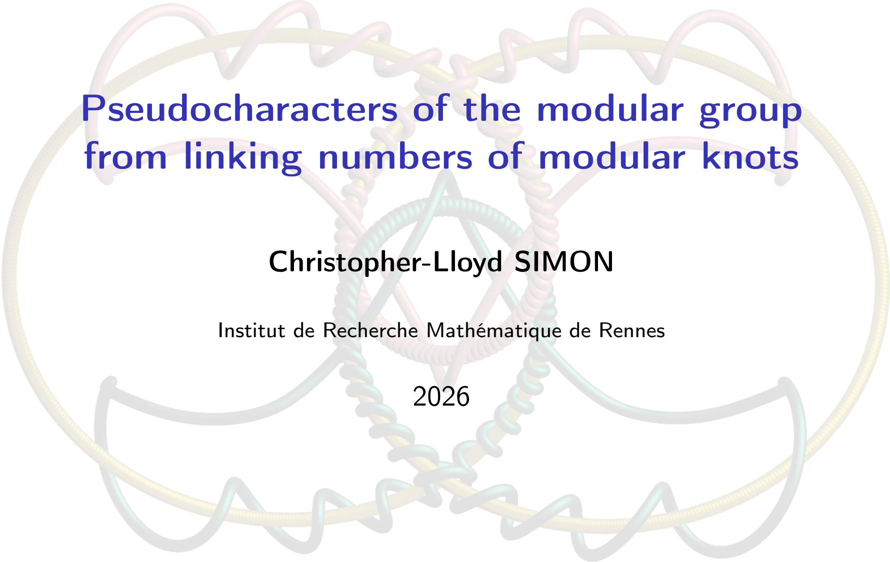

Pseudocharacters of the modular group from linking numbers of modular knots

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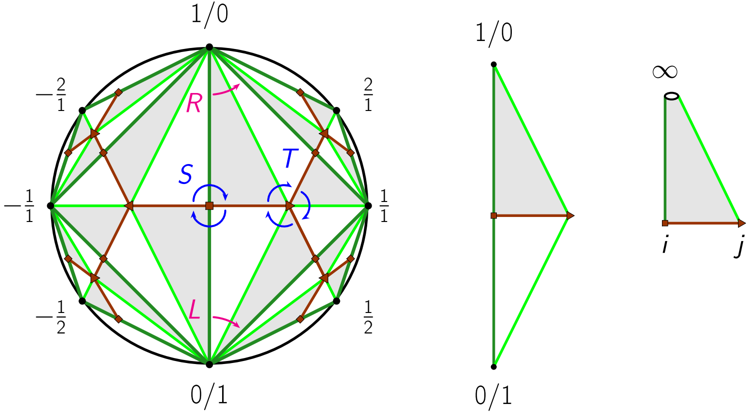
Modular group $\mathrm{PSL}_2(\mathbb{Z})$ acting on the hyperbolic plane $\mathbb{HP}$

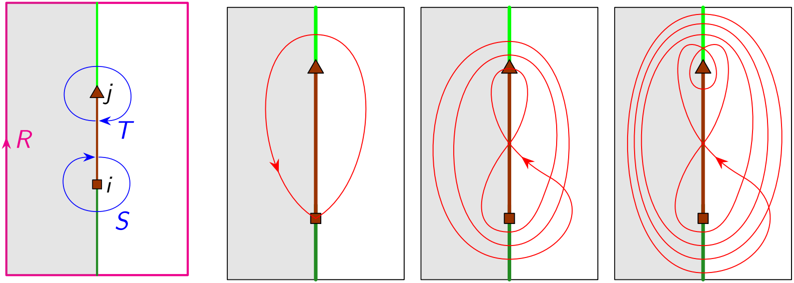
$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ $R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$



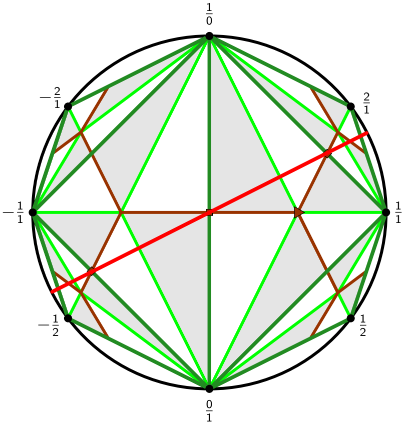
Tiling $\mathbb{HP}$ under the action of the modular group $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$, the basic fundamental tile, and the modular orbifold $\mathbb{M} = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{HP}$.

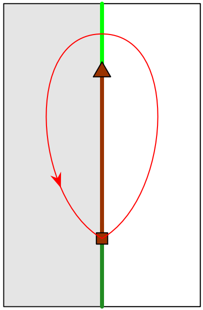
Loops in the modular orbifold $\mathbb{M} = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{HP}$

Free homotopy classes of oriented loops in $\mathbb{M}$	Conjugacy classes in $\pi_1(\mathbb{M}) = \mathrm{PSL}_2(\mathbb{Z})$
Around conic singularity i or j	Elliptic: S or $T^{\pm 1}$
Suround n times the cusp ∞	Parabolic: R^n, $n \in \mathbb{Z}$
$\exists!$ geodesic representative $[\alpha]$ of length λ_A	Hyperbolic: $\mathrm{disc}(A) = \left(2\sinh\frac{\lambda_A}{2}\right)^2$



