} \to \mathbb{C} \backslash \{0, 1, \infty\}$ uniformizes the congruence cover $\mathbb{M}(2)$ of $\mathbb{M}$ associated to the congruence subgroup $\Gamma(2)$ of Γ.
The cover $\mathbb{HP}/\Gamma(2) \to \mathbb{HP}/\Gamma$ has solvable Galois group $\mathfrak{S}_3$
so λ is an algebraic function of $\mathfrak{j}$, namely $\mathfrak{j} = \frac{27(1-\lambda+\lambda^2)^3}{4\lambda^2(1-\lambda)^2}$.

Gauss continued fraction for consecutive hypergeometric ratios yields

With [KZ03] we find $\int_\infty^\tau \eta^4(z)dz = \frac{3}{i2\pi}\left(\frac{1}{2}\lambda\right)^{1/3} \cdot {}_2F_1(1/3, 2/3, 4/3; \lambda(\tau))$ so:

$$\mathrm{hexp}(\tau) = \int_\infty^\tau C\eta^4(z)dz = \frac{3C}{i2\pi}\left(\frac{1}{2}\lambda(1-\lambda)\right)^{1/3} \cdot \cfrac{1}{1 - \cfrac{n_1\lambda}{1 - \cfrac{n_2\lambda}{1 - \dots}}}$$

where for $k \in \mathbb{N}$: $n_{2k+1} = \frac{(k+1/3)}{2(2k+1/3)}$ and for $k \in \mathbb{N}^*$: $n_{2k} = \frac{k}{2(2k+1/3)}$.

A continued fraction expansion for $\tan \circ \mathrm{Shexp} \colon \mathscr{R} \to \mathbb{RP}^1$?

Thank you for your attention and feel free to ask (m)any questions.

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