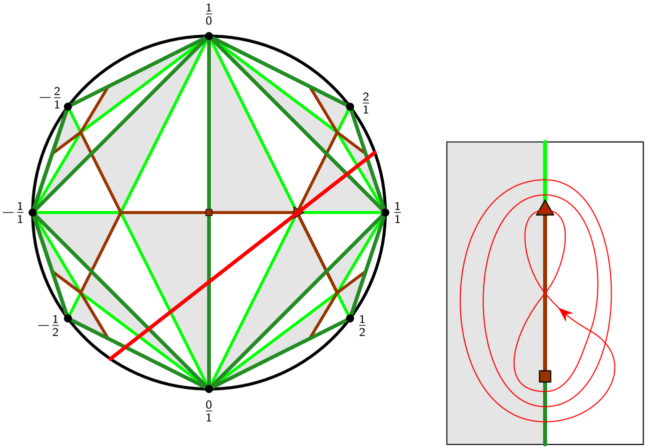
Modular geodesics as projections of hyperbolic axes





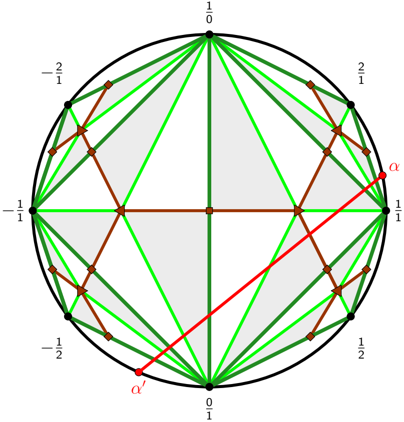
The axis of $A = RL$ in $\mathbb{HP}$ from $\alpha^- = -1/\lfloor\,\overline{1,1}\,\rfloor$ to $\alpha^+ = \lfloor\,\overline{1,1}\,\rfloor$ in $\mathbb{RP}^1$.

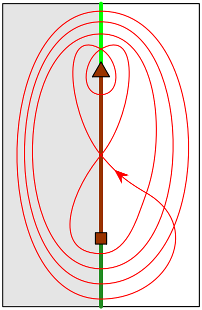
Modular geodesics as projections of hyperbolic axes



The axis of $A = RLL$ in $\mathbb{HP}$ from $\alpha^- = -1/\lfloor \overline{2,1} \rfloor$ to $\alpha^+ = \lfloor \overline{1,2} \rfloor$ in $\mathbb{RP}^1$.

Modular geodesics as projections of hyperbolic axes





The axis of $A = RLLL$ in $\mathbb{HP}$ from $\alpha^- = -1/\lfloor\,\overline{3,1}\,\rfloor$ to $\alpha^+ = \lfloor\,\overline{1,3}\,\rfloor$ in $\mathbb{RP}^1$.

Conjugacy classes and cyclic binary words

The Euclidean monoid $\mathrm{PSL}_2(\mathbb{N}) = \{L, R\}^*$

- The monoid $\mathrm{PSL}_2(\mathbb{N})$ is freely generated by $L = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.
- For $A \in \mathrm{PSL}_2(\mathbb{Z})$ we have $A \in \mathrm{PSL}_2(\mathbb{N}) \iff \alpha^- < 0 < \alpha^+$.
- For $A \in \mathrm{PSL}_2(\mathbb{N})$ its $\{L, R\}$-factorisation has exponents yielding the convergents in the continued fraction expansion of α^+.

The conjugacy class $[A]$ of an infinite order $A \in \mathrm{PSL}_2(\mathbb{Z})$ satisfies:

- $[A] \cap \mathrm{PSL}_2(\mathbb{N}) =$ cyclic permutations of an $L\&R$-word $\neq \emptyset$.
- Class is primitive $\iff$ cyclic word is primitive.
- Class is hyperbolic $\iff$ $\#L > 0$ and $\#R > 0$.

- denote its *combinatorial length* by $\mathrm{len}(A) = \#R + \#L$
- define its *Rademacher cocycle* as $\mathrm{Rad}(A) = \#R - \#L$

The Rademacher invariant in arithmetic, topology, physics...

The Rademacher invariant function Rad: $\mathrm{PSL}_2(\mathbb{Z}) \to \mathbb{Z}$ is the

3 Cocycle for the Eisenstein series E_2 (logarithm of Dedekind function η)

2 Signature defect of the torus bundle with monodromy A

5 Special value at 0 of the Shimizu L-function for $L_A(s) = \sum \frac{\operatorname{sign} Q_A(m,n)}{|Q_A(m,n)|^s}$

Logarithm of the Dedekind η-Funktion 371

It may now be convenient to summarize all our results in an omnibus theorem:

(5.60) **Theorem.** *Let $A \in \mathrm{SL}(2, Z)$ be hyperbolic. Then the following invariants of A coincide.*

1) *Meyer's signature invariant $\phi(A)$ (see (5.3)).*

2) *Hirzebruch's signature defect $\delta(A)$.*

3) *The invariant $\chi(A)$ describing the transformation properties of $\log \boldsymbol{\eta}(\tau)$ under A (see (5.22)).*

4) *$\mu(A)$ the logarithmic monodromy (divided by πi) of Quillen's determinant line-bundle $\mathscr{L}$.*

5) *The value $L_A(0)$ of the Shimizu L-function (see (5.49)).*

6) *The Atiyah-Patodi-Singer invariant $\eta(A)$.*

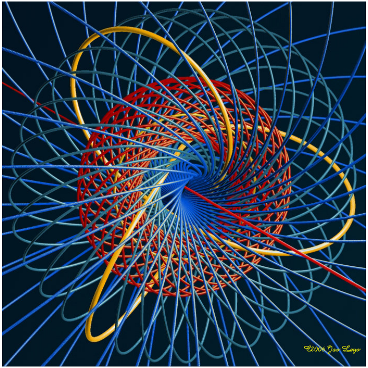
7) *The "adiabatic limit" $\eta^0(A)$.*

Atiyah's omnibus theorem [Ati87]

Unit tangent bundle $\mathbb{U}$ of the modular orbifold $\mathbb{M}$

$$\begin{array}{ccc} \mathsf{PSL}_2(\mathbb{R}) & \xrightarrow[\mathsf{PSL}_2(\mathbb{Z})]{} & \mathbb{U} \\ \Big\downarrow{\scriptstyle \mathbb{S}^1} & & \Big\downarrow{\scriptstyle \mathbb{S}^1} \\ \mathbb{HP} & \xrightarrow[\mathsf{PSL}_2(\mathbb{Z})]{} & \mathbb{M} \end{array}$$

The Seifert fibration $\mathbb{U} \to \mathbb{M}$ reveals that $\mathbb{U} \simeq \mathbb{S}^3 \setminus$ trefoil.

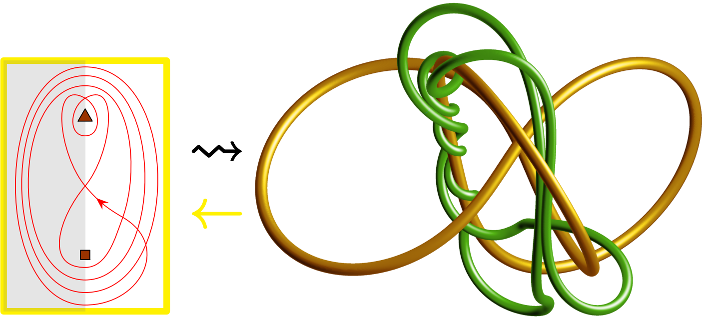


The extension of fundamental group $\mathbb{S}^1 \rightarrowtail \pi_1(\mathbb{U}) \twoheadrightarrow \pi_1(\mathbb{M})$

yields the universal central extension $\mathbb{Z} \rightarrowtail \mathcal{B}_3 \twoheadrightarrow \mathsf{PSL}_2(\mathbb{Z})$ which is the quotient of the braid group $\mathcal{B}_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ by its center generated by $(\sigma_1\sigma_2)^6$ sending $\sigma_1^{-1} \mapsto L$ and $\sigma_2 \mapsto R$.

Primitive modular geodesics in $\mathbb{M}$ lift to modular knots in $\mathbb{U}$

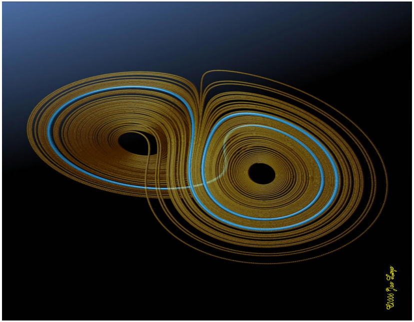
Hyperbolic classes in $\mathrm{PSL}_2(\mathbb{Z})$	Modular geodesics in $\mathbb{M}$	Periodic orbits in $\mathbb{U}$
trace form	intersection form	linking form

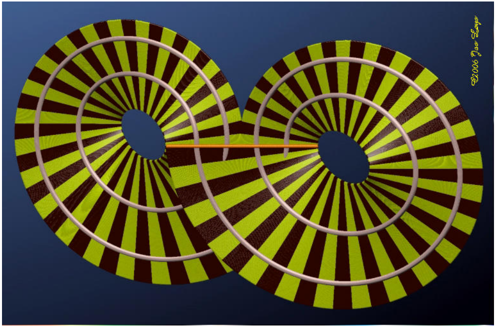


The primitive modular geodesic $[\alpha]$ lifts to the modular knot $\vec{\alpha}$

Master modular link isotopes into the Lorenz template

The master modular link isotopes into the Lorenz template so that the periodic orbit associated to $A \in \mathrm{PSL}_2(\mathbb{N})$ follows the sequence follows that sequence of left and right turns according to its $\{L, R\}$-factorisation.





This is shown in [Ghy07] by opening the cusp of $\mathbb{M}$, deforming the Fuchsian representation $\rho_q \colon \pi_1(\mathbb{M}) \to \mathrm{PSL}_2(\mathbb{R})$ to the boundary, so that the unit tangent bundle of $\mathbb{M}_q$ isotopes to that of the Lorenz template.

Linking the trefoil recovers the Rademacher cocycle

Theorems [BG92, Ghy07] :

For hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$ we have $\mathrm{Rad}(A) = \mathrm{lk}(\text{trefoil}, k_A)$.

The function $\mathrm{Rad} \colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathbb{Z}$ is the pseudocharacter associated to the central extension $\mathbb{Z} \rightarrowtail \mathcal{B}_3 \twoheadrightarrow \mathrm{PSL}_2(\mathbb{Z})$.

$$\begin{array}{ccccc}
0 & \longrightarrow & \pi_1(\mathbb{S}^1) = \mathbb{Z} & \longrightarrow & H_1(\mathbb{S}^1) = \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(\mathbb{M}') = \mathcal{F}_2 & \longrightarrow & \pi_1(\mathbb{U}) = \mathcal{B}_3 & \xrightarrow{\mathrm{lk}} & H_1(\mathbb{U}) = \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(\mathbb{M}') = \mathcal{F}_2 & \longrightarrow & \pi_1(\mathbb{M}) = \mathrm{PSL}_2(\mathbb{Z}) & \xrightarrow{\mathrm{Ab}} & H_1(\mathbb{M}) = \mathbb{Z}/6
\end{array}$$

The Euler class in $H^2(\mathrm{PSL}_2(\mathbb{Z}); \mathbb{Z})$ classifying the central extension of the middle column is the pull back by the abelianisation $\mathrm{PSL}_2(\mathbb{Z}) \to \mathbb{Z}/6$ of the Euler class $H^2(\mathbb{Z}/6; \mathbb{Z})$ classifying the central extension of the right column.

Linking the trefoil recovers the Rademacher cocycle

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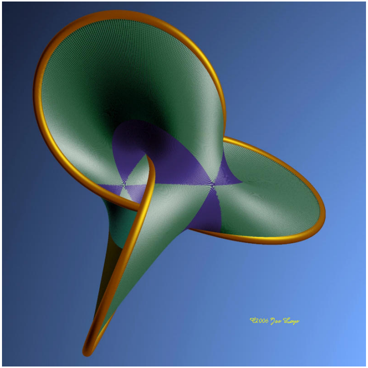
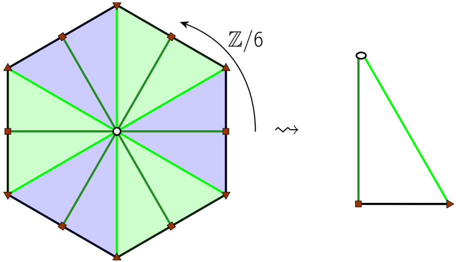
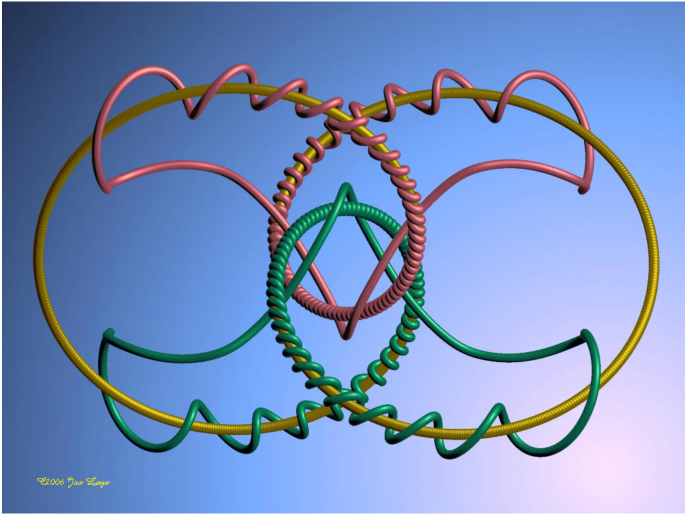




Pages of the open book decomposition ; the universal abelian cover $\mathbb{M}' \to \mathbb{M}$.

What about the linking of modular knots ?



Another question would be to give an arithmetical or combinatorial computation of the linking numbers of two knots k_A and k_B as a function of A, B in $\mathrm{PSL}(2, \mathbb{Z})$

Picture from [GL16] and Question from [Ghy07].

Pseudocharacters of a group Γ and their Banach quotient

Definition: Pseudocharacters are homogeneous quasicharacters

A function $\phi\colon \Gamma \to \mathbb{R}$ is a quasicharacter when it has a bounded derivative:

$$d\phi\colon \Gamma \times \Gamma \to \mathbb{R} \qquad d\phi(A, B) = \phi(B) - \phi(AB) + \phi(A).$$

It is homogeneous when $\phi(A^n) = n\phi(A)$ for all $A \in \Gamma$ and $n \in \mathbb{Z}$.

Theorem ([MM85, Iva88]) : a Banach space

The $\mathbb{R}$-vector space $\mathrm{PX}(\Gamma)$ has weak* topology (pointwize convergence).

- Semi-norm $\|d\phi\|_\infty$ with $\ker \|d\phi\|_\infty = H^1(\Gamma; \mathbb{R}) \subset \mathrm{PX}(\Gamma)$
- The quotient is $H^2_{b,0}(\Gamma) = \mathrm{PX}(\Gamma)/H^1(\Gamma; \mathbb{R})$... a Banach space!

Rotation numbers: from commutative to non-commutative

Characters $H^1(\Gamma; \mathbb{Z}) = \mathrm{Hom}(\Gamma; \mathbb{S}^1)$.
Pseudocharacters $H^2_b(\Gamma; \mathbb{Z}) \supset \mathrm{Hom}(\Gamma; \mathrm{Homeo}(\mathbb{S}^1))$ mod semi-conj [Ghy87].

Brooks cocycles for groups acting on trees

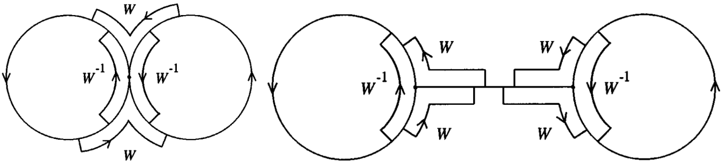
Brooks-Series cocycles ([BS84, BG91]) and [Gri95, Lemma 5.3]

For a *non-overlapping* pattern $P \in \mathrm{PSL}_2(\mathbb{N})$, the measure of P-asymmetry is the conjugacy invariant function $\mathrm{mas}_P \colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathbb{Z}$ defined:

- $A \in \mathrm{PSL}_2(\mathbb{N}) \mapsto \mathrm{mas}_P(A) = \mathrm{occ}_P(A) - \mathrm{occ}_P(A^\star)$

  (In particular for $P = R$ we have $\mathrm{Rad}(A) = \mathrm{mas}_R(A)$.)
- $A \in \mathrm{Tors}(\mathrm{PSL}_2(\mathbb{Z})) \mapsto \mathrm{mas}_P(A) = 0$.

We have $\mathrm{mas}_P \in \mathrm{PX}(\mathrm{PSL}_2(\mathbb{Z}))$ and $\|d\,\mathrm{mas}_P\|_\infty \leq 6$.
If $P \neq P^\star$ then $d\,\mathrm{mas}_P$ yields a non-zero class in $H^2_{b,0}(\mathrm{PSL}_2(\mathbb{Z}))$.



From [Gri95]: some cases in the proof of $\|d\,\mathrm{mas}_P\|_\infty \leq 6$.

From computing the dimension to describing a basis

The dimension of $H^2_b(\Gamma)$ is roughly known for many groups:

1. If Γ is amenable then $H^*_{b,0}(\Gamma;\mathbb{R}) = 0$
2. If Γ is word hyperbolic then $\dim H^2_b(\Gamma) = \infty$ [EF97]
3. If $\Gamma = \mathrm{Mod}(\mathbb{F})$ then $\dim H^2_b(\Gamma) = \infty$ [BF02, Fuj09].

free product of two copies of an infinite cyclic group; that is a free group with two generators. However in [9] there is no discussion about the generation of $H_b^{(2)}(\mathbb{Z} * \mathbb{Z})$ but only the statement that the dimension of $H_b^{(2)}(\mathbb{Z} * \mathbb{Z})$ is infinite. Let us analyze this statement.

From [Gri95]: what about generating PX(Γ)?

Assertion 5.1 *The bounded 2-cocycles $\delta^{(1)} f_W$ are all non-trivial cocycles, and are all linearly independent.*

But the last statement of the assertion is not correct as the following ex-

From [Gri95]: beware of independence!

A Schauder basis for $H_b^2(\Gamma)$

Definition Schauder basis:

For a topological $\mathbb{R}$-vector space V, a sequence $(x_n)_{n\in\mathbb{N}}$ is a *basis* when

$$\forall v \in V, \quad \exists!\,(c_n(v)) \in \mathbb{R}^{\mathbb{N}}, \quad v = \sum c_n(v)\cdot x_n,$$

and a *Schauder basis* when the coefficient functionals c_n are continuous.

$$e_W(g) = \text{number of times } W \text{ occurs in } \overline{g} - \text{number of times } W^- \text{ occurs in } \overline{g},$$

where $\overline{g}$ is the reduced cyclic word corresponding to the element $g \in N$.
Repeating the arguments used when proving Theorem 5.7 we get

Theorem 5.11 *The space $H_{b,2}^{(2)}(N)$ is infinite dimensional. It is isomorphic to the space $PX(N)$ and every element $f \in PX(N)$ can be uniquely written in the form*

$$f = \sum_{W\in P^+} \alpha_W e_W.$$

for some coefficients $\alpha_W \in \mathbb{R}$.

The Schauder base of $\mathrm{PX}(\mathrm{PSL}_2(\mathbb{Z}))$ of [Gri95, Theorem 5.11]

Schauder base: $\mathrm{mas}_P \in \mathrm{PX}(\mathrm{PSL}_2(\mathbb{Z}))$ for non-overlapping $P \in \mathrm{PSL}_2(\mathbb{N})/\star$.

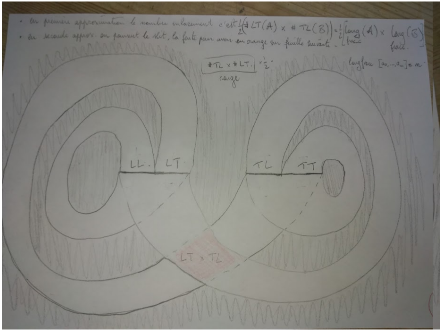
Linking number: sum crossings indexed by linked patterns

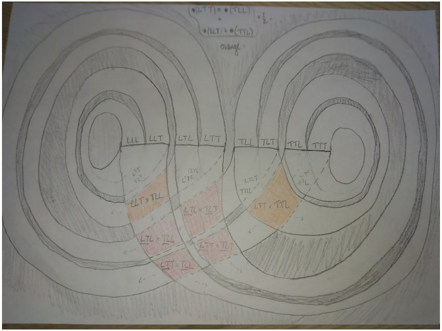
Lemma [Sim22, 4.34] & [Sim25, 7.1]: linking counts sum of crossings

For hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{N})$ the corresponding modular knots have:

$$\mathrm{lk}(A,B) = \frac{1}{2}\sum_{w} \begin{pmatrix} \mathrm{occ}_{RwL}(A)\cdot \mathrm{occ}_{LwR}(B) \\ + \\ \mathrm{occ}_{LwR}(A)\cdot \mathrm{occ}_{RwL}(B) \end{pmatrix}$$

where the summation extends over all words $w \in \mathrm{PSL}_2(\mathbb{N})$.





Pseudocharacters from asymetric linking numbers

Definition [Sim25, 7.6]: The cosign functions

For $A \in \mathrm{PSL}_2(\mathbb{Z})$, define the cosign function $\mathrm{C}_A \colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathbb{Z}$ as the antisymmetric linking number: $\mathrm{C}_A(B) := \frac{1}{2}\left(\mathrm{lk}(A, B) - \mathrm{lk}(A^{-1}, B)\right)$

Proposition [Sim25, 7.6]: decompose C_A in the Schauder basis (mas_P)

For $A, B \in \mathrm{PSL}_2(\mathbb{N})$, we may decompose C_A as:

$$\mathrm{C}_A(B) = \mathrm{lk}(A, B) - \mathrm{lk}(A, B^\star) = \frac{1}{2}\sum_w \begin{pmatrix} \mathrm{occ}_{RwL}(A) \cdot \mathrm{mas}_{LwR}(B) \\ + \\ \mathrm{occ}_{LwR}(A) \cdot \mathrm{mas}_{RwL}(B) \end{pmatrix}$$

the sum over all words $w \in \mathrm{PSL}_2(\mathbb{N})$ with $\mathrm{len}(w) < \max\{\mathrm{len}\, A, \mathrm{len}\, B\}$.

Theorem [Sim25, 7.6, 7.7]: Pseudochracters from linking numbers

- For all $A \in \mathrm{PSL}_2(\mathbb{N})$ we have $\mathrm{C}_A \in \mathrm{PX}(\mathrm{PSL}_2(\mathbb{Z}))$ and $\|d\,\mathrm{C}_A\|_\infty \leq 6$.
- Linear relations spanned by $\mathrm{C}_{A^n} = n\,\mathrm{C}_A$ for $A \in \mathrm{PSL}_2(\mathbb{Z})$ and $n \in \mathbb{Z}$. (In particular C_A is trivial if and only if A is conjugate to A^{-1}.)

Link equivalence

To prove the non-triviality and linear independance of the C_A for $A \in \mathcal{P}_+$, we were led to show the non-degeneracy of the linking form.

Lemma [Sim25] : The linking pairing is non-degenerate

If hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ are link equivalent, then they are conjugate. (Link equivalence: $\forall X \in \mathrm{PSL}_2(\mathbb{Z})$, $\mathrm{lk}(A, X) = \mathrm{lk}(B, X)$.)

Linking recovers intersection

For hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$ we have $\frac{1}{2} I(A, B) = \mathrm{lk}(A, B) + \mathrm{lk}(A^{-1}, B)$.

Intersection from modular cocycles

For hyperbolic $A \in \mathrm{PSL}_2(\mathbb{Z})$, [DIT17] construct a modular function whose symbol recovers $I(A, \cdot) = 2(\mathrm{lk}(A, \cdot) + \mathrm{lk}(A^{-1}, \cdot))$.
What about the cosign $\mathrm{C}_A(\cdot) = \mathrm{lk}(A, \cdot) - \mathrm{lk}(A^{-1}, \cdot)$?

Schauder basis of $PX(PSL_2(\mathbb{Z}))$ from linking numbers

Definition: conjugacy classes modulo powers

The set $\mathcal{P}$ of primitive infinite order conjugacy classes in $PSL_2(\mathbb{Z})$ contains the subset $\mathcal{P}_0$ of those which are stable under inversion.

Partition $\mathcal{P} \setminus \mathcal{P}_0 = \mathcal{P}_- \sqcup \mathcal{P}_+$ in two subsets in bijection by inversion.

We may choose $R \in \mathcal{P}_+$, and note that $C_R = \mathrm{lk}(\mathit{trefoil}, \cdot)$.

Theorem [Sim25]: Schauder basis for $PX(PSL_2(\mathbb{Z}))$

The collection of $C_A \in PX(\Gamma; \mathbb{R})$ for $A \in \mathcal{P}_+$ is a Schauder basis:

- $\forall f \in PX(PSL_2(\mathbb{Z}); \mathbb{R}), \quad \exists!\, (c_A(f))_A \in \mathbb{R}^{\mathcal{P}_+}, \quad f = \sum_A c_A(f) \cdot C_A$
- The period coefficients $c_A \colon f \mapsto c_A(f)$ are continuous

$\Longrightarrow$ Fourier theory of pseudocharacters, with a natural basis of *cosigns*.

Does C_A yield an extremizer in the duality [Bav91, Theorem] ?

- For $A \in \Gamma'$: $\mathrm{scl}(A) = \sup\{|\varphi(A)|/\|d\varphi\|_\infty \colon \varphi \in PX(\Gamma) \setminus H^1(\Gamma)\}$
- The extremal φ obtained by Hanh-Banach is not constructed... C_A?

Sketch of a project to find Schauder bases of PX(Mod($\mathbb{F}$))

Problem 4.10 To construct a basis for $H_b^{(2)}(B_n)$.

Mapping class group Mod($\mathbb{F}$) acts on curve complex $\mathcal{C}(\mathbb{F})$:

- Metric space $\mathcal{C}(\mathbb{F})$ is hyperbolic and $\partial\mathcal{C}(\mathbb{F})$ are ending laminations
- The hyperbolic isometries of $\mathcal{C}(\mathbb{F})$ are pseudo-Anosov elements A
- Fixed points of A in $\partial\mathcal{C}(\mathbb{F})$ are stable-unstable measured laminations.

Schauder bases for PX(Mod($\mathbb{F}$)) from *combinatorics of monoids*:

- Define Brooks–Series Barge–Ghys cocycles mas_P in $H_b^2(\mathrm{Mod}(\mathbb{F}); \mathbb{R})$.
- Find basis of mas_P for P in "train track monoid" of Dehn-twists.

Schauder basis for PX(Mod($\mathbb{F}$)) from *asymetric linking periods*:

- Define $\mathrm{lk}(A, B)$ as sum cross-ratio of fixed points over double cosets.
- Show basis of C_A for classes of pseudo-Anosov A modulo powers.
- Show C_A are extremizers in Bavard duality.

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Moral of the story...

So many mysteries are concealed within a simple trefoil !

Région
Hauts-de-France

Thank you for your attention and feel free to ask unbounded questions.

Character variety $X(\mathrm{PSL}_2(\mathbb{Z}), \mathrm{PSL}_2(\mathbb{R}))$

Caracters of Fuchsian representations :

$$\left\{\begin{matrix}\text{Complete hyperbolic} \\ \text{metrics on } \mathbb{M}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\rho\colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{R}) \\ \rho \text{ faithful \& discrete}\end{matrix}\right\} / \mathrm{PSL}_2(\mathbb{R})$$

- Real algebraic torus of dim 1, parametrized by $q \in \mathbb{R}^*$.
- The matrix $A_q = \rho_q(A)$ is obtained from a factorisation of A into a product of $L\&R$ by replacing $L \rightsquigarrow L_q$ and $R \rightsquigarrow R_q$ where

$$L_q = \begin{pmatrix} q & 0 \\ 1 & q^{-1} \end{pmatrix} \qquad R_q = \begin{pmatrix} q & 1 \\ 0 & q^{-1} \end{pmatrix}.$$

$$\rho_q\colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}[q, q^{-1}])$$

Conjugacy classes of infinite order in $\pi_1(\mathbb{M}_q) = \mathrm{PSL}_2(\mathbb{Z})$	Closed geodesics in $\mathbb{M}_q = \rho_q(\mathrm{PSL}_2(\mathbb{Z}))\backslash\mathbb{HP}$

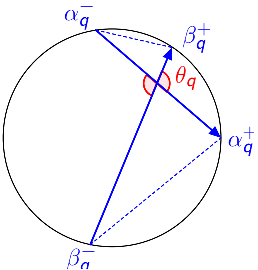
The bivariate Poincaré q-series $\mathsf{L}_q(A, B)$

Definition : "the Alexander-Poincaré q-pairing"

For hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$, we define the sum :

$$\mathsf{L}_q([A],[B]) := \sum \left(\cos \tfrac{1}{2}\theta_q\right)^2 \quad \in \sqrt{\mathbb{Q}(q)}$$

over the intersection angles θ_q of the q-modular geodesics $[\alpha_q], [\beta_q] \subset \mathbb{M}_q$.
This defines a function of $q \in \mathbb{R}^*$, or on $X(\mathrm{PSL}_2(\mathbb{Z}), \mathrm{PSL}_2(\mathbb{R}))$.



Intersection angle $\theta_q \in]-\pi, \pi[$ has $\cos(\theta_q) = [\alpha_q^-, \alpha_q^+; \beta_q^-, \beta_q^+]$.

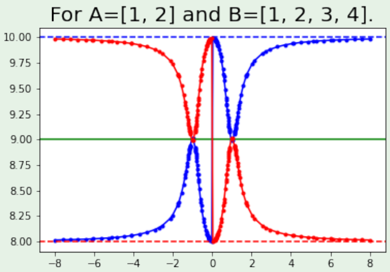
Linking function at the boundary of the character variety

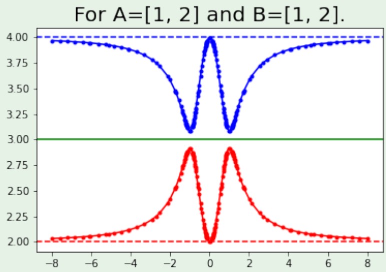
Theorem [Sim25] : Linking number as value of L_q at $+\infty \in \partial X$

For hyperbolic $A, B \in \mathrm{PSL}_2(\mathbb{Z})$, we have the special value :

$$\mathrm{L}_q([A],[B]) \xrightarrow[q \to +\infty]{} 2\,\mathrm{lk}(k_A, k_B).$$

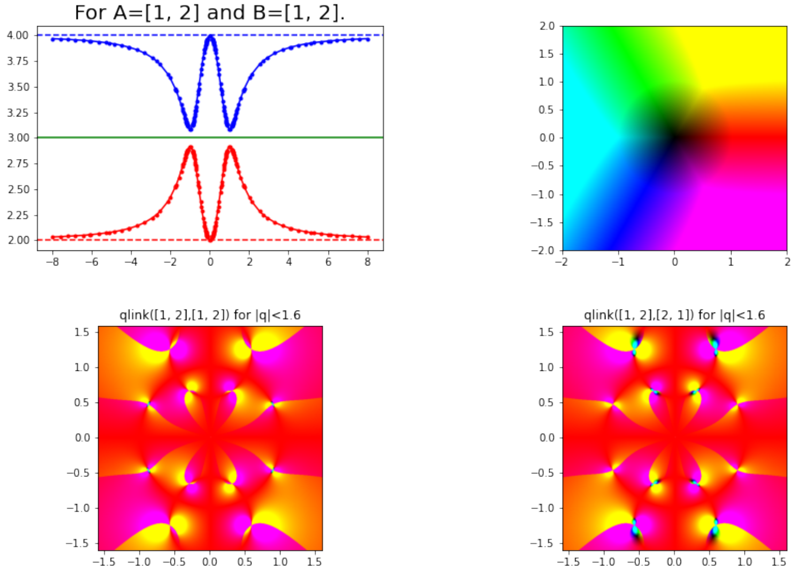
The linking function $q \mapsto \mathrm{L}_q([A],[B])$ interpolates between the arithmetic-geometry at $q = 1$ and the geometric-topology at $q = \infty$.





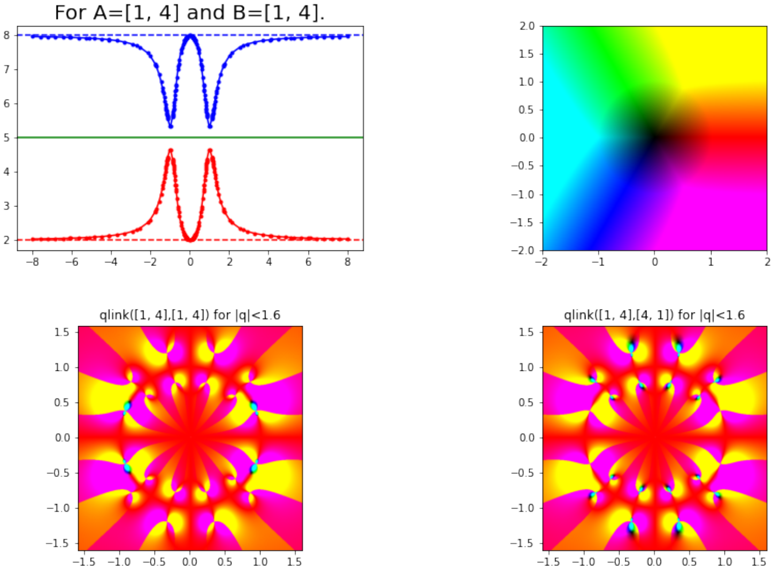
The linking functions $q \mapsto \mathrm{L}_q(A, B)$ and $q \mapsto \mathrm{L}_q(A^{-1}, B)$ have average $I(A, B)$.

Graphs of $q \mapsto \mathsf{L}_q(A, B)$ for real and complex q



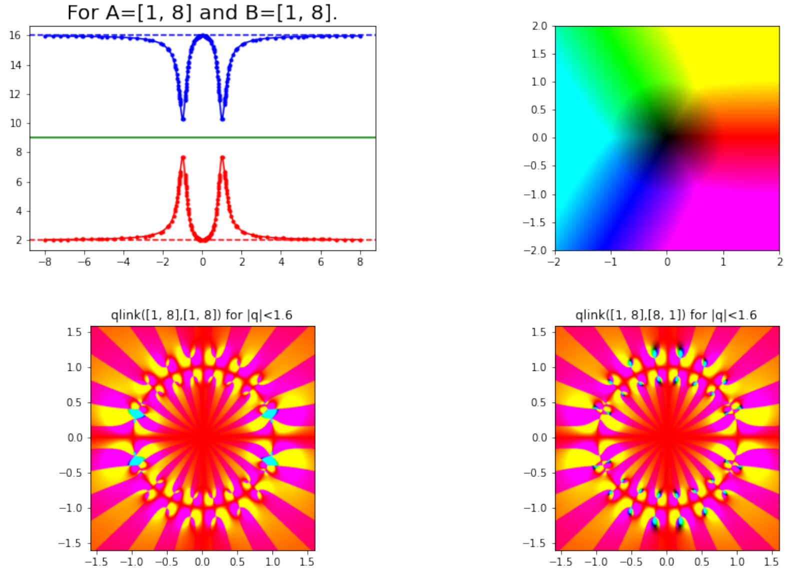
$\mathsf{L}_q(A, B)$ and $\mathsf{L}_q(A, B^\star)$ for $A = B = RLL$ and $B^\star = RRL$.

Graphs of $q \mapsto \mathrm{L}_q(A, B)$ for real and complex q



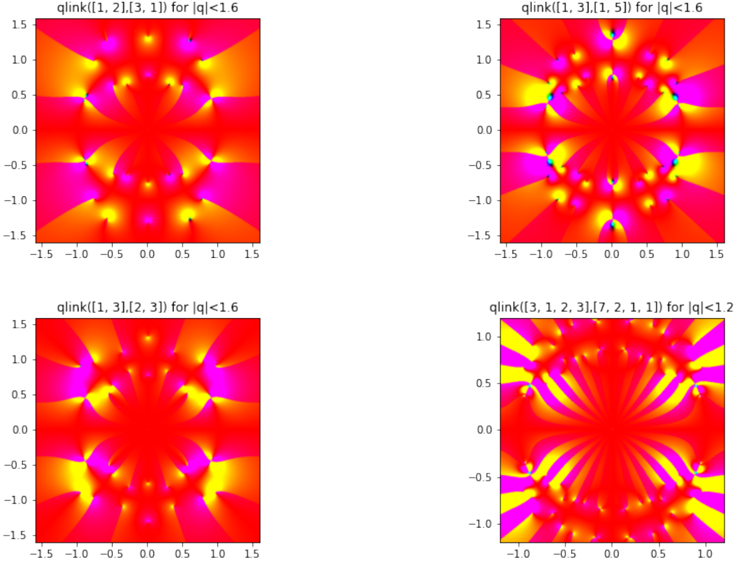
$\mathrm{L}_q(A, B)$ and $\mathrm{L}_q(A, B^\star)$ for $A = B = RL^4$ and $B^\star = R^4 L$.

Graphs of $q \mapsto \mathrm{L}_q(A, B)$ for real and complex q



$\mathrm{L}_q(A, B)$ and $\mathrm{L}_q(A, B^\star)$ for $A = B = RL^8$ and $B^\star = R^8 L$.

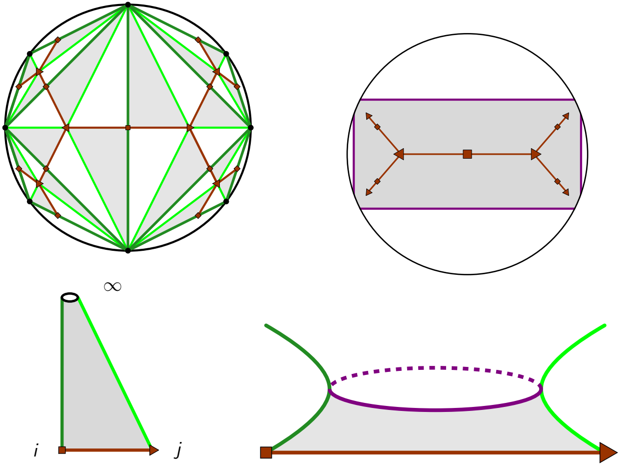
More graphs of $q \mapsto \mathrm{L}_q(A, B)$ for complex q



$\mathrm{L}_q(A, B)$ for various cycles A and B.

Proof using the action of $\mathrm{PSL}_2(\mathbb{Z})$ on the trivalent tree $\mathcal{T}$

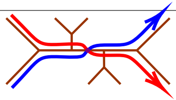
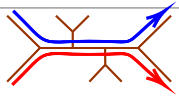
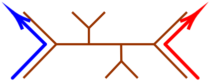
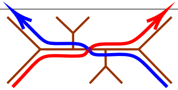
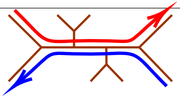
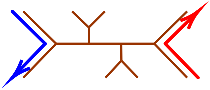
1. Lift the convex core of $\mathbb{M}_q$ in $\mathbb{HP}$: $\frac{1}{q^2}$-neighbourhood of $\mathcal{T}_q$.



2. The representation ρ_q tends to the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{T}$.

Proof using the action of $\mathrm{PSL}_2(\mathbb{Z})$ on the trivalent tree $\mathcal{T}$

3. The angles $\theta_q \to 0 \bmod \pi$ thus $\cos(\theta_q) \to \pm 1$.
4. The sum $\mathrm{L}_q(A, B)$ counts the pairs of axes $(+1, +1)$:

cosign \ across	$+1$	0	0
$+1$			
-1			

5. In the unit tangent bundle of $\mathbb{M}_q$, the master q-modular link is isotoped into a branched surface called the Lorenz template
6. In the limit, we recover an algorithmic formula for linking numbers in terms of the $L\&R$-cycles, using the topology of the Lorenz template.